Radiation transfer in quantum electrodynamics

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A space-time kinetic equation for electromagnetic fields is proposed, which describes variation of the statistical field properties on passage through a resonant medium. As examples, spontaneous emission with allowance for re-emission and a single-mode laser with distributed parameters are considered.

INTRODUCTION

The statistical properties of electromagnetic radiation can be exhibited in the most natural manner within the framework of quantum electrodynamics. For the description of these properties the method of the field density matrix is widely used, a method which allows to exhibit all the correlation characteristics of the field which are of interest. One can indicate as examples some of the most interesting papers[1-3] in which this method has been applied successfully. Their authors dealt with a density matrix which describes one of several modes of an ideal resonant cavity. This approach turned out very fruitful in many laser problems. However, in essence this method is temporal and is completely useless for those cases when one is interested in the spatial variation of statistical properties of radiation.

In the present paper we propose a method where in addition to the temporal aspects the spatial aspect is explicitly present. We shall show that one can introduce the density matrix $\rho_{\mathbf{m}}(\mathbf{r}, t)$ describing a field oscillator with propagation vector $\mathbf{m}$, situated in the vicinity of the point $\mathbf{r}$ (the linear dimensions of the vicinity are much larger than the wavelength of the radiation). The equation of motion for such a density matrix has the form

$$\frac{\partial\rho_{\mathbf{m}}}{\partial t}+c\frac{\mathbf{m}}{m}\frac{\partial\rho_{\mathbf{m}}}{\partial\mathbf{r}}=Q.$$

The quantity Q in the right-hand side of the equation is determined by the state of the medium through which the radiation propagates. The second term on the left describes the transport of radiation from one space point to another.

In form and physical content the equation coincides with the transport equation of classical electrodynamics.[4] Therefore it makes sense to talk of a quantum equation and a quantum theory of radiation transport (or radiation transfer).

It is most convenient to analyze the quantum transport equation in the diagonal representation of the density matrix introduced by Glauber.[5] This is related to two circumstances. First, in the diagonal representation the kinetic equation becomes a partial differential equation, for which the usual methods of mathematical physics are adequate. Second, the description in the diagonal representation is in a language close to the classical, which considerably facilitates posing the boundary value problems.

As an illustration of the method we shall solve the following problem. In a traveling-wave resonant cavity one introduces a resonant medium, consisting of two-level atoms. The losses of the resonator are concentrated on one of the mirrors. We show that in each resonator mode the radiation is equilibrium radiation with a temperature depending on the frequency of the mode and on the point of space where we measure it. The spectral line of the radiation is Lorentzian, with a width which does not depend on the spatial coordinates. If we reduce the losses in the resonator to such a degree that the number of photons decreases more slowly than the increase of the radiation of the atoms, then the spontaneous emission will become generation. At the same time the statistical and spectral properties of the radiation change radically. In the same manner as in the papers of Lamb and Scully,[1] Kazantsev and Surdutovich,[2] we find that the line-broadening is related to phase diffusion, the speed of diffusion being uniform along the generator. At the same time the field intensity may vary substantially from point to point. The formula for the spectral width of the line was here the same form as in[1,2], with the difference that the total power accumulated in the resonator must be replaced with the output power of the generator. In the paper by Malakhov and Sandler[6] a similar problem was solved for the generator. It was formulated in classical language with a phenomenological introduction of a source of noise. However, this paper contains certain inaccuracies, which explains the discrepancies between its results and ours.

1. RADIATION TRANSPORT IN THE ABSENCE OF A RESONANT MEDIUM

We describe the most common scheme of the quantum theory of the electromagnetic field (cf., e.g.,[7]). In the normalization volume L^3 we expand the field into harmonic oscillators. The electric field strength operator has the following expansion

$$E(\mathbf{r},t)=\sum_{\mathbf{k}} i\left(\frac{k\hbar}{2L^3}\right)^{1/2}[a_{\mathbf{k}}(t)e^{i\mathbf{k}\mathbf{r}}-\text{h.c.}]$$

(we use a system of units where $c = 1$). The summation is over all eigenvalues of the propagation vector of the normalization box, forming a cubic lattice with lattice constant $2\pi L^{-1}$ in the wave number space.

The operators $a_{\mathbf{k}}^{+}$ and $a_{\mathbf{k}}$ are the creation and annihilation operators for photons of propagation vector $\mathbf{k}$. They satisfy the commutation relations

$$[a_{\mathbf{k}}(t),\ a_{\mathbf{k}'}(t)]=[a_{\mathbf{k}}^{+}(t),\ a_{\mathbf{k}'}^{+}(t)]=0,$$
$$[a_{\mathbf{k}}(t),\ a_{\mathbf{k}'}^{+}(t)]=\delta_{\mathbf{k}\mathbf{k}'}. \quad (1)$$

An arbitrary operator F satisfies the Heisenberg equation

$$i\hbar\dot{F}=[F,\ H]. \quad (2)$$

In order to be able to write this equation in explicit form one must know how F depends on all the $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{+}$. The Hamiltonian is of the following form

$$H=\sum_{\mathbf{k}}\hbar k\left(a_{\mathbf{k}}^{+}a_{\mathbf{k}}+\frac{1}{2}\right). \quad (3)$$

For the special case $F = a_k$ it is easy to see that

$$i\dot{a}_k = ka_k.$$

This scheme is very simple and intuitive. However, its use leads to considerable difficulties in problems with spatial inhomogeneities. Therefore we propose to change to a slightly different description. We split the space of propagation vectors into cells of linear dimensions l^{-1} which are much smaller than the reciprocal wavelength of the radiation (we assume that we are interested in a sufficiently narrow frequency interval, emitted by the optical transitions of the atom) and is much larger than the reciprocal length L^{-1} (the dimensions of the normalization box are assumed to be considerably larger than the size of the inhomogeneities of the field), i.e., $\lambda \ll l \ll L$. The position of these cells will be determined by the vectors $\mathbf{m}$. Each of the vectors $\mathbf{m}$ coincides with one of the eigenvectors of the normalization volume, so that the set of vectors $\mathbf{m}$ forms in the space of wave vectors a cubic lattice of lattice constant $2\pi l^{-1}$. In place of the variables $a_k(t)$ and $a_k^+(t)$ we introduce the new set of variables $a_m(\mathbf{r}, t)$ and $a_m^+(\mathbf{r}, t)$ according to the relations

$$a_\mathbf{m}(\mathbf{r},t)=\sum_{\mathbf{k}\sim\mathbf{m}} L^{-3/2}a_\mathbf{k}(t)\exp[i(\mathbf{k}-\mathbf{m})\mathbf{r}],$$
$$a_\mathbf{m}^+(\mathbf{r},t)=\sum_{\mathbf{k}\sim\mathbf{m}} L^{-3/2}a_\mathbf{k}^+(t)\exp[-i(\mathbf{k}-\mathbf{m})\mathbf{r}]. \quad (4)$$

Here the summation is over the wave eigenvectors of the normalizing volume which are situated in the cell $\mathbf{m}$.

The commutation relations for the new dynamical variables are the following

$$[a_\mathbf{m}(\mathbf{r},t), a_{\mathbf{m}'}(\mathbf{r}',t)] = [a_\mathbf{m}^+(\mathbf{r},t), a_{\mathbf{m}'}^+(\mathbf{r}',t)] = 0,$$
$$[a_\mathbf{m}(\mathbf{r},t), a_{\mathbf{m}'}^+(\mathbf{r}',t)] = \delta_{\mathbf{m}\mathbf{m}'}f(\mathbf{r}-\mathbf{r}'), \quad (5)$$

where the function $f(\mathbf{r}-\mathbf{r}')$ is defined as follows:

$$f(\mathbf{r}-\mathbf{r}')=\sum_{\mathbf{k}\sim\mathbf{m}} L^{-3/2}\exp[i(\mathbf{k}-\mathbf{m})(\mathbf{r}-\mathbf{r}')].$$

Functions $\Phi(\mathbf{r})$ which are sufficiently smooth can be approximately or exactly written in the form (4). For such functions one can derive the following integral equation.

$$\int_{(L^3)} d^3r'\,\Phi(\mathbf{r}')f(\mathbf{r}-\mathbf{r}')=\Phi(\mathbf{r}).$$

The commutation relations with $\mathbf{r} \neq \mathbf{r}'$ are essential for writing down the equations for an arbitrary operator F if we wish to find the explicit form of the commutator $[F, H]$. One can show that only functions of the indicated type will be involved. Therefore in the sequel we shall assume that $f(\mathbf{r}-\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}')$. This makes it possible to interpret the operators $a_m^+(\mathbf{r}, t)$ and $a_m(\mathbf{r}, t)$ as creation-annihilation operators of photons with propagation vectors $\mathbf{m}$ at the point $\mathbf{r}$. It will become clear in the sequel that in fact one should not talk about the point $\mathbf{r}$, but about a vicinity of the point, the dimensions of which are much larger than the wavelength of the radiation.

All operators of the dynamical variables have to be expressed in terms of the new variables. For the Hamiltonian we obtain the expression

$$H \approx H_0 + H_1,$$
$$H_0=\sum_\mathbf{m}\int_{(L^3)} d^3r\left(a_\mathbf{m}^+a_\mathbf{m}+\frac{1}{2}\right)\hbar m,$$
$$H_1=\sum_\mathbf{m}\int_{(L^3)} d^3r\frac{\mathbf{m}}{2m}\left(-i\hbar a_\mathbf{m}^+\frac{\partial a_\mathbf{m}}{\partial\mathbf{r}}+i\hbar a_\mathbf{m}\frac{\partial a_\mathbf{m}^+}{\partial\mathbf{r}}\right).$$

This is an approximate expression, since in the transition from the equation (3) to the latter expression we have made use of the relation $k \approx m + \mathbf{m}\cdot(\mathbf{k}-\mathbf{m})/m$, which is valid in the cell $\mathbf{m}$ up to terms of order $\lambda/l \ll 1$. All other dynamical variables can be similarly expressed. The approximation which was used means that we neglect the variation of the field over distances of the order of the wavelength. This leads to the appearance of space cells in place of space points, as indicated above.

The operator $\hbar m a_m^+(\mathbf{r}, t)a_m(\mathbf{r}, t)$ is the energy operator of an oscillator with wave number $\mathbf{m}$ located at point $\mathbf{r}$. The Hamiltonian H_0 defines the energy of the noninteracting oscillators at different space points and with different wave vectors. The Hamiltonian H_1 determines the interaction of oscillators with the same wave number but situated in neighboring space points.

The representation of quantum electrodynamics considered here is the Heisenberg picture. The entire time-dependence is concentrated in the operators (cf. (2)). In order to complete the picture we must still introduce the state vector of the field. It will be useful to consider the density matrix $\rho_m(\mathbf{r}, t)$ describing the oscillator at the point $\mathbf{r}$ with propagation vector $\mathbf{m}$. The equation for the density matrix is $\partial\rho_m/\partial t = 0$. A practically more convenient picture is related to the Heisenberg picture by means of the unitary transformation

$$F=\exp\left[\frac{i}{\hbar}H_1\left(t-\frac{\mathbf{mr}}{m}\right)\right]F'\exp\left[-\frac{i}{\hbar}H_1\left(t-\frac{\mathbf{mr}}{m}\right)\right].$$

Since the operators H_0 and H_1 commute, the new (primed) picture is the interaction picture, with interaction H_1. The new equations of motion are

$$i\hbar\dot{F}'=[F',H_0],\qquad \frac{\partial\rho_\mathbf{m}'}{\partial t}+\frac{\mathbf{m}}{m}\frac{\partial\rho_\mathbf{m}'}{\partial\mathbf{r}}=0.$$

We have assumed the electromagnetic field to be localized in an auxiliary normalization volume. Nothing prevents us from considering this volume to be infinite or finite, but with nonideally reflecting walls. Then the definition of the variables (4) will change (the summation over a cell will be replaced by an integration with weight $(L/2\pi)^{3/2}$). The rest of the reasoning remains the same.

2. BOUNDARY CONDITIONS IN THE QUANTUM THEORY OF RADIATION TRANSPORT

In the preceding section we have written the kinetic equation for the field density matrix. It is clear that this equation involves spatial derivatives. And this means that we must pose for it a boundary condition. Here the problem is not as clear cut as in classical electrodynamics. In order not to complicate the reasoning with unnecessary calculations we consider a more special problem. We shall assume that the field is in a traveling-wave resonant cavity, which can be considered one-dimensional (the x axis is directed along the perimeter of the resonator). At the point $x = 0$ there is a semitransparent mirror which can be characterized by a reflection coefficient $R = |R|\exp(i\varphi_R)$ and a transmission coefficient $T = |T|\exp(i\varphi_T)$ (there is the relation $|T|^2 + |R|^2 = 1$). The length of the perimeter of the resonator is L. If we consider waves traveling in the positive direction of the x axis, the wave goes away from the mirror at the point $x = 0$ and arrives at the mirror at the point $x = L$.

In the classical formulation the boundary condition is in this case of the following form: $\mathscr{E}(x=0, t) = R\,\mathscr{E}(x=L, t)$, which implies

$$\langle \mathscr{E}(x=0,t)\rangle = R\langle \mathscr{E}(x=L,t)\rangle, \quad \langle |\mathscr{E}(x=0,t)|^2\rangle = |R|^2\langle |\mathscr{E}(x=L,t)|^2\rangle$$

etc. (here $\mathscr{E}$ is the complex amplitude of the field). We can use this condition if we go over to the diagonal representation of the density matrix,[5] introduced according to

$$\rho_m(\mathbf{r},t) = \int d^2\alpha\, P_m(\alpha,\mathbf{r},t)\,|\alpha\rangle\langle\alpha|,$$

$$d^2\alpha = d\alpha_1\, d\alpha_2, \qquad \alpha_1 = \mathrm{Re}\,\alpha, \qquad \alpha_2 = \mathrm{Im}\,\alpha;$$

here α and $|\alpha\rangle$ are the eigenvalue and eigenvector of the photon annihilation operator, i.e., $a|\alpha\rangle = \alpha|\alpha\rangle$.

The introduction of the diagonal representation allows one to carry out the reasoning in a language close to the classical one. The role of the complex amplitude is taken here by the quantity α. It is easy to check that if we require for the density matrix the validity of the condition

$$P(\alpha, x=0, t) = \frac{1}{|R|^2} P\left(\frac{\alpha}{R}, x=L, t\right), \tag{6}$$

then all the relations imposed above on the complex amplitude will be automatically valid. This entitles us to use the condition (6) as a boundary condition for our transport equation.

The use of the diagonal representation of the density matrix allows us to solve the problem of finding the boundary conditions in more complicated three-dimensional problems and also in the presence of an external signal.

3. THE QUANTUM RADIATION TRANSPORT EQUATION IN A RESONANT MEDIUM

Let us now fill our one-dimensional resonator with a resonant medium. We shall assume that the medium consists of immobile two-level atoms, situated at a sufficient distance from each other, so that one may neglect the dipole-dipole interaction. The two working levels of these atoms (a is the upper level, b is the lower one) decay into lower levels with the constants γ_a and γ_b, respectively. At the same time there is stationary homogeneous pumping into these levels, so that in each spatial cell of the size $\sim l^3$ in the absence of a strong field there are N_a atoms on the upper level and N_b atoms on the lower level. We shall assume that in each spatial cell $\sim l^3$ one can take into account the interaction of the atoms only with one field oscillator with index m. Such a way of posing the problem coincides with that of Lamb and Scully.[1] This allows us to avoid constructing the variation of the field density matrix of the oscillator on account of its interaction with the resonant medium, and to make use of the results of [1]. Our equation for the density matrix will coincide with the one in [1] is we change in the latter the time-derivative of the density matrix to the expression $(\partial/\partial t + \partial/\partial x)\rho_m$. We write the equation in the diagonal representation for the density matrix:

$$\frac{\partial P_m}{\partial t} + \frac{\partial P_m}{\partial x} = -\frac{A}{2}\frac{\partial}{\partial\alpha}\left[\frac{1+i\Delta/\gamma_{ab}}{Z}\alpha P_m\right] + \frac{A_1}{2}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}\left[\frac{1}{Z}P_m\right] + \text{c.c.} \tag{7}$$

Here we have used the notations:

$$A_{1,2} = 2N_{a,b}|g|^2\gamma_{ab}(\gamma_{ab}^2+\Delta^2)^{-1}, \quad B_{1,2} = A_{1,2}\frac{4|g|^2}{\gamma_a\gamma_b}\gamma_{ab}^2(\gamma_{ab}^2+\Delta^2)^{-1},$$

$$Z = 1 + \frac{B}{A}\left(|\alpha|^2 - \frac{1}{2}\alpha\frac{\partial}{\partial\alpha} - \frac{1}{2}\alpha^*\frac{\partial}{\partial\alpha^*}\right).$$

The quantity $A = A_1 - A_2$ has the meaning of a linear gain factor per unit length of the resonant medium; $B = B_1 - B_2$ determines the nonlinear properties of the resonant medium; $\gamma_{ab} = (\gamma_a + \gamma_b)/2$ is the transverse relaxation constant; $\Delta = \omega_0 - \omega$ (ω_0 is the frequency of the working transition of the atom, ω is the frequency of the field oscillator);

$$g = i\left(\frac{m\hbar}{2l^3}\right)^{1/2}(\mathbf{d}_{ab}\boldsymbol{\varepsilon})\,e^{imr}$$

is the coupling constant of the field oscillator at the point r with index m to the atom situated at the same point r. Later we will encounter the quantity $N = N_a - N_b$, the difference in populations of the working levels in the cell $\sim l^3$.

4. TRANSPORT OF SPONTANEOUS RADIATION

In this section we will be interested in the case when the nonlinear properties of the medium may be neglected. Then in Eq. (7) the coefficient B has to be set equal to zero. Information on spontaneous radiation from a resonator can be found, e.g., in [1,2,8] It follows from them that into each mode of the resonator a photon gas is emitted whose statistical properties are determined by the equation

$$\rho_{nn} = \exp\left(-\frac{n\hbar\omega}{kT}\right)\left[1 - \exp\left(-\frac{\hbar\omega}{kT}\right)\right], \tag{8a}$$

where

$$T = -\hbar\omega\left[k\ln\frac{A_1}{C+A_2}\right]^{-1}$$

is the temperature of the photon gas (the quantity C determines the speed with which the field disappears from the resonator). Thus the radiation differs from the equilibrium radiation through the fact that the temperature depends on the frequency in a manner more complicated than a linear dependence ($A_{1,2} = A_{1,2}(\omega)$). The spectral composition of each mode is determined by a Lorentz shape centered at the mode frequency and with width $C - A$.

Let us investigate to what degree these data change on account of the spatial nonhomogeneity in the form of the semitransparent mirror.

One can verify that the solution of (7) for $B = 0$ with the boundary condition (6) can be written in the form

$$P_m(\alpha x t) = \int d^2\alpha'\, P_m(\alpha' x t = 0)\frac{1}{\pi\sigma_m}\exp\left(-\frac{|\alpha - \alpha' v_m|^2}{\sigma_m}\right) \tag{8}$$

where

$$v_m = R^{t/L}\exp\left(\frac{1}{2}At - i\Delta\frac{A}{\gamma_{ab}}t\right),$$

$$\sigma_m = \frac{N_a}{N}\frac{|R|^2e^{AL} + |T|^2e^{Ax} - 1}{1 - |R|^2e^{AL}}(1 - |R|^{2t/L}e^{At}).$$

In the case when the constant $C = \ln|R|^2/L$, having the interpretation of a loss coefficient per unit length, is larger than the gain coefficient of the medium per unit length, A, there exists a stationary solution of the problem, in which we are usually interested. In the opposite situation, $A > C$, the parameters v_m and σ_m increase without bound as functions of time; which invalidates the solution (8).

The properties of the distribution (8) are well known (cf., e.g., [5,9]). This makes it possible for us to make the following assertions without further justification. In a generator of sufficiently high Q (for which the intermode distance is considerably larger than $\Gamma = C - A$) on each pulled mode frequency $\omega + \Delta' + \Delta''$ ($\Delta' = A/\gamma_{ab}$ is the frequency shift due to the dispersion of the resonator) there is centered a Lorentz line-shape with width independent of the coordinate and coinciding with the width given in papers which do not take into account the spatial inhomogeneity[1,2,8]. The integral power of the circuit is determined by the quantity σ_m for $t \to \infty$:

$$\sigma_m = \frac{N_a}{N} \frac{|R|^2 e^{AL} + |T|^2 e^{Ax} - 1}{1 - |R|^2 e^{AL}}.$$

Thus, taking into account the spatial inhomogeneity does not lead to a change of the width of the spectral shape on each resonator mode, but leads to a dependence of the total power on the coordinate. This dependence is the stronger, the closer the state of the medium is to the threshold (the closer the quantity A is to C).

The statistics of photons emitted into a certain mode retains its form (8a). However, the dependence of the temperature on the frequency becomes more complicated and is defined by the following equation:

$$T = \hbar\omega [k \ln(\sigma_m^{-1} + 1)]^{-1}.$$

In addition, as can be seen, the temperatures are different at different points of the resonator.

In the limiting case of small gains and losses (AL, CL $\to$ 0) all our equations go over into the ones already known.

Our results are valid for arbitrary values of the parameters R and T. In particular, for $R = 0$ and $T = 1$ we obtain the formulas for the radiation of a resonant medium into free space. The quantity σ_m then determines the well known spectral line shape of spontaneous emission:

$$\sigma_m = \frac{N_a}{N}(e^{Ax} - 1).$$

As can be seen, due to the propagation of radiation through the medium, the spectral line stops being Lorentzian (we recall that the quantity A is proportional to the Lorentz spectral shape).

5. TAKING INTO ACCOUNT THE SPATIAL INHOMOGENEITY IN THE THEORY OF LINEWIDTH OF LASER GENERATION

As is well known, the reason of broadening of the emission line of a laser are the phase fluctuations which occur on account of the quantum nature of the interacting atoms and the electromagnetic field (we shall not deal here with the problem of the technical linewidth). In the paper of Lamb and Scully,[1] which by its method is closest to ours, it was found that the lineshape is Lorentzian with a width determined by the equation

$$\Delta\nu = {}^1\!/_2 C (N_a/N) \langle n \rangle^{-1},$$

$\langle n \rangle$ is the number of photons accumulated in the resonant cavity in the stationary regime.

We would like to find out how this result is modified when the losses of the resonant cavity are concentrated in the mirror.

We have to solve Eq. (7) with a boundary condition of the form (6). This cannot be done analytically. We use the following simplifying circumstance. In a single-mode laser the photon statistics is close to a Poisson distribution. Since this distribution is characterized by a small relative dispersion, the quantity can be written in the form $|\alpha| = \sqrt{n(x)} + \epsilon$ where $n(x)$ is the photon number distribution of stationary generation along the axis of the resonator and $\epsilon \ll \sqrt{n(x)}$. In Eq. (7), written in terms of the variables r and φ ($\alpha = re^{i\varphi}$, $\alpha^* = re^{-i\varphi}$) one can separate the variables in the form

$$P_m(\alpha x t) = \mathcal{R}(\varepsilon x t)\,\Phi(\varphi x t).$$

The function $\mathcal{R}$ describes the amplitude fluctuations. We shall not discuss them here, since they determine only a small background in the spectral line shape.

The function Φ determines the phase fluctuations. Its equation has the form

$$\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x} = \frac{N_a}{N}\frac{C(x)}{4n(x)}\frac{\partial^2\Phi}{\partial\varphi^2}. \tag{9}$$

The boundary conditions follow from the boundary condition (6):

$$\Phi(\varphi, x=0, t) = \Phi(\varphi - \varphi_R, x = L, t).$$

In the writing of the equation we have assumed that the mode frequency coincides with the frequency of the atomic transition ($\Delta = \omega_0 - \omega = 0$). We introduce the quantity $C(x) \equiv n^{-1}(x)dn(x)/dx$. It is easy to find that for stationary generation we have the condition $C(x) = A[1 + BA^{-1}n(x)]^{-1}$.

The solution of the equation (9) is written in the form

$$\Phi(\varphi, x, t) = \int d\varphi'\, \Phi(\varphi', x, t=0)(\pi\sigma)^{-1/2} \exp\left[-\frac{(\varphi - v)^2}{\sigma}\right],$$

where

$$v = \varphi' + \varphi_R t / L,$$

$$\sigma = \frac{t}{L}\frac{N_a}{4N}\int_0^L C(x) n^{-1}(x)\, dx = \frac{C}{4n(L)}\frac{N_a}{N}t = \frac{1}{2}t\Delta\nu.$$

The dependence of the quantity v on time determines the shift of the generation frequency owing to the dispersion of the resonator (the shift due to the dispersion of the medium is absent since $\Delta = 0$). The quantity σ determines the correlation law for the phase

$$\langle[\varphi(t) - \varphi(t-\tau)]^2\rangle_{t\to\infty} = 2\sigma = \tau\Delta\nu,$$

which leads to a broadening of the monochromatic line and to the appearance of a Lorentz shape with width

$$\Delta\nu = {}^1\!/_2 C(N_a/N) n^{-1}(L).$$

This formula differs from the analogous formula of [1] in that the power $\langle n \rangle$ accumulated in the whole resonator is replaced by the output power $n(L)$. Taking into account spatial inhomogeneity did not lead to a nonuniformity of the linewidth along the length of the cavity.

As was already noted in the introduction, Malakhov and Sandler[6] have considered a phenomenological model of broadening of laser lines taking into account the spatial inhomogeneity, but they have not taken into account the influence of the strong field of generation on the sources of stochastically acting forces. This led to a linewidth formula which differs from ours.

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