"Antidynamo"—A possible mechanism of phenomena occurring in neutral layers of a magnetic field

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A mechanism of rapid annihilation of a magnetic field due to the presence of a velocity field is considered. The conditions for operation of such a mechanism are quite simple: the presence of a certain simple velocity profile v, $v > v_A$ (v_A is the Alfven velocity) and a magnetic Reynolds number $Rm \gg 1$ ("freezing in" condition). Two models are considered. In one the velocity field is potential and in the other $\operatorname{div} v = 0$. The problem is regarded as one of the eigenvalue type, the equations for the eigenvalues being of the Schrödinger type with a complex potential. For $\mathbf{v} = \nabla\psi$ the potential is real and the lower harmonics are determined by the usual methods. The WKB method is employed when $\operatorname{div} v = 0$. It is shown that the field annihilation time may be very small $\approx L/v$, where L is the characteristic dimension. This time is comparable with that of a solar flare.

Great interest in phenomena connected with rapid changes of magnetic fields, and most frequently in their annihilation, has been recently evinced in geophysics, astrophysics, and also in the theory of Z-pinches. Thus, for example, it is assumed that the energy release in solar flares is connected with realization of magnetic energy in a neutral layer. It is well known that the magnetic energy concentrated in a region having a dimensions on the order of the distance between sunspots would be quite sufficient for a flare. The problem is how to release this energy within such a short time, on the order of 10 minutes (the time of ohmic dissipation amounts to hundreds of years!). A similar situation obtains the theory of solar wind, of the frontal point of the magnetosphere, and of the tail of the earth's magnetosphere. It is known that in the aforementioned phenomena the magnetic field is "frozen-in" in the plasma, but nevertheless much is said in the literature on the "rejoining" of the force line of the magnetic field (separation of "loops" in solar wind, "uncovering" of the magnetosphere during the time of the storm, formation of globules in the transition layer of the magnetic field of the magnetosphere tail). The plasma of the solar wind and of the magnetosphere is collisionless, and therefore practically nondissipative. In view of this, the explanations of these phenomena are based on mechanisms in which the following play an important role: a) the coming together of oppositely directed magnetic fields (shock waves—the Petchek mechanism etc.); b) the decrease of the effective electric conductivity (for which turbulence, predominantly of the ion-acoustic type, is brought into consideration); c) different plasma instabilities.

This article proposes one more mechanism that yields a very rapid decay of the field and is physically extremely simple. The simplicity lies primarily in the fact that the conditions for realization of such rapid decay are very easily attained, namely, they require the presence of a macroscopic (hydrodynamic) motion of the plasma with a certain simple profile. The problem will be posed in analogy with the problem of magnetic-field generation ("dynamo"), namely, the velocity field is specified. This can be done if the kinetic energy $\rho^2/2$ exceeds the magnetic energy $H^2/8\pi$ or, equivalently,

$$v > v_A,$$

where v_A is the Alfven velocity. This condition is realized on the sun because of the large density; in a collisionless plasma it can be realized if the motion is produced not by a magnetic field, but by the plasma pressure. We note that the velocity of flow of solar wind around the earth's magnetosphere exceeds v_A. Of course, we are interested here not in the generation of the field, but, to the contrary, in its rapid decay, and therefore this mechanism is called "antidynamo." The idea of such a mechanism was proposed to the author by R. Z. Sagdeev.

Obviously, this mechanism should result in an effective decrease of the scale of the field, and then in its decay as a result of the finite electric conductivity. We note that it causes a faster damping of the field than magnetohydrodynamic turbulence does, namely, it will be shown below that the damping occurs within a time L/v (L is the characteristic dimension of the field), whereas turbulent damping occurs in a time L^2/χ_T, where $\chi_T = lv/3$ (l is the scale of the pulsations)[1]. Moreover, magnetohydrodynamic turbulence must still be excited. It should also be borne in mind that the transfer of energy from the regular magnetic field to the small-scale field, which occurs in the presence of magnetohydrodynamic turbulence, differs radically from the process considered here. First, the turbulence is a stochastic phenomenon, in contrast to the given dynamic process. Second, the energy transfer in turbulence is connected with the very presence of an entire spectrum of scales of the velocity field, whereas in our problem the initial scale of the magnetic field is the same as that of the regular velocity field.

The very possibility of accelerated field dissipation under the influence of motion is quite obvious and has been discussed in the literature. Chandrasekhar[2] has considered damping of an axially-symmetric field under the influence of meridional circulation. Gershuni, Zhukhovitskiĭ, and Rudakov[3] have considered the damping of thermal perturbations. The solution and **v** were expanded in terms of several basis functions of the diffusion equation (the Galerkin method), corresponding to a Reynolds number Rm (or a Peclet number Pe) not much larger than unity. We consider below the asymptotic behavior of the field at very large Rm, more accurately $Rm^{1/3} \gg 1$ (condition (33)); it is therefore no accident that the WKB method is used for solenoidal fields; it is precisely in this case that the fastest field damping is obtained.

1. FORMULATION OF PROBLEM. POTENTIAL VELOCITY

If the velocity is given, then the equation of motion need not be used, and the dynamics of the magnetic field is described by the induction equation

$$\partial \mathbf{H} / \partial t = \mathrm{rot}[\mathbf{vH}] + \nu_m \Delta \mathbf{H}. \quad (1)^*$$

Henceforth $\mathbf{v}$ will not depend on t, so that we can substitute the eigenvalue problem

$$\mathbf{H} = e^{-Et}\mathbf{h}(\mathbf{r}), \quad -E\mathbf{h} = \hat{L}\mathbf{h} = \mathrm{rot}\,[\mathbf{vh}] + \nu_m \Delta \mathbf{h}. \quad (2)$$

The boundary conditions must be such that the field has no external sources or sinks; otherwise the dynamics would be determined by external factors. These conditions are easiest to formulate if the outer region is vacuum; then

$$\mathbf{H}_n^{(1)} = \mathbf{H}_n^{(2)}, \quad \mathrm{rot}_n^{(1)} \mathbf{H} = 0, \quad \mathbf{H}_t^{(1)} = \mathbf{H}_t^{(2)}, \quad \mathbf{H}^{(2)} = \nabla \Phi, \quad (3)$$

the superscripts (1) and (2) correspond to the internal and external fields, while n and t are the normal and tangential components of the field. Other simple boundary conditions correspond to the case when the outer region is superconducting; in this case we have

$$\mathbf{H}_n = 0, \quad \mathrm{rot}_t \mathbf{H} = 0. \quad (4)$$

In addition, $\mathbf{v}_n = 0$. We shall henceforth use the boundary conditions (4), since they are the simplest. The operator $\hat{L}$, while linear, is not trivial, since it is neither Hermitian nor anti-Hermitian. This constitutes the principal difficulty. In the present section we use a transformation which reduces the problem to a Hermitian operator for the potential velocity.

Thus, let

$$\mathbf{v} = \nabla \psi. \quad (5)$$

It is easy to show that the transformation

$$\mathbf{h} = \mathbf{f}(\mathbf{r}) \exp({}^1/_2 \psi / \nu_m)$$

leads to the equation

$$\Delta f_i + \frac{f_i}{\nu_m}\left\{\left[E - \frac{v^2}{4\nu_m} - \frac{1}{2}\mathrm{div}\,\mathbf{v}\right]\delta_{ij} + \frac{\partial}{\partial x_j} v_i\right\} = 0. \quad (6)$$

The boundary conditions (4) reduce to

$$\mathbf{f}_n = 0, \quad \mathrm{rot}_t \mathbf{f} = 0. \quad (7)$$

It can be shown that the operator L′ in (6)

$$\hat{L}' = \frac{1}{\nu_m}\left(\frac{v^2}{4\nu_m} + \frac{1}{2}\mathrm{div}\,\mathbf{v}\right)\delta_{ij} - \frac{1}{\nu_m}\frac{\partial}{\partial x_j} v_i - \Delta \delta_{ij}, \quad (8)$$

is Hermitian, i.e.,

$$\int \bar{f} \hat{L}' f \, d\mathbf{r} = \int f L' \bar{f} \, d\mathbf{r} \quad (9)$$

(integration over the entire volume), the superior bar denotes complex conjugation. It follows therefore that f can be expanded in a complete system of eigenfunctions of the operator $\hat{L}'$, and all the eigenvalues are real:

$$f(\mathbf{r}, t) = \sum_n f_n(\mathbf{r}) \exp(-E_n t), \quad \mathrm{Im}\, E_n = 0. \quad (10)$$

From the form (8) of $\hat{L}'$ it is incidentally clear that no transition to the limit $\nu_m \to 0$ can be made, and this is why it is incorrect to expand in terms of Rm^{-1}. At first glance this seems strange: if $\mathrm{Rm} \gg 1$, then the first term of the right-hand side of (2) is much larger than the second. Why is it impossible to discard this last term in first order? The point is that by putting $\nu_m = 0$ we change the order of the system (2). On the other hand, as will be shown below, E_n with small n depends little on ν_m, as can be expected from physical considerations.

We shall now prove that the operator $\hat{L}'$ is positive-definite. To this end it suffices to verify that all $E_n > 0$. We take the scalar product of (2) with $\mathbf{h}\exp(-\psi/\nu_m)$ and integrate over the entire volume. After obvious transformations, using the boundary conditions, we obtain

$$E \int h^2 \exp(-\psi/\nu_m)\, d\mathbf{r} = \int \left(\nu_m^{1/2} \mathrm{rot}\,\mathbf{H} - \frac{[\mathbf{vH}]}{\nu_m^{1/2}}\right)^2 \exp\left(\frac{-\psi}{\nu_m}\right) d\mathbf{r}. \quad (11)$$

It is seen from (11) that $E_n \geq 0$, and equality to zero is attained only if

$$[\mathbf{vH}] = \nu_m \,\mathrm{rot}\,\mathbf{H}. \quad (12)$$

We thus arrive at the following theorem: stationary potential flow does not generate a field, with the possible exception of a stationary (undamped) magnetic field.

We note that a nonstationary potential velocity generates a field (see, for example, the acoustic turbulence in[4]). Equation (12) was analyzed by Pichakhchi[5] and called short-circuited dynamo (in view of the fact that (12) is simply Ohm's law, and curl $\mathbf{E} = 0$ because of the stationarity and because div $\mathbf{E} = 0$, i.e., there is no space charge). Pichakhchi[5] has shown that (12) has no solution under suitable boundary conditions. Thus, all $E_n > 0$.

In concluding this section, let us formulate the eigenvalue problem for the temperature field. If T is the temperature then, as is well known, we have for T the equation

$$\partial T / \partial t + \mathrm{div}\,\mathbf{v}T = \chi \Delta T, \quad (13)$$

χ is the coefficient of molecular thermal conductivity. Let $T = f(\mathbf{r})e^{-Et}$, and then the transformation

$$T(\mathbf{r}) = h(\mathbf{r}) \exp(\psi / 2\chi) \quad (14)$$

leads to a Schrödinger equation without the time:

$$\Delta h + \frac{1}{\chi}(E - U)h = 0, \quad U = \frac{1}{2}\mathrm{div}\,\mathbf{v} + \frac{v^2}{4\chi}. \quad (15)$$

The quantity $1/\chi$ plays the role of $m/\hbar$ in the Schrödinger equation, and has the same dimensionality. In the absence of heat flow through the boundary we have

$$\nabla_n T = 0, \quad \nabla_n h = 0 \quad (16)$$

and the operator $\chi^{-1}U - \Delta$ is Hermitian. It is also easy to show that all $E_n \geq 0$. The transformation (14) may be useful for different heat-exchange problems in the presence of a potential velocity and large Peclet numbers, for example in potential flow around a body. Indeed, the principal role in such processes is played by the lowest harmonics, and these can be obtained from (15) by various methods.

2. "ANTIDYNAMO" IN A POTENTIAL FIELD

The "antidynamo" problem is more convenient to solve than the inverse "dynamo" problem because the latter is essentially three-dimensional, whereas the former can also be two-dimensional. This is precisely the problem which we consider.

Let

$$\mathbf{v} = \{v_x, v_y, 0\}, \quad \mathbf{H} = \{H_x, H_y, 0\}, \quad \partial / \partial z = 0;$$

For the vector potential $\mathbf{A}$ (curl $\mathbf{A} = \mathbf{H}$) we then have $\mathbf{A} = \{0, 0, A\}$ and

$$\partial A / \partial t + \mathbf{v}\nabla A = \nu_m \Delta A. \quad (17)$$

Equation (17) does not differ at all from (13) if div $\mathbf{v}$ = 0. We, however, consider the velocity field (5). The conditions (4) then take the form

$$\nabla_t A = 0, \quad \Delta A|_s = 0. \tag{18}$$

Further, using (15), we obtain (we recall that $v_n = 0$)

$$A|_s = 0, \tag{19}$$

where S is the boundary. A transformation of the type (14) leads to the Schrödinger equation

$$\Delta h + \frac{1}{\nu_m}(E - U)h = 0,$$
$$U = -{}^1/_2 \mathrm{div}\,\mathbf{v} + v^2/4\nu_m, \quad h|_s = 0. \tag{20}$$

The potential U at Rm $\gg 1$ has a very peculiar form: in the main, U is positive, $v^2/4\nu_m \gg (1/2)\mathrm{div}\,\mathbf{v}$, and takes on negative values in small regions where $\mathbf{v} = 0$ and div $\mathbf{v} > 0$. These regions have dimensions $\approx \mathrm{Rm}^{-1/2} l$, and the plasma flows out of them (div $\mathbf{v} > 0$).

Were we to have $\mathbf{v} \equiv 0$, then $U \equiv 0$ and the lower E_n would be of the order of ν_m/L^2, where L is the dimension of the region; the presence of such a large potential (20), on the order of $v^2/4\nu_m$, raises all the lower levels appreciably. This is indeed the "antidynamo," since E_n is the decrement of the field decay. Thus, the presence of U accelerates the field damping. We shall illustrate the statements made in the last paragraph later on, by means of an example, but we show first that all $E_n > 0$ (there is no dynamo). In fact, using (20), we obtain

$$E \int h^2\, d\mathbf{r} = \int \left(\nu_m^{1/2} \nabla h + \frac{\mathbf{v} h}{2\nu_m^{1/2}}\right)^2 d\mathbf{r}. \tag{21}$$

The right-hand side vanishes only if

$$\nu_m \nabla h + \frac{\mathbf{v} h}{2} = 0, \text{ i.e.,}$$
$$h = h_0 \exp\left(-\frac{\psi}{2\nu_m}\right);$$

this solution does not satisfy the boundary condition $h|_S = 0$, and is therefore not suitable. Thus, all $E_n > 0$ (analog of the theorem that no two-dimensional dynamo can exist[6] for a potential velocity field).

We now determine the lower level. We consider for concreteness a region in the form of a square. We demonstrate the raising of the levels with a one-dimensional example. Let ψ be a function of x only; in this case

$$h(x, y) = h_1(x) e^{iky}, \tag{22}$$

k is determined from the boundary conditions, $k \neq 0$, and then there is added to the potential a quantity $\nu_m k^2 \approx \nu_m L^{-2}$, which is small for the lower harmonics and which we introduce into the energy $E_1 = E - \nu_m k^2$. Let furthermore U have the form shown in Fig. 1. This form of U is obtained, for example, if $\psi = a \cos x\pi/L$. To find the lower level, we consider a potential well in the vicinity of x = L. We continue temporarily the potential into the region $L \leq x \leq 2L$, so that ψ is there also equal to

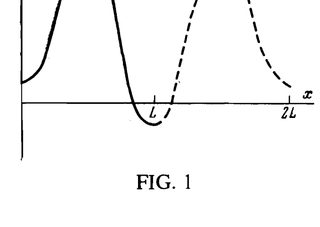

FIG. 1

$a \cos x\pi/L$ (dashed curve in Fig. 1), and stipulate that the continued h vanish at h(2L). Near x = L we have the problem of a bounded harmonic oscillator, shifted into the region of negative energies by an amount $a\pi^2/2L^2$. We put $E_2 = E_1 + a\pi^2/2L^2$; we expand U about x = L:

$$h'' + \frac{1}{\nu_m}\left[E_2 - \frac{a^2\pi^4}{2L^4\nu_m}(x-L)\right]h = 0, \quad E_2^{(n)} = (2n+1)\frac{a^2\pi^4}{\sqrt{2}L^2} \tag{23}$$

for not too large values of n, for which the boundary exerts no influence[7]. It must be borne in mind, however, that the boundary conditions for (23) do not correspond to a quantum oscillator, for which the wave function need not vanish at x = L. It follows therefore that the eigenvalue with n = 0, which reaches a maximum at x = L, is not suitable for our purposes. It is known that at n = 1 the wave function $h^{(1)}(L) = 0$ (the functions $h(L-x)$ are either all even or all odd) and, as follows from the theorem concerning the zeroes of eigenfunctions, it has no other zeroes. This $h^{(-1)}(x)$, which satisfies the fundamental equation in the interval [0, L], vanishes nowhere in this interval, and satisfies the boundary conditions (20), is indeed the ground state (the wave function of the ground state should have no zeroes inside the region). We write out finally the lower eigenvalue:

$$E^{(1)} = \nu_m k^2 + \frac{1}{2}(3\sqrt{2} - 1)\frac{a\pi^2}{L^2}. \tag{24}$$

Recognizing that $k^2 = \pi^2/L^2$, so that the first term of (24) is much smaller than the second term, we find the estimate

$$E^{(1)} \approx v/L. \tag{25}$$

The presence of the second minimum of U at x = 0 does not affect substantially the estimate (25), since $U(0) = 8\pi^2/2L^2$, and for this well we have $E > U(0)$.

Thus, instead of the Ohmic slow damping we obtained a vanishing of the field within a time L/v, which is shorter than the Ohmic time by a factor Rm. This indeed is the "antidynamo" in a potential field.

A few words concerning the physical realization of the model. In the presence of the assumed velocity field, an arbitrary initial field will attenuate rapidly, the higher harmonics vanish very rapidly, while the lower one will correspond to the current layer in the usual flare models. In spite of the effectiveness of the solution, the physical meaning of such a rapid annihilation of the field is quite trivial. The described potential flow compresses the field to small dimensions. Since the reaction of the magnetic field is not taken into account here, the final thickness of the current layer is determined by the Ohmic damping. This follows formally from the solution, namely, expression (24) yields $E^{(1)} > U(0)$, so that the wave function is large not only at $x \approx L$, but also in the vicinity of x = 0. To obtain the vector potential, on the other hand, it is necessary, in accordance with (14), to multiply $h(\mathbf{r})$ by the expression $\exp(\psi/2\nu_m)$, which has a sharp maximum at x = 0. As a result we obtain that A, meaning also the magnetic field, is concentrated mainly at $x \approx 0$, where div $\mathbf{v} < 0$, i.e., where the plasma is compressed. Obviously, under real conditions it is impossible to disregard the electromagnetic forces, the role of which increases with decreasing scale of the field. This procedure is therefore apparently of greater interest for the calculation of the nonstationary temperature field from Eq. (13). Nonetheless, it is probably possible to find a velocity $\mathbf{v} = \nabla\psi$ which is not directed opposite to the electromagnetic field and nevertheless causes rapid dissipation of the field. We shall show in the next section that the antidynamo is produced even if div **v** = 0, and the velocity is perpendicular in this case to the electromagnetic forces.

3. ANTIDYNAMO IN SOLENOIDAL FIELD

We consider now Eq. (17) under the assumption

$$\operatorname{div} \mathbf{v} = 0.$$

We refer here to a paper by Weiss[8], who solved Eq. (1) numerically with **v** stationary. It was shown there that a magnetic field that is homogeneous at the initial instant becomes forced out of the region in which solenoidal motion takes place. The field remains quasihomogeneous, being somewhat distorted on going around the solenoidal region. Since the magnetic field energy is not decreased as a whole, it would be stretching the point quite a bit to interpret this result as the vanishing of the magnetic field inside the region as a result of solenoidal motion. In any case, for a correct formulation of our problem, we need boundary conditions (3) and (4) corresponding to a situation in which the magnetic field inside the solenoidal region is not maintained by external sources, whereas in[8] the field was homogeneous at infinity, meaning that it maintains the field inside the vortex. We consider, however, an even simpler solenoidal motion.

Let $\mathbf{v} = \{0, v_y(x)\}$, so that a solution of (17) in the form $A_1(x)\exp(-Et + iky)$ is meaningful; as a result we obtain

$$A_1'' + \nu_m^{-1}(E - iv(x)k - \nu_m k^2)A_1 = 0. \tag{26}$$

We have arrived at a Schrödinger equation with a complex potential. We introduce next $E_k = E - \nu_m k^2$. The eigenfunctions $A_1(x)$ must be sought in complex form, and the operator is not Hermitian, so that $E^{(n)}$ are likewise generally speaking complex. We represent the general solution in the form

$$A(x, y, t) = \sum_{n,k} A_1^{(n)} \exp(-E_n t + iky) + \bar{A}_1^{(n)} \exp(-\bar{E}_n t - iky). \tag{27}$$

There exists a theorem according to which the eigenfunctions of (26) form a complete system[9] (A should be real). It is easily seen from (26) that $\bar{A}_1^{(n)}$ also satisfies the same equation (26) with E replaced by E and k replaced by $-k$. If we formulate the eigenvalue problem, then we must bear in mind that the flow parallel to y, namely the flow in the channel, cannot be limited in terms of y. If we thus assume that $A(x, y, \psi) = 0$ at $x = \pm L$, then it follows from (27) that

$$A_1^{(n)}(\pm L) = 0 \tag{28}$$

(the real as well as the imaginary part). We assume as before that $Rm \gg 1$. This means

$$v / k\nu_m \gg 1, \quad vL / \nu_m \gg 1. \tag{29}$$

In our problem ν_m is a small quantity, and it is therefore natural to use the WKB method, in which case Rm is precisely the large parameter needed to justify the WKB approach.

Assume that the velocity can be represented in the form

$$v = ax, \quad ak > 0 \tag{30}$$

(Couette flow). With the aid of the transformation

$$Z = x\left(\frac{iak}{\nu_m}\right)^{1/3} - \frac{E_k}{\nu_m^{1/3}(iak)^{2/3}} \tag{31}$$

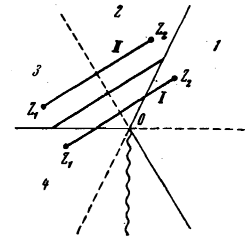

FIG. 2

we change over to an equation in the Airy-Stokes dimensionless variables:

$$A_1'' = ZA_1. \tag{32}$$

From now on we follow Heading[10]. Figure 2 shows the Stokes lines (dashed) and the conjugate Stokes lines (solid) emerging from a point, while the wavy line corresponds to a cut in the Z plane. One of the Stokes lines crosses in Fig. 2 a series of straight lines perpendicular to it, and the solution (26) is defined on one of the straight lines where, namely, x is real on these lines (we call them the solution lines). We put $E_k = E_1 + iE_2$; then the distance to the solution line is $E_1/\nu_m^{1/3}(ak)^{2/3}$. If $E_2 = 0$, then the solution is defined at equal intervals $L(ak)^{1/3}\nu_m^{-1/3}$ from the point x = 0 (intersection of the Stokes line with the solution line). On the other hand if $E_2 \neq 0$, then the region where the solution is defined is shifted along the solution line. We assume that k is not too small ($k \approx 1/L$), and then we have for the entire length interval

$$2L(ak)^{1/3}/\nu_m^{1/3} \gg 1, \tag{33}$$

which is precisely the reason why the asymptotic WKB approximation is used, for no matter where the solution line is located, the region where the solution is defined is always located mainly at values $|Z| > 1$, where the WKB solution is valid. We use Heading's notation

$$(0, Z) = Z^{-1/4} \exp({}^2/{}_3 Z^{3/2}), \qquad (Z, 0) = Z^{-1/4} \exp(-{}^2/{}_3 Z^{3/2}), \tag{34}$$

where d (or $(0, Z)_d$) refers to a solution in which the real part in a given region of the Z plane is positive and $|\exp(2Z^{3/2}/3)|$ is an exponentially large quantity, while s (or respectively $(0, Z)_s$) is ascribed to a solution that is exponentially small. In the regions of interest to us, the general solution takes the form

$$\begin{aligned} &1)\quad a(0, Z)_d + b(Z, 0)_s, \\ &2)\quad a(0, Z)_s + b(Z, 0)_d, \\ &3)\quad (a + ib)(0, Z)_s + b(Z, 0)_d, \\ &4)\quad (a + ib)(0, Z)_d + b(Z, 0)_s. \end{aligned} \tag{35}$$

We denote the Z-plane points corresponding to $x = -L$ and $x = L$ by Z_1 and Z_2. It follows from the boundary conditions that the solution should vanish at these points. Different placements of Z_1 and Z_2 are possible, some of which are shown in Fig. 2; we shall not write out all of them. We note only that some of them can be discarded immediately as not satisfying the boundary conditions. By way of example, we consider a case when Z_1 is in region 3 and Z_2 is in region 1. Then

$$\begin{aligned} &a(0, Z_2)_d + b(Z_2, 0)_s = 0, \\ &a(0, Z_1)_s + b(i(0, Z_1)_s + (Z_1, 0)_d) = 0. \end{aligned} \tag{36}$$

Equating the determinants to zero, we obtain

$$i(0, Z_2)_d(0, Z_1)_s + (0, Z_2)_d(Z_1, 0)_d - (Z_2, 0)_s(0, Z_1)_s = 0; \tag{37}$$

expression (37) cannot be satisfied in any way, since the second term is $|(0, Z_2)_d(Z_1, 0)_d| \gg 1$, and none of the remaining terms can offset it, being all much smaller than the second. By reviewing all the possible variants, we obtain the following conditions: either

$$i(0, Z_2)_s(0, Z_1)_s + (0, Z_2)_s(Z_1, 0)_d - (Z_2, 0)_d(0, Z_1)_s = 0, \quad (38)$$

or

$$(0, Z_2)(Z_1, 0) - (Z_2, 0)(0, Z_1) = 0. \quad (39)$$

It is clear that they are equivalent, since the first term of (38) is exponentially small. We thus have the condition (39), or in Heading's notation

$$[Z_1, Z_2] = [Z_2, Z_1]. \quad (40)$$

We note that we can use also other roots ($i^{1/3}$ and $i^{2/3}$) of expression (31), and then the solution lines make different angles, and the point corresponding to x = 0 lies on another Stokes line. The conclusions, however, remain the same as before.

We rewrite the condition (40) in the form

$$\operatorname{sh} i \int_{-L}^{L} p\,dx = 0, \quad p = \nu_m^{-1/2}(E_k - ivk)^{1/2}. \quad (41)$$

This leads to the "quantization condition"

$$\int_{-L}^{L} p\,dx = n\pi. \quad (42)$$

It should be added that expressions (38)—(42) appear only in the case when Z_1 is in region 3 and Z_2 in region 2, or else when both Z_1 and Z_2 are in the same region. Expanding (42), we obtain

$$\operatorname{Im} \int_{-L}^{L} p\,dx = \frac{1}{\nu_m^{1/2}} \int_{-L}^{L} \left[-\frac{1}{2}E_1 + \frac{1}{2}(E_1^2 + (E_2 - vk)^2)^{1/2} \right]^{1/2} dx = 0, \quad (43a)$$

$$\operatorname{Re} \int_{-L}^{L} p\,dx = \frac{1}{\nu_m^{1/2}} \int_{-L}^{L} \left[\frac{1}{2}E_1 + \frac{1}{2}(E_1^2 + (E_2 - vk)^2)^{1/2} \right]^{1/2} dx = n\pi. \quad (43b)$$

The condition (43a) can be satisfied only if Im p reverses sign. Im p can vanish if

$$-aLk < E_2 < aLk, \quad (44)$$

but this means, as follows from (31), that the region where the solution is defined passes over both sides of the Stokes line and crosses the latter at $x_0 = E_2/ak$. Thus, we are left with the only possibility, namely Z_1 in region 3 and Z_2 in region 2 (Fig. 2, solution line II). Using the condition (43) and (30), we can easily refine (44), namely: (43a) is satisfied under the only condition

$$E_2 = 0. \quad (45)$$

Finally, from an examination of Fig. 2 it is clear that Z_1 and Z_2 do not fall in regions 4 and 1 if L is smaller than the distance from the Stokes line to the conjugate Stokes line along the solution line. Elementary geometry yields

$$E_1 \geqslant Lak/\sqrt{3}. \quad (46)$$

This is already in essence the "antidynamo" and it remains only to show that such $E_1^{(n)}$ exist (moreover, that there exists an infinite denumerable set of $E_1^{(n)}$). We turn therefore to the second "quantization condition" (43b). Re p does not reverse sign, so that we assume that Re p > 0 and n > 0 (the inverse is equivalent to the given case). Integrating, we obtain an equation for E_1

$$\sin \frac{3}{2} \operatorname{Arctg} \frac{Lak}{E_1} = \frac{3}{4} \frac{ak\nu_m^{1/2} n\pi}{(E_1^2 + (Lak)^2)^{3/4}}. \quad (47)$$

The character of the roots can be assessed from the plots of the left-hand side of (47) (curve I of Fig. 3) and of the right-hand side part at n = 1 (curve II) and at n ≫ 1 (curve III); only the roots satisfying Eq. (46) need be considered. Since the right-hand side of (47) increases more rapidly than the left-hand side at large E_1, roots will always exist when n exceeds a certain n_0. It is seen from Fig. 3, incidentally, that $E_1(n) > 0$, which is natural, since a two-dimensional dynamo is impossible (see[6]). To estimate n_0 (the lower harmonic) we can equate the right-hand side of (47) to unity at $E_1 = 0$; we then obtain

$$n_0 \approx (L^3ak/\nu_m)^{1/2} \approx \mathrm{Rm}^{1/2} \gg 1, \quad (48)$$

and the lower eigenvalues are

$$E_1^{(n_0)} \approx Lak/\sqrt{3} \approx v/L. \quad (49)$$

We have obtained an estimate that coincides with (25). So far we have assumed, for simplicity, that $k \approx 1/L$, but actually the estimate $E_1 \approx vk/L^2$ is valid for k in the next limits (one of them follows from (29)):

$$\nu_m/L^2v < k < v/\nu_m. \quad (50)$$

The lower limit is clear from (26): the potential has no significance at values of k that are too small. Thus, the WKB solution is not suitable for the lowest level, and it is the presence of the large parameter Rm which affords such a possibility. The asymptotic solution

$$A_1 = \operatorname{sh} i \int_{-L}^{x} p\,dx_1 \quad (51)$$

(the imaginary part of p reverses sign at x = 0) is characterized by a sharp maximum at x = 0. The width of the current layer is $\sim L/\mathrm{Rm}^{1/2}$, which coincides with the estimates of the preceding section. The force lines of the harmonic that attenuates most slowly are shown in Fig. 4 ($k \approx 1/L$). The ellipses of Fig. 4 tend to turn into circles, and the resultant electromagnetic forces act in the direction of the arrows. Such forces do not prevent motion along the y axis, and apparently can be compensated for by pressure.

4. APPLICATIONS

1. It can be assumed that such a procedure can be used also to calculate the magnetic field in magnetohydrodynamic flow at Rm ≫ 1 and at not too strong magnetic fields. We have in mind the nonstationary problem. The boundary conditions, of course, should be modified.

2. In essence, any dynamo mechanism incorporates the antidynamo. In fact, in these mechanisms there are unwanted fields that must be annihilated in some manner. This situation arises not only in an oscillatory dynamo (solar cycle), but also in models where the field increases exponentially, for example in the "doughnut" model[1]. The model of the preceding section shows how simple it is to annihilate a large-scale field. We note that Piddington, assuming that such a field cannot be annihilated in any way within a short time interval, "negates" the possibility of a dynamo for the sun and for the galaxy[11].

3. In all cases we deal with a dilemma: the velocity field either generates or annihilates a magnetic field.

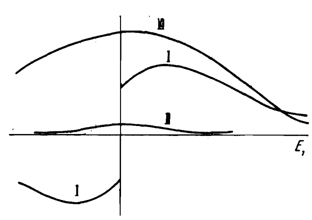

FIG. 3

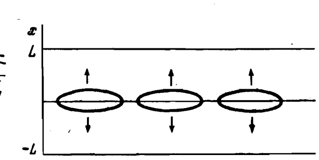

FIG. 4

Exceptions are possible, to be sure. Thus, for example, differential rotation does not annihilate an axially-symmetrical field (the number corresponding to k in our problem is equal to zero) and does not generate it. Whereas the dynamo is essentially three-dimensional, the antidynamo can also be two-dimensional, and furthermore the velocity can be very simple. This indicates that antidynamos should be encountered much more frequently, which raises with new urgency the question of how the magnetic fields are maintained, since it is too easy to annihilate them.

4. Let us calculate the rate of annihilation of the magnetic field under the conditions of a solar flare. We assume a characteristic length $L \sim 10^{10}$ cm; velocities up to 2×10^7 cm/sec are observed in flares [12]. Under such conditions, the entire magnetic energy is released within 8 minutes if, of course, the given mechanism does operate in the flare. It is assumed that annihilation of a field of 50—100 Oe compensates for the energy released in the flare. Of course, this statement makes no claim that the model of the flare is indeed such.

5. Under the "quiescent" conditions of the solar photosphere, it is feasible to annihilate without difficulty an extended field measuring 10^{10} cm by a constantly existing differential rotation with a velocity drop $\sim 2 \times 10^4$ cm/sec in five days.

6. In the collisionless solar-wind plasma, there is practically no ohmic decay, so that Eq. (1) is not realistic. At the same time, there exists a quasihydrodynamic description in which it is possible to justify the equation for the "freezing-in" of the magnetic field in the plasma, $\partial \mathbf{H}/\partial t = \mathrm{curl}[\mathbf{v}\times\mathbf{H}]$ (see, for example, [13]). The field dissipation can be resonant, i.e., it can be due to collisions between waves and particles. The fact that the dissipation is not contained in the damping decrement of the field in the given mechanism gives grounds for hoping that an analogous phenomenon takes place also in a collisionless plasma. An estimate based on formula (49) results in a rather short time of rejoining of the magnetic-field force lines.

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*$[\mathbf{vH}] \equiv \mathbf{v} \times \mathbf{H}$.

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