# Nonstationary structure of a collisionless shock wave

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Solutions of the Korteweg-de Vries equation which are asymptotically correct for $t \to \infty$ are obtained for the problem of decay of the initial discontinuity and for the problem of overturning of the front of a simple wave. In both cases there is an oscillation region which expands in time and is bounded at two sides in space by singularities.

## 1. FORMULATION OF THE PROBLEM. THE BASIC EQUATIONS

As is well known, processes in nondissipative media with weak nonlinearity and weak dispersion (e.g., in a plasma with $T_e \gg T_i$) are described quite well by the Korteweg-de Vries equation[1]:

$$\frac{\partial \eta}{\partial t} + \eta \frac{\partial \eta}{\partial x} + \frac{\partial^3 \eta}{\partial x^3} = 0. \tag{1}$$

So far, the solutions to (1) which have been most thoroughly investigated are those corresponding to problems in which the perturbation at the initial moment of time sufficiently rapidly decreases with increasing $|x|$:

$$\eta \to 0, \quad |x| \to \infty$$

(so that $\int_{-\infty}^{\infty} \eta dx$ has a finite value). In particular, in [2] Karpman determines the number and the amplitudes of the solitons into which such a "local" initial perturbation breaks up in the limit at $t \to \infty$. A convenient method of solving such problems, which is based on the connection between the Eqs. (1) and the Schrödinger equation, is expounded by Garner et al.[3] (see also [4]).

In our opinion, one of the most interesting questions connected with Eq. (1) is the question as to what corresponds in the collisionless case to the shock wave of ordinary hydrodynamics. As was shown by Sagdeev, such a collisionless shock wave has an oscillatory character (see [5,6]). The problems that have thus far been solved cannot however give a quantitative answer to this question. Indeed, it is a priori clear that any local perturbation will asymptotically, as $t \to \infty$, disperse in the same way as it would attenuate in ordinary hydrodynamics because of viscosity. The mean (over the period of the oscillations) value of $\eta$ in a local perturbation tends to zero in time.

In order to investigate the structure of the "shock front" with the aid of (1), it is necessary to consider the problems in which the perturbation is sustained at all t by appropriate boundary conditions at $x \to \pm\infty$. In other words, we must consider those problems in which in ordinary hydrodynamics there exists a shock wave of temporally nondecreasing intensity.

The aim of the present paper is to solve two such problems. First, we shall consider the problem in which the quantity $\eta$ undergoes at the initial moment of time a finite jump at the point $x = 0$. In hydrodynamics a shock wave of constant intensity would be produced in this case (see, for example, [7], Sec. 93). Secondly, we shall consider the situation in the vicinity of the simple-wave breaking point. The exact formulation of this problem will be given below, at the beginning of Sec. 3. In hydrodynamics, a shock wave whose intensity increases in time is produced behind this point (see [7], Secs. 94 and 95).

Let us emphasize that each of these two formulations of the problem is important in its own right. The first formulation is often realized under real experimental conditions, while the second describes the general case of collisionless shock wave "generation."

Qualitatively, the picture of the solution has the same nature in both cases. There is at any $t \gg 1$ a finite region in space occupied by oscillations, the solution being smooth to the right and left sides of this region (see Figs. 2 and 9 below).

In Sagdeev's terminology,[5] the above-described picture corresponds to a "collisionless laminar shock wave." In this case, however, a shock front of definite width is not produced. The resulting oscillation region expands in self-similar fashion in time. The region is bounded by first-order discontinuities, the first-order discontinuity at the leading edge of such a collisionless shock wave being of a singular nature. Let us emphasize in this connection that our formulation of the problem with the assumed total nondissipation differs radically from the problem of the effect of dispersion on the structure of a dissipative shock wave. In the latter case, according to Sagdeev,[5] a stationary oscillating shock front of width determined by the particle mean free path is produced (see also [8]).

We shall construct the solution for large t, when the length of the oscillating region is much larger than the characteristic wavelength of the oscillations. In this case it is natural to use Witham's quasiclassical method [9]. In the limit as $t \to \infty$, the solution obtained in this way will be asymptotically exact.

Let us summarize the results of Witham's work[9] that will be required below. As is well known, Eq. (1) has an exact solution in the form of a stationary running wave:

$$\eta(x, t) = \frac{2a}{s^2} dn^2 \left[ \left( \frac{a}{6s^2} \right)^{1/2} (x - Vt), s \right] + \gamma, \tag{2}$$

where dn(y, s) is the Jacobi elliptic function of modulus $0 \le s \le 1$ (see [10], pp. 910–914). The parameters a, s, and $\gamma$ are arbitrary constants, a determining the amplitude of the oscillations:

$$2a = \eta_{max} - \eta_{min}.$$

The velocity V is equal to

$$V = 2a \frac{2 - s^2}{3s^2} + \gamma. \tag{3}$$

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According to the properties of the elliptic functions, the wave vector of the wave, defined as $2\pi/\lambda$, where $\lambda$ is the wavelength, is given by

$$k=\frac{\pi}{K(s)}\left(\frac{a}{6s^2}\right)^{1/2}. \quad (4)$$

The value of $\eta$ averaged over the period is equal to

$$\bar{\eta}=\gamma+\frac{2aE(s)}{s^2K(s)}. \quad (5)$$

Here K and E are the complete elliptic integrals of the first and second kind respectively.

We shall seek the solution in the form (2), assuming, however, that the parameters a, s, and $\gamma$ (and, consequently, V, k, and $\bar{\eta}$) are slowly varying functions of x and t. According to Witham, these functions should satisfy definite equations. It is convenient to write these equations for three new functions:

$$r_3\geqslant r_2\geqslant r_1,$$

connected with a, s, and $\gamma$ according to

$$a=r_2-r_1,\quad s^2=\frac{r_2-r_1}{r_3-r_1},\quad \gamma=r_2+r_1-r_3. \quad (6)$$

The velocity V and the maximum and minimum values of $\eta$ in the wave are then equal to:

$$V={}^1/_3(r_1+r_2+r_3),$$
$$\eta_{max}=r_3+r_2-r_1,\quad \eta_{min}=r_3+r_1-r_2. \quad (7)$$

The equations for $r_\alpha$ have the form[1)]

$$\frac{\partial r_\alpha}{\partial t}+v_\alpha\frac{\partial r_\alpha}{\partial x}=0,\quad \alpha=1,2,3. \quad (8)$$

(There is no summation over $\alpha$ in this formula and everywhere below.)

The three "group velocities" $v_\alpha$ are equal to:

$$v_1=\frac{r_1+r_2+r_3}{3}-\frac{2}{3}(r_2-r_1)\frac{K(s)}{K(s)-E(s)}, \quad (9)$$
$$v_2=\frac{r_1+r_2+r_3}{3}-\frac{2}{3}(r_2-r_1)\frac{(1-s^2)K(s)}{E(s)-(1-s^2)K(s)},$$
$$v_3=\frac{r_1+r_2+r_3}{3}-\frac{2}{3}(r_3-r_1)\frac{(1-s^2)K(s)}{E(s)}$$
$$v_3\geqslant v_2\geqslant v_1.$$

The Korteweg-de Vries equation (1) is invariant under the transformations

$$\eta\to C\eta,\quad x\to x/C^{1/2},\quad t\to t/C^{3/2} \quad (10)$$

(C is an arbitrary constant) and

$$x\to x+Ct,\quad \eta\to\eta+C. \quad (11)$$

And the averaged equations (8) are also invariant under the transformation

$$x\to Cx,\quad t\to Ct. \quad (12)$$

Let us write out for reference the approximate formulas for the functions K(s) and E(s) for $s\to 0$ and $s\to 1$. We have

$$K(s)\approx\frac{\pi}{2}\left(1+\frac{s^2}{4}+\frac{9}{64}s^4+\ldots\right), \quad (13)$$
$$E(s)\approx\frac{\pi}{2}\left(1-\frac{s^2}{4}-\frac{3}{64}s^4+\ldots\right),\quad s\ll 1$$

and

$$K(s)\approx\frac{1}{2}\ln\frac{16}{1-s^2},$$
$$E(s)\approx 1+\frac{1}{4}(1-s^2)\left(\ln\frac{16}{1-s^2}-1\right),\quad (1-s^2)\ll 1.$$

As has already been noted, the parameter $r_2$ can vary from $r_1$ to $r_3$. Therefore, it is natural to assume that the oscillation region is bounded on one side by the point $x^-(t)$ at which $r_2=r_1$. The amplitude a and the parameter s vanish at this point. On the other side, the oscillation region is bounded by the point $x^+(t)$ at which $r_2=r_3$, so that $s=1$ and, according to (4), the wave vector k vanishes. In both problems considered by us $x^+>x^-$ and therefore we shall call the point with $a=0$, $s=0$ the trailing edge and the point with $s=1$ the leading edge. This corresponds to the usual picture, when the higher-amplitude solitons move with higher velocity. It is necessary, however, to emphasize that the existence of solutions with the indicated properties is by no means obvious: it is a hypothesis, the validity of which can be verified only by the actual construction of the solution.

## 2. DECAY OF THE INITIAL DISCONTINUITY[2)]

Let $\eta(x)$ have at the initial moment of time $t=0$ a finite discontinuity, i.e., $\eta=\eta_-$ for $x<0$ and $\eta=\eta_+$ for $x>0$. We shall first assume that $\eta_->\eta_+$. We shall discuss the opposite case at the end of this section. Notice that we can with the aid of the transformations (10) and (11) always reduce the initial conditions to the form

$$t=0,\quad \eta=\begin{cases}1, & x<0\\ 0, & x>0\end{cases}. \quad (14)$$

For the averaged equations (8), such an initial condition should be imposed on $\bar{\eta}$ and the amplitude a should be assumed to be zero at $t=0$. The most important point of the solution is the fact that the initial conditions (14) remain invariant under the substitution $x\to Cx$. Then, as is easy to understand, in virtue of the invariance under (12), the solution should contain x and t only in the combination

$$\tau=x/t. \quad (15)$$

The situation here is the same as in the ordinary hydrodynamics of an ideal fluid.

Thus, we shall assume that

$$r_\alpha=r_\alpha(\tau). \quad (16)$$

The boundary conditions expressed in terms of the variables $\tau$ should, according to the above-described general picture, have the following form. At the "leading edge" of the oscillation region, where $\tau=\tau^+$ we should have

$$s(\tau^+)=1,$$

i.e.,

$$r_3(\tau^+)=r_2(\tau^+)=r_3^+. \quad (17)$$

On account, however, of the continuity of the quantity $\bar{\eta}(\tau)$, we should have

$$\bar{\eta}(\tau^+)=0. \quad (18)$$

Further, for $\tau>\tau^+$, we simply have $r_\alpha=0$. At the trailing edge, where $\tau=\tau^-$, we should have $a=0$, so that

$$r_2(\tau^-)=r_1(\tau^-)=r_1^-, \quad (19)$$
$$\bar{\eta}(\tau^-)=1. \quad (20)$$

For $\tau<\tau^-$, $a=0$ and $\bar{\eta}=1$.

Substituting (16) into (8), we obtain

$$(v_\alpha-\tau)\frac{dr_\alpha}{d\tau}=0. \quad (21)$$

It is easy to understand that a sensible solution can be obtained if we equate to zero the expression in one of the pairs of round brackets in (21) and assume $r_\alpha$ = const. in two other equations. It is evident that the boundary conditions (17) and (19) can be fulfilled only if $r_2$ depends on $\tau$. Therefore the solution of interest to us has the form[3)]

$$v_2 = \tau, \; r_1 = \text{const}, \quad r_3 = \text{const}. \tag{22}$$

Notice first of all that it follows from (5) and the fact that $a/s^2 = r_3 - r_1$ that for $a \to 0$

$$\bar{\eta} = r_3. \tag{23}$$

Then, according to (20), $r_3 = 1$. At the leading edge, we have for $s \to 1$

$$\bar{\eta} = r_1, \tag{24}$$

so that $r_1 = 0$ and $s^2 = r_2$. As a result

$$a = s^2, \quad V = \frac{1+s^2}{3}, \quad \gamma = -(1-s^2).$$

Substituting these values for the parameters into (2), we obtain

$$\eta(x,t) = 2dn^2\left[\left(\frac{1}{6}\right)^{1/2}\left(x - \frac{1+s^2}{3}t\right), s\right] - (1-s^2), \tag{25}$$

and the formula (22) can be rewritten in the form

$$\frac{1+s^2}{3} - \frac{2}{3}\frac{s^2(1-s^2)K(s)}{E(s)-(1-s^2)K(s)} = \frac{x}{t}. \tag{26}$$

The formulas (25) and (26) in the parametric form (the parameter s) completely solve the set problem. Let us emphasize that the quantity $\eta(x, t)$ itself certainly does not depend on $\tau$ only. On $\tau$ only depends the slowly varying parameter s.

Let us specially investigate the behavior of the solution in the vicinity of the leading and trailing edges. At the trailing edge we easily find for $a \to 0$, using the formulas (13), that

$$-1 + {}^3/_2 s^2 = \tau,$$

or

$$\tau^- = -1, \quad a = s^2 \approx {}^2/_3\tau' \quad (\tau = \tau^- + \tau' = -1 + \tau'), \tag{27}$$

so that the amplitude tends to zero as $\tau \to \tau^-$ according to a linear law. The mean value $\bar{\eta}$, on the other hand, tends, as can be seen from (5), to unity quadratically.

At the leading edge, instead of (26), we obtain for $s \to 1$

$$\frac{2}{3} - \frac{1-s^2}{3}\ln\frac{16}{1-s^2} \approx \tau.$$

Hence we find that

$$\tau^+ = {}^2/_3.$$

The law according to which $1 - s^2$ tends to zero has the form

$$(1-s^2)\ln\frac{16}{1-s^2} = 3|\tau''|, \tag{28}$$

where

$$\tau = \tau^+ + \tau'', \quad \tau'' < 0,$$

or, with logarithmic accuracy,

$$1 - s^2 \approx \frac{3|\tau''|}{\ln(1/|\tau''|)}.$$

We see that as $s \to 1$, the wave vector k tends to zero according to the law

$$k \approx \frac{1}{\pi}\left(\frac{2}{3}\right)^{1/2}\frac{1}{\ln(1/|\tau''|)} \tag{29}$$

(neglecting the terms $\sim \ln\ln|\tau''|/\ln|\tau''|$).

Finally, as $s \to 1$, the mean value $\bar{\eta}$ behaves as

$$\bar{\eta} = 4/\ln\frac{1}{|\tau''|}.$$

The function $\bar{\eta}(\tau)$ has at $\tau = \tau^+$ an infinite derivative, so that this point is a peculiar singular discontinuity.

Figure 1 shows the plots of the quantities $\bar{\eta}$, $a = s^2$, and the wave vector k given by (25) and (26). In Fig. 2 we present the values of the unaveraged quantity $\eta(x, t)$ for $t = 50$ and $t = 100$.[4)] It can be seen that the width of the region occupied by the oscillations increases rapidly in time. Here the expansion occurs owing mainly to an increase in the number of oscillations, while the wavelength changes insignificantly. Nevertheless, in the vicinity of the leading edge the period increases gradually and isolated solitons are discharged. According to (29), the distance between these solitons logarithmically increases in time, and this can be seen in Fig. 3, which shows the leading edge for t = 100, 1000, and 10 000. (The scale along the x axis is given at the bottom of the figure.)

We now note that the transition from a "unit" discontinuity to a discontinuity with arbitrary $\eta_-$ is, according to (10), realized by multiplying the values of $\eta$ by $\eta_-$ and simultaneously dividing the values of x by $\eta_-^{1/2}$ and the values of t by $\eta_-^{3/2}$. (Such a scaling law is valid for all t and not only in our asymptotic region.) The values of $\tau$ are then multiplied by $\eta_-$. For example, for a discontinuity with $\eta_- = 2$, the plots in Fig. 2 correspond to $t = 17.7$ and $t = 35.4$ with a corresponding change in the scale along the axes. We point out in this connection that the condition of applicability of the obtained solution for a discontinuity with an arbitrary $\eta_-$ has the form

$$\eta_-^{3/2} t \gg 1 \tag{30}$$

(actually the solution is fairly accurate if $\eta_-^{3/2} t \gtrsim 6$).

The evolution of the initial discontinuity was con-

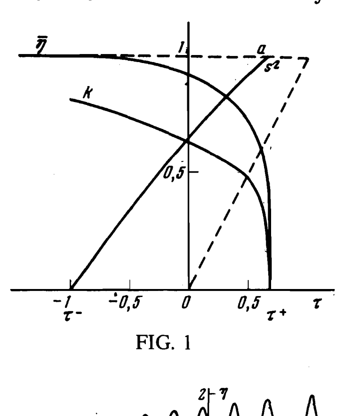


FIG. 1

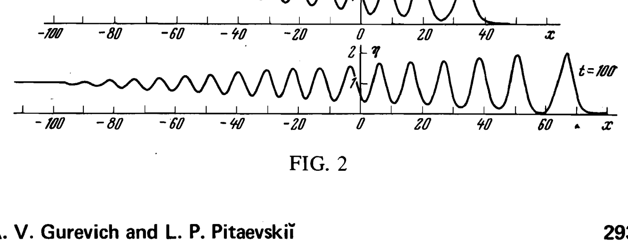


FIG. 2

sidered by Washini and Taniuti[12] on the basis of Eq. (1), which they linearized by dropping the second term. The solution obtained by these authors, however, describes only the initial phase of the process when $\eta_-^{3/2}t \ll 1$ and the nonlinear effects are weak. Subsequently, the process should be described by the above-obtained solution, in which the dispersion and nonlinearity effects are of the same order of magnitude.

Let us now discuss briefly the evolution of a discontinuity of the opposite sign with $\eta_- < \eta_+$. (We can, without loss of generality, set $\eta_- = 0$ and $\eta_+ = 1$.) It is clear that we cannot describe such a discontinuity with a solution of the form (26), since $r_3 > r_1$ and it follows at once from (23) and (24) that $\eta_- > \eta_+$. However, Eqs. (8) then have a trivial nonoscillating solution.

$$\eta = \bar{\eta} = \begin{cases} 0, & \tau < 0 \\ \tau, & 0 < \tau < 1. \\ 1, & \tau > 0 \end{cases} \qquad (31)$$

This solution has two first-order discontinuities—at $\tau = 0$ and $\tau = 1$—and corresponds to the neglect of dispersion, i.e., to the neglect of the term with the third derivative in Eq. (1). Dispersion will be important only for the determination of the structure of these discontinuities. We shall not give an account of the computation. We only note that this structure has an oscillatory character. The width of the discontinuity can be estimated by comparing the terms $\partial\eta/\partial t$ and $\partial^3\eta/\partial x^3$ in this equation. It turns out to be $\sim t^{1/3}$ (see similar arguments at the end of Sec. 89 in [7]). Since, according to (31), the whole picture expands linearly with the time t, the discontinuity can be assumed to be infinitely narrow at sufficiently large t.

We note that if we tried to apply the formula (31) to the discontinuity with $\eta_- > \eta_+$ considered earlier, we would obtain the meaningless ambiguous solution represented by the dashed curve in Fig. 1.

We recall in conclusion that the Korteweg-de Vries equation (1) describes perturbations running in only one direction. If an initial density discontinuity (the dashed curve in Fig. 4) is assigned in some real medium—for example, in a plasma—then two perturbations are generated which propagate in opposite directions. The structure of one of them will be described by the formulas (25) and (26), while the structure of the other will be described by (31) (see Fig. 4). It is obvious that in order for the Korteweg-de Vries equation to be applicable at all, the initial discontinuity should be sufficiently small.

To the experimental investigation of the decay of a small discontinuity in a plasma at $T_e/T_i \sim 20$ is devoted Taylor, Baker, and Ikezi's work.[13] The observed picture is qualitatively similar to our solution. A quantitative comparison is, however, impossible due, in particular, to the insufficiency of the observation time, which, under the conditions of [13], was limited by collisions [14].

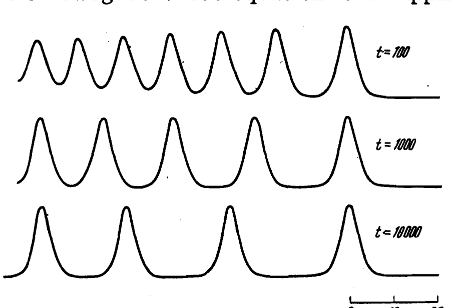


FIG. 3

FIG. 4

## 3. THE APPEARANCE OF OSCILLATIONS IN THE VICINITY OF THE FRONT-BREAKING POINT

Let us consider some perturbation whose initial dimension $L \gg 1$. Then dispersion can be neglected in the beginning in Eq. (1). The nonlinear effects will in due course lead to the growth of the slope of the perturbation front, so that $\partial\eta/\partial x$ will become infinite at some moment of time (see the curves a and b in Fig. 5). At this moment there occurs "a breaking of the front," so to speak, and upon the subsequent increase of t the solution of (1) obtained upon the neglect of dispersion formally becomes three-values (see the curve c in Fig. 5).

In ordinary hydrodynamics, there arises after the breaking of the front a second-order discontinuity—a shock wave (the dashed line in Fig. 5c). In Eq. (1), on the other hand, dispersion becomes important in the vicinity of the breaking point and there is formed after the breaking an expanding region filled with oscillations (see Fig. 5d).

To describe the phenomena in the vicinity of the breaking point, we note first of all that we can, after neglecting dispersion, i.e., the term with the third derivative, write Eq. (1) as

$$\frac{\partial\eta/\partial t}{\partial\eta/\partial x} = -\left(\frac{\partial x}{\partial t}\right)_\eta = -\eta,$$

so that the general solution has the form

$$x = \eta t + P(\eta), \qquad (32)$$

where $P(\eta)$ is an arbitrary function. At the breaking moment, when the point with

$$(\partial x/\partial\eta)_t = 0 \qquad (33a)$$

appears first, we should also have at that same point

$$(\partial^2 x/\partial\eta^2)_t = 0, \qquad (33b)$$

it being always possible to choose the reference points for t, x, and $\eta$, such that $t = x = \eta = 0$ at this point. Expanding $P(\eta)$ near this point and taking (33a) and (33b)

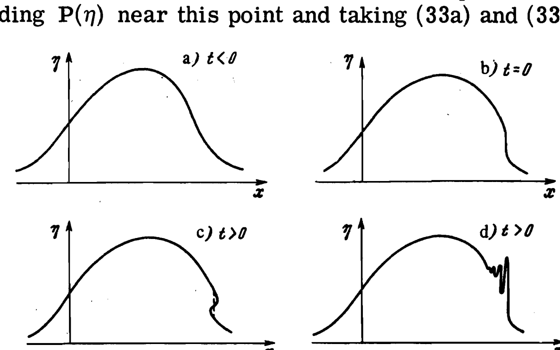


FIG. 5

into account, we find that

$$P(\eta) \approx -\mu\eta^3,$$

it being always possible to choose $\mu = 1$ with the aid of the transformation (10).[5)]

Finally, the solution in the vicinity of the breaking point is described by the formula

$$x = \eta t - \eta^3. \qquad (34)$$

It is shown in Fig. 6. It becomes nonunique at $t > 0$ (dashed curve). In fact, oscillations appear at $t > 0$. We shall again assume that they occupy a finite region of space. In this region, the solution is described by Eqs. (8). Outside the oscillation region, however, (34) is valid as before, so that $\eta$ there has the form

$$\eta = t^{1/2}\theta(z), \quad z = x / t^{3/2}, \qquad (35)$$

where $\theta$ is determined by the equation

$$z = \theta - \theta^3. \qquad (36)$$

In order to join the solutions (8) and (36) at the boundary of the oscillation region, we must set[6)]

$$r_\alpha = t^{1/2} l_\alpha (x / t^{3/2}) = t^{1/2} l_\alpha (z). \qquad (37)$$

Thus, in this case the region occupied by the oscillations expands $\sim t^{3/2}$, while the amplitude of the oscillations increases for a fixed $z$ as $t^{1/2}$. Substituting (37) into (8), we obtain for $l_\alpha$ a system of three ordinary differential equations:

$$\frac{dl_\alpha}{dz} = \frac{l_\alpha}{3z - 2u_\alpha}, \quad \alpha = 1, 2, 3, \qquad (38)$$

where $u_\alpha = v_\alpha / t^{1/2}$, so that the $u_\alpha$ can be expressed in terms of the $l_\alpha$ by the same formulas (9) that give the $v_\alpha$ in terms of the $r_\alpha$.

In order for the self-similar solution (37) to be valid, the size of the region occupied by the oscillations should be large compared to the wavelength and small compared to the dimension L of the initial perturbation. As we shall see (see formula (46) below), to the boundaries of the oscillation region correspond values of $z \sim 1$. Returning to the case $\mu \neq 1$, the condition of applicability can be written in the form

$$\mu^{1/4} \ll x \sim \frac{t^{3/2}}{\mu^{1/2}} \ll L. \qquad (39)$$

The system (38) cannot, of course, be solved in quadratures. It requires a numerical integration. We shall now investigate the form of the solution near the trailing and leading edges of the oscillation region.

Let us begin with the trailing edge. At the trailing edge, as $z \to z^-$, we should have $a = 0$, i.e., $l_1 = l_2$. Then $u_1 = u_2$. It is a priori clear that the denominators of the right hand side of (38) in the equations for $l_1$ and $l_2$ should vanish at this point. In the opposite case, we would obtain for small values of the difference $l_2 - l_1$ a homogeneous linear equation with constant coefficients whose exponential solution cannot vanish. The vanishing, however, of the denominators creates in the equations a situation of the type of the quantum mechanical "fall to the center," as a result of which the wave amplitude vanishes at such a singular point.

For $l_2 = l_1 = l_1^-$, we have

$$u_1 = u_2 = 2l_1^- - l_3^-,$$

so that

$$z^- = {}^4/_3 l_1^- - {}^2/_3 l_3^-. \qquad (40)$$

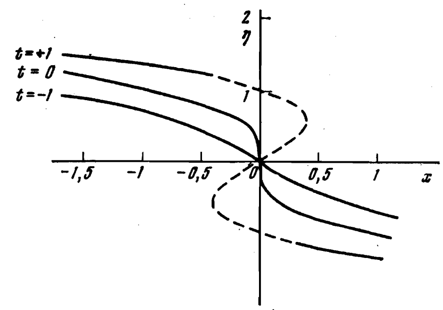

FIG. 6

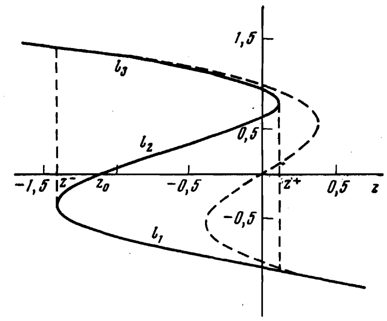

FIG. 7

On the other hand, according to (23), for $a \to 0$,

$$\bar{\theta}^- = l_3^-,$$

and from the condition for matching with the solution (36), we find

$$z^- = l_3^-[1 - (l_3^-)^2], \qquad (41)$$

it being necessary to choose values of $l_3^- > 0$, since the solution should be matched with that branch of the function $\theta(z)$ that stretches to $z = -\infty$ (see Fig. 7).

To determine the law according to which the amplitude of the oscillations tends to zero, we write out the equations for the small quantities $l_1'$ and $l_2'$:

$$l_1 = l_1^- + l_1', \quad l_2 = l_1^- + l_2'.$$

Investigation shows that the vicinity of the point $z = z^-$

$$|l_2' - l_1'| \gg |l_2' + l_1'|, \; z' \quad (z = z^- + z').$$

Expanding in terms of $l_2'$ and $l_1'$ and setting $l_2' \approx -l_1'$, we obtain

$$\frac{dl_2'}{dz'} = -\frac{l_1^-}{2l_2'},$$

whence

$$(l_2')^2 \approx -l_1^- z', \quad s^2 = \frac{2|l_1^- z'|^{1/2}}{l_3^- - l_1^-}. \qquad (42)$$

Since at the trailing edge we should have $z' > 0$, it follows from (42) that $l_1^- < 0$.

Further, it can be seen from (42) that the amplitude of the oscillations tends to zero as $z \to z^-$ according to a root law. However, since, as can easily be verified with the aid of (5), only the square of the amplitude enters into the expression for the mean value $\bar{\theta}$ for $s \to 0$, $\bar{\theta}$ tends to its limiting value according to a linear law.

Let us turn to the investigation of the leading edge, where at some value of $z = z^+$ we should have $s = 1$, i.e., $l_2 = l_3 = l_3^+$. Besides, it is easy to show that at this point $u_2 = u_3$ and the denominators of the right hand side of the second and third equations of (38) should vanish. Since $u_2 = u_3 = \frac{2}{3}l_3^+ + \frac{1}{3}l_1^+$ when $s = 1$, we obtain from this relation a relation connecting $l^+$ and $l_3^+$ with $z^+$:

$$z^+ = {}^4/_9 l_3^+ + {}^2/_9 l_1^+. \tag{43}$$

The condition of continuity of $\overline{\theta}$ with allowance for (24) gives

$$z^+ = l_1^+(1 - l_1^{+2}), \quad l_1^+ < 0. \tag{44}$$

To determine the law according to which $1-s$ tends to zero, we must write out the equations for

$$l_3'' = l_3 - l_3^+, \quad l_2'' = l_2 - l_3^+.$$

It can be shown that again for $s \to 1$

$$|l_3'' - l_2''| \gg |l_3'' + l_2''|, \ |z''|$$
$$(z'' = z - z^+ < 0).$$

In this case it is sufficient to restrict ourselves in the equations to the first two terms of the expansion in powers of $(1-s^2)\ln[1/(1-s^2)]$. We obtain

$$\frac{dl_{2,3}''}{dz} = \frac{3}{2}\frac{l_3^+}{l_3^+ - l_1^+}\left[\pm\left((1-s^2)\ln\frac{16}{1-s^2}\right)^{-1} - \frac{1}{4}\right].$$

The equation for $l_3'' - l_2''$ then has the form

$$\frac{d(l_3'' - l_2'')}{dz''} = -\left\{4(l_3'' - l_2'')\ln\left[\frac{16(l_3^+ - l_1^+)}{l_3'' - l_2''}\right]\right\}^{-1}.$$

Integrating, we find the law according to which $1-s^2$ tends to zero:

$$(1-s^2)^2\left(\ln\frac{16}{1-s^2} + \frac{1}{2}\right) = -\frac{6l_3^+ z''}{(l_3^+ - l_1^+)^2}, \quad z'' < 0, \ l_3^+ > 0. \tag{45}$$

As in the preceding case, the wave vector $k$ tends to zero in proportion to $(\ln|z''|)^{-1}$ as $z'' \to 0$, The mean value $\overline{\theta}$ also tends to its limiting value, as $z'' \to 0$, according to the law $(\ln|z''|)^{-1}$. Therefore, $\theta(z)$ again has at the leading edge a singularity with an infinite derivative. Adding the equations for $l_3''$ and $l_2''$, we obtain

$$l_3'' + l_2'' = -\frac{3}{4}\frac{l_3^+ z''}{l_3^+ - l_1^+}.$$

The Eqs. (38) for the determination of the values of $l_\alpha$ in the whole oscillation region were solved numerically. The results are shown in Fig. 7. We see that $l_3 > 0$ and $l_1 < 0$ at all values of z. The parameter $l_2$, on the other hand, changes its sign at some point $z_0$. This point is a singular point. Indeed, it can be seen from the equation for $l_2$ that this quantity can vanish only at the point where the denominator $3z - 2u_2$ vanishes. It is easy to show, however, that the singularity at the point $z_0$ is quite weak. Not only the quantities $l_\alpha$, but also their first derivatives, are continuous at this point. Nevertheless, the existence of such a singularity is important: the boundary conditions, it turns out, can be satisfied only because of its existence. The point is that the formulas (40)–(41) and (43)–(44) impose four conditions on the solution, whereas the general solution of the system (38) contains only three arbitrary constants. The missing constant is provided owing to the existence of the singular point $z_0$. In order to understand how this occurs, we shall construct the solution from both sides—from the a priori unknown points $z^-$ and $z^+$. After satisfying (40)–(41) and (43)–(44), we shall be left with two arbitrary constants, and as these constants we can choose, for example, the values $l_3^-$ and $l_1^+$. Let us set one of them so that the vanishing of $l_2$ from both sides occurs at one and the same point $z_0$ and choose the second so that the quantity $l_3$ is continuous at $z = z_0$. Then the continuity of $l_1$ will be guaranteed automatically due to the fact that

$$3z_0 = 2u_2 \text{ for } z = z_0.$$

The solution was in fact constructed precisely in this way.

The curious similarity between the graph of $l_\alpha(z)$ and the formal three-valued solution to Eq. (36) should be noted. (The graph of the three-valued function $\theta(z)$ is shown in Fig. 7—the dashed curve.) Such a similarity between the graph for $r_\alpha$ and the three-valued solution for $\eta$ can also be seen in Fig. 1. Thus, the existence of solutions with oscillations is intimately connected with the existence of a region where the solution of the dispersionless equation is nonunique.

The graphs of Fig. 7, together with the formulas (2), (6), and (37), solve the set problem of the determination of $\eta(x, t)$ for the case when the front breaks.

The computations yield the following values for the coordinates of the singular points of the solution:

$$z^- = -1{,}41, \quad z^+ = 0{,}117, \quad z_0 = -1{,}11. \tag{46}$$

The amplitude of the leading soliton is equal to

$$2a^+ = 2(l_2^+ - l_1^+)t^{1/2} = 3{,}69t^{1/2}.$$

It can be seen from the self-similarity relations (37) that the mean value $\overline{\eta}$, the amplitude a, and the wave vector k can be represented in the form

$$\overline{\eta} = t^{1/2}\overline{\theta}(z), \quad a = t^{1/2}b(z), \quad k = t^{1/4}\varkappa(z). \tag{47}$$

The plots of the functions $\overline{\theta}(z)$, $s^2(z)$, $b(z)$, and $\kappa(z)$ are shown in Fig. 8 (let us emphasize that the function $\kappa(z)$ tends smoothly to zero as $z \to z^+$). It can be seen that as $z \to z^+$, the quantities $\overline{\theta}$, and consequently $\overline{\eta}$, undergo a singular jump at the leading edge of the wave. Figure 9 shows the dependence of $\eta$ on x for $t = 5.06$ and 8.35. It can be seen that in time, there occur a rapid expansion of the oscillation region, the growth of the amplitude of the oscillations, and the decrease of the wavelength.

The breaking of the wave front has been experimentally investigated by Alikhanov, Belan, and Sagdeev.[15] Again, there is in this case a qualitative agreement with the theory, while a quantitative comparison is impossible.

In conclusion, let us note that the case investigated in this section of the appearance of oscillations in the vicinity of the point where the front breaks in the Korteweg-de Vries equation is of quite general importance, since this equation is always valid in the vicinity of the breaking point provided the dissipative effects are unimportant. In particular, the obtained solution also describes the breaking of a simple wave constructed in kinetics using the self-similar solution of the expansion of a plasma into a vacuum (see [16]).

Notice also that the constructed solution is a necessary step in the determination of the solution that is valid during an arbitrarily long interval of time after the breaking moment, when the condition (39) is already broken and the self-similarity relations (37) are not valid. In this case our self-similar solution should be taken as an initial condition for the Eqs. (8) at some moment of time $t > 0$ close to the breaking moment. (A rough, but qualitatively correct, estimate of the solution for all t can be obtained if we formally construct the nonunique solution (32) and identify the upper branch

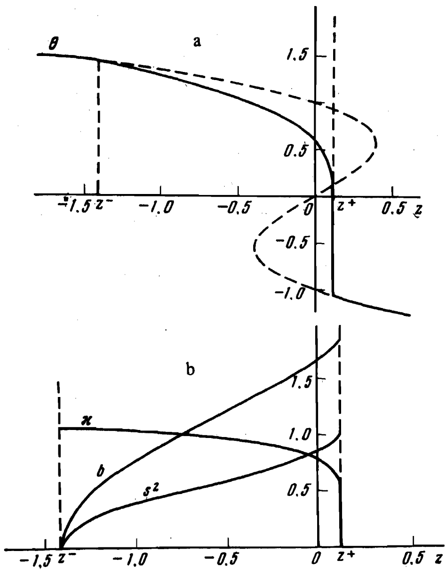

FIG. 8

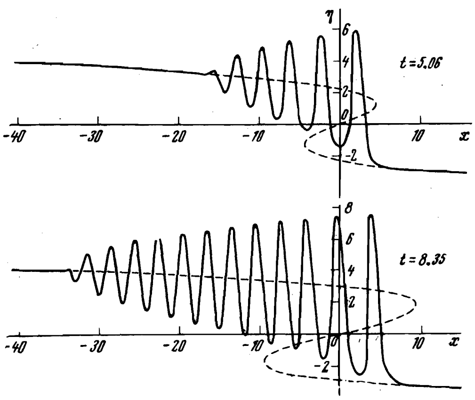

FIG. 9

with $r_3$, the middle branch with $r_2$, and the bottom branch with $r_1$.) It is clear that as long as the Eqs. (8) have a unique solution, the oscillation region will be bounded by singular discontinuities similar to those which are described in our case by the formulas (42) and (45). We recall in this connection that the possibility of the existence of solutions to the system (8) which have second-order discontinuities of the type of shock waves in ordinary hydrodynamics is postulated in Witham's paper.[9] The reality of such discontinuities is, however, quite problematic. It can be seen from the foregoing that in the considered range of problems first-order singular discontinuities at $s = 1$ and $s = 0$ are sufficient and that second-order discontinuities do not arise.

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[1)]In [9] Eq. (1) was written with a coefficient 6 in front of the nonlinear term, whence the slight difference in the form of the formulas.

[2)]A brief account of the results of this section has been published in [11]. In the present paper, in comparison with [11], we make the change of notation: $r_\alpha \to 2r_\alpha$.

[3)]The system (21) also has solutions of the more general form: $r_3$ = const., $r_1$ = const., and $x = v_2 t + P(r_2)$, where $P(r_2)$ is an arbitrary function.

[4)]In the computations with the formula (25), $\eta = 2$ for $\tau = \tau^+$, but the shape of the first "hump" turns out to be distorted in that the front half is "sheared." A more detailed investigation shows that at the leading edge the vibration waves always have the shape of solitons. Therefore, in the plots the leading hump is constructed like a soliton of amplitude $2a = 2$. This circumstance is connected with the insufficient accuracy of the formula (25): the asymptotic nature of the theory does not allow the determination of the exact phase of the wave (25) for not too large values of t. Therefore, the whole picture shown in Figs. 2 and 3 can be moved through a distance of the order of a soliton width (the same thing pertains to Fig. 8).

[5)]To return to the case $\mu \neq 1$, it is sufficient to multiply in the final result the values of $\eta$ by $\mu^{-2/7}$, the values of x by $\mu^{1/7}$, and the values of t by $\mu^{3/7}$. The values of z are then multiplied by $\mu^{-1/2}$.

[6)]In the general case the Eqs. (8) have self-similar solutions of the form $r_\alpha = t^p l_\alpha(x/t^{1+p})$, where p is an arbitrary exponent.

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