# Effect of long-range forces on the phase transition in the two-dimensional model of an antiferromagnetic substance

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The free energy and thermodynamic parameters of a two-dimensional decorated model of an antiferromagnet with a long-range potential $a$ (positive or negative) is calculated. It is shown that the introduction of long-range forces leads to a significant change in the behavior of the system near the phase-transition point. A first-order transition in an ideal antiferromagnet becomes a third-order transition upon introduction of long-range interaction if the potential is positive and a first-order transition if the potential negative.

1. A number of recent papers describe the influence of weak perturbations on a system with a second-order phase transition. The turning on of such perturbations can alter significantly the behavior of the system. Allowance for the long-range particle-attraction forces in the hard-sphere model[1] leads to an equation of state of the Van der Waals type. Introduction of a nonmagnetic impurity in an Ising ferromagnet[2,3] leads to a third-order phase transition. Allowance for the influence of the elasticity of the system on the second-order transition in a ferromagnet[4] leads to a first-order transition.

When such systems are considered, one uses as a rule the idea of the isomorphism of phase transitions[5,6]. It is assumed that the singularities of the thermodynamic quantities retain their functional form when a new thermodynamic variable ("generalized coordinate") is added, if the generalized force g conjugate to this variable is fixed. In experiments, however, it is frequently impossible or exceedingly difficult to fix the generalized force, so that it is therefore necessary to perform the experiment with other fixed parameters. In terms of the experimental variables, a strong effect on the behavior of the system is exerted by the regular part of the thermodynamic potential. If the application of the perturbation leads to a change in the main interaction then, as will be shown below, the second-order phase transition in the unperturbed system becomes a first-order phase transition (in terms of the experimental variables) if $\partial^2 F_{reg}/\partial g^2 > 0$, and a third-order transition if $\partial^2 F_{reg}/\partial g^2 < 0$.

We have examined an Ising antiferromagnet with additional long-range potential, both positive and negative. When the signs of the short-range and long-range potentials are the same $\partial^2 F_{reg}/\partial g^2 < 0$ in this case, we obtain a third-order phase transition, and if the signs are different ($\partial^2 F_{reg}/\partial g^2 > 0$) we obtain a first-order transition.

2. We consider a two-dimensional decorated model of the antiferromagnet (superexchange model), which has an exact solution[7]. Only the interaction of the nearest neighbors is taken into account in the lattice (Fig. 1). We add an interaction of constant magnitude between all the "atoms" belonging to antiferromagnetic sublattice ($\sigma_{(ij)}$). The Hamiltonian of this system is

$$\mathscr{H}=\varepsilon_0-H\Sigma+\frac{a}{2N}\Sigma^2;\qquad \varepsilon_0=\sum_{ij}J_{ij}(s_i+s_j)\sigma_{(ij)},\quad \Sigma=\sum_{ij}\sigma_{(ij)}. \tag{1}$$

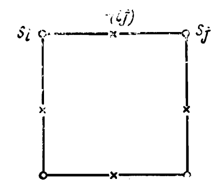


FIG. 1

Here $s_i = \pm 1$ is the "spin" of the lattice site that realizes the superexchange interaction between the "atoms" $\sigma_{(ij)} = \pm 1$ of the antiferromagnetic sublattice, (ij) are the coordinates of the bond on which the "atom" is located. The interaction energy is $J = J_{ij} > 0$ for horizontal bonds and $J_{ij} < 0$ for vertical bonds (Fig. 1). The solutions for the model without allowance for the long-range forces[7] are given in the Appendix.

Let us estimate the region in which the influence of the long-range interaction is most significant. Comparing the second and third terms of the Hamiltonian (1) we find that in this region $a\Sigma/2N > H$. The magnetic moment of the antiferromagnet near an arbitrary point on the phase-transition line (A.4) is given by

$$M\equiv\sum\sigma_{(ij)}\sim N[M_c(T_c,H_{0c})+B|H-H_{0c}|^{1-\alpha}],$$

where $M_c(T_c, H_{0c})$ is the magnetic moment at the point $T_c(H_{0c})$ of the unperturbed system. Then

$$a(M_c+B|H-H_{0c}|^{1-\alpha})\geqslant(H-H_{0c})+H_{0c},$$

and the principal terms in this inequality $aM_c \sim H_c$ determine the shift of the point of the transition when the additional interaction is turned on (the terms of this type will be considered later on in greater detail); the most important change in the behavior of the system should be expected in the region $|H - H_{0c}| \leq |aB|^{1/\alpha}$.

The partition function of the system with Hamiltonian (1) is equal to

$$Z=\mathrm{Sp}\left\{\exp\left[-\frac{\varepsilon_0}{T}+\frac{H}{T}\Sigma-\frac{a}{2NT}\Sigma^2\right]\right\}. \tag{2}$$

For the calculation we use the method of Berlin and Katz[8]:

$$\exp\frac{|a|}{2NT}(\gamma\Sigma)^2=\sqrt{\frac{N}{2\pi T|a|}}\int_{-\infty}^{+\infty}\exp\left[-\frac{NH_0^2}{2|a|T}+\frac{H_0\gamma}{T}\Sigma\right]dH_0. \tag{3}$$

Here $\gamma = 1$ for $a < 0$ and $\gamma = i$ for $a > 0$. We have

$$Z=\exp\left(-\frac{F_n}{T}\right)=\sqrt{\frac{N}{2\pi|a|T}}\int_{-\infty}^{+\infty}\exp\left(-\frac{NH_0^2}{2|a|T}\right)\mathrm{Sp}\exp\left[-\frac{\varepsilon_0}{T}\right.$$

$$+\frac{H+\gamma H_0'}{T}\Sigma\Big]dH_0'=\frac{1}{\gamma}\sqrt{\frac{N}{2\pi|a|T}}\int_{-\gamma\infty}^{+\gamma\infty}\exp\left(\frac{N(H-H_0)^2}{2aT}\right)\operatorname{Sp}\exp\Big[-\frac{\varepsilon_0}{T}$$

$$+\frac{H_0}{T}\Sigma\Big]dH_0=\frac{1}{\nu}\sqrt{\frac{N}{2\pi|a|T}}\int_{-\gamma\infty}^{+\gamma\infty}\exp\left[\frac{N(H-H_0)^2}{2aT}-\frac{NF(T,H_0)}{T}\right]dH_0. \tag{4}$$

Calculating the integral (4) by the saddle-point method, we find that the free energy of the system, with allowance for the long-range force $F_n(T, H)$ is determined in terms of the free energy of the ideal system

$$F_n(T,H)\equiv F_n(T,H;H_0)=F(T,H_0)-\frac{1}{2a}(H-H_0)^2, \tag{5a}$$

where $H_0$ is determined from the equation for the saddle point

$$\partial F_n/\partial H_0=0. \tag{5b}$$

The internal field $H_0$ is the "generalized force" in this equation, and at constant $H_0$ the phase transition in the system with long-range action is isomorphic to the phase transition in the ideal system.

Taking expression (A.1) for $F(T, H_0)$ into account and expanding the nonsingular quantities ln A and $T^*$ in a Taylor series near the transition point ($T_c$, $H_{0c}$), we obtain an expression for the free energy of the antiferromagnet:

$$F(T,H_0)=-B|\tau^*|^{2-\alpha}-TB_1h_0+TB_2\tau, \tag{6}$$

where

$$\tau=\frac{T-T_c}{T_c},\quad h_0=H_0-H_{0c},\quad \tau^*=\frac{T^*-T_c^*}{T_c^*}=ch_0+b\tau. \tag{7}$$

All the constants in these expressions are positive, while $B_1$ and C are proportional to tanh ($H_{0c}/T_c$) and tend to zero as $H_{0c}\to 0$. Differentiating (6) with respect to $H_0$, we find that $TB_1 = M_c$ (the magnetic moment of an ideal antiferromagnet on the transition curve ($T_c$, $H_{0c}$)) and that $M_c$ is equal to zero in a zero field.

Let the internal critical field $H_0(T_c)$ correspond to an external critical field $H_{cr}(T_c)$. Substituting (6) and (7) in (5a) and putting $h = H_{cr}$, we obtain the free energy of the system:

$$F(T,H)=-B|\tau^*|^{2-\alpha}-M_ch_0+TB_2\tau-\frac{[(h-h_0)+(H_{cr}-H_{0c})]^2}{2a}. \tag{8a}$$

From (5b) we have

$$\frac{\partial F_n}{\partial H_0}=-B(2-\alpha)(\operatorname{sign}\tau^*)|\tau^*|^{1-\alpha}\frac{\partial\tau^*}{\partial H_0}-M_c+\frac{1}{a}(h-h_0)+\frac{1}{a}(H_{\text{кр}}-H_{0c}). \tag{8b}$$

We see therefore that at the transition point ($\tau^* = 0$, $h = 0$, $h_0 = 0$) the external critical field is equal to $H_{cr} = H_{0c} + aM_c$.

We shall consider henceforth two different approaches to the critical point (two sets of experimental variables): 1) $T = T_c$ = const, 2) $H = H_{cr}$ = const.

In the case T = const, $\tau = 0$, and $\tau^* = ch_0$, Eqs. (8a) and (8b) take the form

$$F_n(T,H)=-Bc^{2-\alpha}|h_0|^{2-\alpha}-M_ch_0+[(h-h_0)+aM_c]^2/2a, \tag{9a}$$

$$(h-h_0)/a=Bc^{2-\alpha}(2-\alpha)(\operatorname{sign}h_0)|h_0|^{1-\alpha}. \tag{9b}$$

Starting from (6) and (9), we obtain the magnetic moment, the susceptibility, and $\partial\chi/\partial H$:

$$M_0(T,H_0)=-\frac{\partial F(T,H_0)}{\partial H_0}=M_c+Bc^{2-\alpha}(2-\alpha)(\operatorname{sign}h_0)|h_0|^{1-\alpha},$$

$$M(T,H)=-\frac{\partial F_n}{\partial H}=M_c+\frac{h-h_0}{a}=M_c+Bc^{2-\alpha}(2-\alpha)(\operatorname{sign}h_0)|h_0|^{1-\alpha}=M_0(T,H_0), \tag{10}$$

$$\chi_0=\partial M_0(T,H_0)/\partial H_0=Bc^{2-\alpha}(1-\alpha)(2-\alpha)|h_0|^{-\alpha}.$$

From (9b) we determine the value of $\partial h_0/\partial h$:

$$\frac{\partial h_0}{\partial H}=\frac{1}{1+ac^{2-\alpha}(1-\alpha)(2-\alpha)B|h_0|^{-\alpha}}=\frac{1}{1+a\chi_0(T,H_0)}.$$

Then

$$\chi=\frac{\partial M(T,H)}{\partial H}=\frac{1}{a}\left(1-\frac{\partial h_0}{\partial h}\right)=\frac{\chi_0(T,H_0)}{1+a\chi_0(T,H_0)}, \tag{11}$$

$$\frac{\partial\chi}{\partial H}=\frac{\partial\chi_0}{\partial H_0}\frac{1}{[1+a\chi_0]^3}. \tag{12}$$

The subscript 0 labels here quantities pertaining to the ideal system. The connection between quantities pertaining to the complete and ideal systems (10) and (11) has the same functional form as for a ferromagnet with allowance for the demagnetizing factors[9]. The values of the internal field $H_0$, which enter in $M_0(T, H_0)$ and $\chi_0(T, H_0)$ are determined by the steepest-descent equation (9b).

We consider now the case H = const (h = 0). The deviation of $h_0$ from zero is due to the deviation of the temperature from the critical value. From (7) we have $h_0 = (\tau^* - b\tau)/c$. We write down the free energy (5a) and the steepest-descent equation (5b) in terms of the variables $\tau$ and $\tau^*$:

$$F_n(T,H)=-B|\tau^*|^{2-\alpha}-M_c\frac{\tau^*-b\tau}{c}+TB_2\tau+\left[aM_c-\frac{\tau^*-b\tau}{c}\right]\Big/2a, \tag{13a}$$

$$\frac{b\tau-\tau^*}{a}=Bc^2(2-\alpha)(\operatorname{sign}\tau^*)|\tau^*|^{1-\alpha}. \tag{13b}$$

We obtain the expressions for the entropy, specific heat, and $\partial C/\partial T$ (just as in the preceding case, the subscript 0 labels quantities pertaining to the ideal system):

$$S_0(T^*)=-\frac{\partial F(T,H_0)}{\partial T^*}=B(2-\alpha)(\operatorname{sign}\tau^*)|\tau^*|^{1-\alpha}-\frac{M_c}{c},$$

$$C_0(T^*)=T^*\frac{\partial S_0(T^*)}{\partial T^*}=T^*(1-\alpha)(2-\alpha)B|\tau^*|^{-\alpha}, \tag{14}$$

$$S(T,H)=-\frac{\partial F_n}{\partial T}=\frac{b}{ac^2}(b\tau-\tau^*)-TB_2=bB(2-\alpha)(\operatorname{sign}\tau^*)|\tau^*|^{1-\alpha}-TB_2.$$

Obtaining $\partial\tau^*/\partial\tau = b/[1 + aC_0(T^*)]$, from the saddle-point equation (13b), we get

$$C=T\frac{\partial S}{\partial T}=\frac{Tb}{a}\left(b-\frac{\partial\tau^*}{\partial\tau}\right)=Tb^2\frac{C_0(T^*)}{1+aC_0(T^*)}, \tag{15}$$

$$\frac{\partial C}{\partial T}=Tb^3\frac{\partial C_0}{\partial T^*}\frac{1}{[1+aC_0]^3} \tag{16}$$

The quantities in these expressions are determined by solving (13b).

3. The steepest-descent equations (9b) and (13b) have the same functional form, which coincides with the steepest-descent equation in the magnetic-impurity model (3) at $a > 0$ and with the steepest-descent equation for an elastic ferromagnet[4] at $a < 0$.

At $a > 0$, the solution of the steepest-descent equation (9b) is

$$h_0\approx h+aQ_1(\operatorname{sign}h)|h|^{1-\alpha}\quad\text{for } |h|>(aQ_1)^{1/\alpha}; \tag{17}$$

$$|h_0|\approx\left|\frac{h}{aQ_1}-\frac{1}{aQ_1}\left|\frac{h}{aQ_1}\right|^{1/(1-\alpha)}\right|^{1/(1-\alpha)},\ \operatorname{sign}h_0=\operatorname{sign}h\ \text{for } |h|<(aQ_1)^{1/\alpha}, \tag{18}$$

where $Q_1 = (2-\alpha)c^{2-\alpha}B > 0$. The solutions for (13b) are analogous to the solutions (17) and (18), with the substitutions $h\to b\tau$, $h_0\to\tau$ and $Q_1\to Q_2=(2-\alpha)Bc^2$.

Substituting (17) and (18) in (10)–(12) and (14)–(16), we obtain far from the transition point ($|h| > (aQ_1)^{1/\alpha}$)

$$M(T,H)\approx M_c+Q_1(\operatorname{sign}h)|h|^{1-\alpha},\quad S\approx bQ_2(\operatorname{sign}\tau)|\tau|^{1-\alpha}, \tag{19}$$

$$\chi(T,H)=\left[a+\frac{|h|^\alpha}{(1-\alpha)Q_1}\right]^{-1},\quad C\approx Tb^2\left[a+\frac{|b\tau|^\alpha}{(1-\alpha)Q_2}\right]^{-1};$$

and near the transition point $(|h| < (aQ_1)^{1/\alpha})$

$$\begin{gathered}M\approx M_c+Q_1(\operatorname{sign} h)\left|\frac{h}{aQ_1}\right|^{1/(1-\alpha)},\quad S\approx bQ_2(\operatorname{sign}\tau)\left|\frac{b\tau}{aQ_2}\right|^{1/(1-\alpha)},\\ \chi=\frac{1}{a}\left(1-\frac{|h/aQ_1|^{1/(1-\alpha)}}{a(1-\alpha)Q_1}\right),\quad C\approx Tb^2\left(1-\frac{|b\tau/aQ_2|^{1/(1-\alpha)}}{a(1-\alpha)Q_2}\right),\\ \frac{\partial\chi}{\partial H}\approx\frac{\alpha}{[(1-\alpha)Q_1]^2}\left|\frac{h}{aQ_1}\right|^{-(1-2\alpha)/(1-\alpha)},\\ \frac{\partial C}{\partial T}\approx\frac{Tb^3\alpha}{[(1-\alpha)Q_2]^2}\left|\frac{b\tau}{aQ_2}\right|^{-(1-2\alpha)/(1-\alpha)}.\end{gathered}\tag{20}$$

Just as in the case of constant T, in the case of constant H the changeover to the experimental variables leads to a third-order phase transition. Far from the transition point, the behavior of an antiferromagnet with additional interaction is close to the behavior of an ideal system. The renormalization region $|h| < (aQ_1)^{1/\alpha}[|\tau| < (aQ_2)^{1/\alpha}]$ has the same form in both cases, and as $H_{0c}\to 0$ its width tends to zero (in both cases the expression for the renormalization region contains the constant $c \sim \tanh(H_{0c}/T_c)$. Figure 2 shows a schematic picture of the renormalization region.

At $a < 0$, in analogy with the case of an elastic ferromagnet[4], we obtain a first-order phase transition. The solution of the saddle-point equations (9b) and (13b) for a stable phase is

$$\begin{gathered}h_0=|aQ_1|^{1/\alpha}+h/|a|,\quad 0<|h|<|h_1|,\\ \tau^*=|aQ_2|^{1/\alpha}+\tau/|a|,\quad 0<|\tau|<|\tau_1|,\end{gathered}\tag{21}$$

and far from the transition point we have

$$\begin{gathered}|h_0|=|h|+|aQ_1|^{(1+\alpha)/\alpha},\quad \operatorname{sign} h_0=\operatorname{sign} h,\quad |h|>|h_1|,\\ |\tau^*|=|\tau|+|aQ_2|^{(1+\alpha)/\alpha},\quad \operatorname{sign}\tau^*=\operatorname{sign}\tau,\quad |\tau|>|\tau_1|.\end{gathered}\tag{22}$$

The quantities $h_1$ and $\tau_1$ coincide with the boundaries of the metastable state

$$|h_1|=\left|\frac{a}{1-\alpha}\right||aQ_1(1-\alpha)|^{1/\alpha},\quad |\tau_1|=\left|\frac{a}{1-\alpha}\right||aQ_2(1-\alpha)|^{1/\alpha}.$$

It is seen from (22) that far from the transition point the behavior of the system differs little from the behavior of an ideal system. Near the transition point we have

$$\begin{gathered}M_1=M_a+Q_1\left|\frac{h}{|a|}+|aQ_1|^{1/\alpha}\right|^{1-\alpha},\quad h>0,\\ M_2=M_a-Q_1\left|\frac{h}{|a|}+|aQ_1|^{1/\alpha}\right|^{1-\alpha},\quad h<0,\\ \chi=\frac{(1-\alpha)}{|a|}Q_1\left|\frac{h}{|a|}+|aQ_1|^{1/\alpha}\right|^{-\alpha}.\end{gathered}\tag{23}$$

The jump of the magnetic moment on the transition curve is $\Delta M = 2Q_1|aQ_1|^{(1-\alpha)/\alpha}$. Analogously, at constant H we have

$$\begin{gathered}S_1=S_0+bQ_2\left|\frac{b\tau}{|a|}+|aQ_2|^{1/\alpha}\right|^{1-\alpha},\quad \tau>0,\\ S_2=S_0-bQ_2\left|\frac{b\tau}{|a|}+|aQ_2|^{1/\alpha}\right|^{1-\alpha},\quad \tau<0,\\ C=\frac{Tb^2(1-\alpha)}{|a|}\left|\frac{b\tau}{|a|}+|aQ_2|^{1/\alpha}\right|^{-\alpha}.\end{gathered}\tag{24}$$

The jump of the entropy at the transition point is $\Delta S = 2bQ_2|aQ_2|^{(1-\alpha)/\alpha}$.

4. One can expect the described behavior of an antiferromagnet with long-range action to be characteristic also under more general assumptions. We have used in the calculations the isomorphism of two-dimensional models: the decorated model of an antiferromagnet, and the ferromagnetic Ising model without a field. A similar isomorphism holds also for three-dimensional decorated lattices (only the constants in Eqs. (A.1), (A.5) and in the final results are altered). It can be assumed that in more general forms the phase transition in the antiferromagnet is likewise isomorphic to the phase transition in the ferromagnet without a field, and that an equation of the type (A.1) holds, and can be used to obtain the same result as for the two-dimensional decorated model.

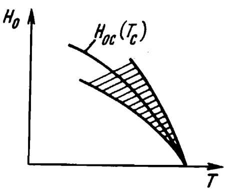


FIG. 2

We consider now a more general potential of the type $v_{ij} \sim 1/r_{ij}^{d+\sigma}$ (d is the dimensionality of the space and $\sigma$ is a small positive quantity). We regard the influence of the long-range forces as a small perturbation. We add and subtract from the Hamiltonian the term $H_0\Sigma s_i$. The Hamiltonian of the system takes the form

$$\begin{gathered}\mathscr{H}=\mathscr{H}_0+\mathscr{H}_{\text{int}},\\ \mathscr{H}_0=\sum_{ij}J_{ij}^a s_i s_j-H_0\sum s_i,\quad \mathscr{H}_{\text{int}}=\sum_{ij}v_{ij}s_is_j-(H-H_0)\sum s_i,\end{gathered}\tag{25}$$

$J_{ij}^a$ is the antiferromagnetic nearest-neighbors interaction and $H_0$ is the internal field determined from the equilibrium condition.

In first-order approximation, the free energy takes the form

$$\exp\{-F_n(T,H)/T\}=\int dH_0\exp\left\{-\frac{F_0(T,H_0)}{T}-\frac{1}{T}\langle\mathscr{H}_{\text{int}}\rangle\right\},\tag{26}$$

where $\langle\ \rangle$ denotes averaging over the ideal system with the Hamiltonian $\mathscr{H}_0$ that depends on $H_0$,

$$\langle\mathscr{H}_{\text{int}}\rangle=\sum v_{ij}\langle s_is_j\rangle-(H-H_0)\sum\langle s_i\rangle$$

and at $H_0 \neq 0$ we have in the molecular-field approximation

$$\langle\mathscr{H}_{\text{int}}\rangle\sim\sum v_{ij}\langle s_i\rangle\langle s_j\rangle-(H-H_0)\sum\langle s_i\rangle=N\left(M_0^2\sum_j v_{ij}-(H-H_0)M_0\right),\tag{27}$$

where $M_0 = M_0(H_0)$ is the magnetic moment in the ideal system. Substituting (27) in (26) and evaluating the integral by the saddle-point method, we obtain

$$F_n(T,H)=F_0(T,H_0)+\sum_j v_{ij}M_0^2-(H-H_0)M_0,\tag{28a}$$

$$\frac{\partial F_n(T,H)}{\partial H_0}=\frac{\partial M_0}{\partial H_0}\left[2\sum_j v_{ij}M_0-(H-H_0)\right].\tag{28b}$$

These equations correspond to formulas (5a) and (5b). We see from (28a) that an antiferromagnet with long-range forces can be regarded as two subsystems: 1) an antiferromagnet in the field $H_0$, and 2) an antiferromagnet in the field $(H - H_0)[F_{add}]$. At $\Sigma v_{ij} < 0$, the second subsystem is stable $(\partial^2 F_{add}/\partial H_0^2 > 0)$, and a first-order transition is observed, whereas at $\Sigma v_{ij} > 0$ the system is unstable $(\partial^2 F_{add}/\partial H_0^2 < 0)$ and a third-order transition is observed.

5. We have thus shown that if introduction of a long-range leads to a change in the principal interaction in the ideal system, then the isomorphism between the

ideal system and the system with the additional interaction in terms of the experimental variables is violated. If the additional subsystem is stable ($\partial^2 F_{add}/\partial H_0^2 > 0$), then a first-order transition takes place in terms of the experimental variables, and if it is unstable ($\partial^2 F_{add}/\partial H_0^2 < 0$), a third-order transition is produced. A similar behavior is exhibited by a ferromagnet with nonmagnetic impurity ($\partial^2 F_{imp}/\partial g^2 < 0$)[3], where the introduction of the impurity leads to a third-order phase transition, and by a ferromagnet with allowance for the elasticity of the system[4], in which a first-order phase transition is observed ($\partial^2 F_{el}/\partial^2 p > 0$).

In conclusion, the author thanks M. A. Mikulinskiĭ for suggesting the topic and for discussions during the course of the work, and E. E. Gorodetskiĭ and E. A. Shapoval for a discussion of the results.

## APPENDIX

We consider a two-dimensional decorated (the so-called superexchange) model of an antiferromagnet[7] (Fig. 1). As shown by Fisher[7], the phase transition in this model, in a nonzero magnetic field, is isomorphic to the phase transition in the two-dimensional Ising model in a zero field. The free energy of the superexchange antiferromagnet is given by

$$F(T, H_0) = F^*(T^*, 0) - T \ln A, \tag{A.1}$$

where $F^*$ is the free energy of the Ising model and $T^*$ is the renormalized temperature, which depends both on T and on the external magnetic field H:

$$\frac{|J|}{2T^*} = \frac{1}{4} \ln \frac{\operatorname{ch}(H_0/T + 2|J|/T)\operatorname{ch}(H_0/T - 2|J|/T)}{\operatorname{ch}^2(H_0/T)}. \tag{A.2}$$

The quantity A is defined by the equation

$$A^4 = \operatorname{ch}\left(\frac{H_0}{T} + \frac{2|J|}{T}\right)\operatorname{ch}\left(\frac{H_0}{T} - \frac{2|J|}{T}\right)\operatorname{ch}^2\frac{H_0}{T}. \tag{A.3}$$

The second-order phase transition to the antiferromagnetic state occurs on the line $T_c(H_{0c})$ defined by the equation

$$\operatorname{ch}\left(\frac{H_{0c}}{T_c} - \frac{2|J|}{T_c}\right)\operatorname{ch}\left(\frac{H_{0c}}{T_c} + \frac{2|J|}{T_c}\right) = \frac{\operatorname{ch}^2(H_{0c}/T_c)}{(\sqrt{2}-1)^2}. \tag{A.4}$$

Near the phase-transition line, the dependence of the magnetic moment on the field is given by

$$M \sim N[M_c(T_c, H_{0c}) + B|H - H_{0c}|^{1-\alpha}], \tag{A.5}$$

where $M_c(T_c, H_{0c})$ is the value of the magnetic moment on the phase-transition line $T_c(H_{0c})$, B is a chosen, and $\alpha$ is the specific-heat exponent ($\alpha \to 0$ in the case of a logarithmic singularity).


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