Oscillation spectrum of the von Karman vortex street

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Small perturbations of the von Karman vortex street are considered. The dependence of the frequency of the oscillations on the wavelength of the perturbation is found and also the shape of the normal oscillations. Four types of normal oscillations correspond to each wavelength. For long waves, these are longitudinal and transverse oscillations of the street, propagating along it either forward or backward, with definite velocities.

The von Karman street is a system of rectilinear parallel vortices of unit intensity in an ideal fluid. The vortices with positive circulation form a linear chain, with a constant distance between neighboring vortices; this distance is the unit of length. Vortices with negative circulation form a similar chain, located at a distance k from the previous one, and the vortices of one chain lie opposite the center of the segment connecting two neighboring vortices of the other chain. It has been shown[1,2] that such a street is stable at a certain k.

It is convenient for our purposes to introduce the complex coordinate z on the plane perpendicular to the vortices, and correspondingly the complex velocity v. We can then assume that the coordinates of the positive vortices will be $z_m = m$, and the coordinates of the negative ones $z'_m = m + \frac{1}{2} - ik$. For the velocity field v(z) of a positive vortex at a point z_m, we have (the bar denotes the complex conjugate)

$$\overline{v(z)} = -i/(z - z_m).$$

For the velocity field of the entire street we then have

$$\overline{v(z)} = -i\sum_{m=-\infty}^{\infty}\frac{1}{z-z_m} + i\sum\frac{1}{z-z_m'} = -i\pi\operatorname{ctg}\pi z - i\pi\operatorname{tg}\pi(z+ik) + \text{const.}$$

The value of the constant determines the velocity of motion of the fluid far from the street, and the velocity of the street itself will be $v = \pi\tanh\pi k$ + const. We set const $= -\pi\tanh\pi k$. Then the street will be at rest and the velocity field will not depend on the time. For $k = 0.2805$ ($\cosh^2\pi k = 2$), for which the street is stable, $\pi\tanh\pi k = 2.221\ldots$.

If we now displace the vortices by small distances ϵ_m and ϵ'_m from their respective stationary positions, then they are set into motion. The effect of the vortex z_m on the vortex z_0 can be described in the form

$$\dot{\bar\varepsilon}_0 = -i\frac{1}{\varepsilon_0 - z_m - \varepsilon_m} + i\frac{1}{0 - z_m} \approx -i\frac{\varepsilon_m - \varepsilon_0}{z_m^2}.$$

Summing the contributions of all the vortices, we find

$$\dot{\bar\varepsilon}_0 = -i\sum_{m\neq 0}\frac{\varepsilon_m-\varepsilon_0}{z_m^2} + i\sum\frac{\varepsilon_m'-\varepsilon_0}{z'^2} = -i\sum_{m\neq 0}\frac{\varepsilon_m-\varepsilon_0}{m^2} + i\sum\frac{\varepsilon_m'-\varepsilon_0}{(m+\frac{1}{2}-ik)^2} \quad (1)$$

and similarly for the vortex z'_0

$$\dot{\bar\varepsilon}_0' = i\sum_{m\neq 0}\frac{\varepsilon_m'-\varepsilon_0'}{m^2} - i\sum\frac{\varepsilon_m-\varepsilon_0'}{(m-\frac{1}{2}+ik)^2}. \quad (2)$$

It would not be difficult to determine the velocities of the remaining vortices. The solution of the equation of motion is sought in the form of a superposition of periodic waves of the form $\epsilon_m = \gamma(t)e^{im\varphi}$. But, inasmuch as complex conjugation enters in (1) and (2), we set

$$\varepsilon_m = \gamma_+ e^{im\varphi} + \gamma_- e^{-im\varphi},\quad \varepsilon_m' = \gamma_+' e^{i(m+\frac{1}{2})\varphi} + \gamma_-' e^{-i(m+\frac{1}{2})\varphi}. \quad (3)$$

Upon substitution of (3) in (1) and (2), the following series arise, the values of which are known[1,2] (the formulas are correct for $0 \le \varphi \le 2\pi$):

$$\sum_{n\neq 0}\frac{e^{in\varphi}-1}{n^2} = -\frac{(2\pi-\varphi)\varphi}{2},\quad \sum\frac{1}{(n+\frac{1}{2}-ik)^2} = \frac{\pi^2}{\operatorname{ch}^2\pi k},$$

$$\sum\frac{e^{i(n+\frac{1}{2})\varphi}}{(n+\frac{1}{2}-ik)^2} = \frac{\pi e^{-\varphi k}}{\operatorname{ch}\pi k}\left(\frac{\pi}{\operatorname{ch}\pi k} - \varphi e^{\pi k}\right) \equiv D_\varphi.$$

Introducing the notation $B_\varphi = \frac{1}{2}(2\pi-\varphi)\varphi - \pi^2/\cosh^2\pi k$, we can write down the result of substitution of (3) in (1) and (2) in the form

$$\dot{\bar\varepsilon}_0 = iB_\varphi\gamma_+ + iD_\varphi\gamma_+' + iB_\varphi\gamma_- - iD_{2\pi-\varphi}\gamma_-',$$
$$\dot{\bar\varepsilon}_0' = -iB_\varphi\gamma_+'e^{i\varphi/2} + iD_{2\pi-\varphi}\gamma_+e^{i\varphi/2} - iB_\varphi\gamma_-'e^{-i\varphi/2} - iD_\varphi\gamma_-e^{-i\varphi/2}.$$

The general formulas are as follows:

$$\begin{aligned}\dot{\bar\varepsilon}_m &= i(B_\varphi\gamma_+ + D_\varphi\gamma_+')e^{im\varphi} + i(B_\varphi\gamma_- - D_{2\pi-\varphi}\gamma_-')e^{-im\varphi},\\ \dot{\bar\varepsilon}_m' &= -i(B_\varphi\gamma_+' - D_{2\pi-\varphi}\gamma_+)e^{i(m+\frac{1}{2})\varphi} - i(B_\varphi\gamma_-' + D_\varphi\gamma_-)e^{-i(m+\frac{1}{2})\varphi}.\end{aligned} \quad (4)$$

Substituting (3) in (4), we find the equations for γ (if $\varphi \neq \pi$):

$$\dot{\bar\gamma}_+ = iB_\varphi\gamma_- - iD_{2\pi-\varphi}\gamma_-',\quad \dot{\bar\gamma}_{+}' = -iB_\varphi\gamma_-' - iD_\varphi\gamma_-; \quad (5)$$

$$\dot{\bar\gamma}_- = iB_\varphi\gamma_+ + iD_\varphi\gamma_+',\quad \dot{\bar\gamma}_-' = -iB_\varphi\gamma_+' + iD_{2\pi-\varphi}\gamma_+. \quad (6)$$

To find the characteristic frequencies ω, we eliminate the complex conjugate, substituting (6) in (5):

$$\begin{aligned}\ddot\gamma_+ &= (B_\varphi^2 - D_{2\pi-\varphi}^2)\gamma_+ + B_\varphi(D_\varphi + D_{2\pi-\varphi})\gamma_+',\\ \ddot\gamma_+' &= -B_\varphi(D_\varphi + D_{2\pi-\varphi})\gamma_+ + (B_\varphi^2 - D_\varphi^2)\gamma_+'.\end{aligned} \quad (7)$$

It is now not difficult to write out the equation for ω—the determinant of the set (7):

$$\omega^4 + (2B_\varphi^2 - D_\varphi^2 - D_{2\pi-\varphi}^2)\omega^2 + (B_\varphi^2 - D_{2\pi-\varphi}^2)(B_\varphi^2 - D_\varphi^2) + B_\varphi^2(D_\varphi + D_{2\pi-\varphi})^2 = 0.$$

Its solution is

$$\omega = \tfrac{1}{2}\{\pm(D_\varphi + D_{2\pi-\varphi}) \pm \sqrt{(D_\varphi - D_{2\pi-\varphi})^2 - 4B_\varphi^2}\}.$$

All the frequencies will be real only if $B_\pi = 0$, since the first term under the square root equals zero for $\varphi = \pi$. This also means that $\cosh^2\pi k = 2$ or $k = 0.2805$ (see the definition of B_φ). We limit ourselves below to this value of k. The integrand will then be non-negative for all φ. To each $0 < \varphi < \pi$ there correspond four frequencies, the positive pair of which we denote by ω_1 and ω_2 ($\omega_1 > \omega_2$):

$$\omega_{1,2} = \tfrac{1}{2}\{-(D_\varphi + D_{2\pi-\varphi}) \pm \sqrt{(D_\varphi - D_{2\pi-\varphi})^2 - 4B_\varphi^2}\},$$

inasmuch as $D_\varphi + D_{2\pi-\varphi} \le 0$ for all φ. As $\varphi \to 0$, the dependence of ω on φ becomes linear: $\omega_j = v_j\varphi$, where $v_1 = 5.376$ and $v_2 = 1.836$.

We now consider how the normal oscillations of the street appear. We limit ourselves here to the interval

TABLE.

Oscillation type	γ_+	γ_-	γ_+'	γ_-'
1	$\xi e^{i\omega_1 t}$	$-\alpha_1\bar{\xi}e^{-i\omega_1 t}$	$\alpha_1\xi e^{i\omega_1 t}$	$-\bar{\xi}e^{-i\omega_1 t}$
2	$\eta e^{i\omega_2 t}$	$-\alpha_2\bar{\eta}e^{-i\omega_2 t}$	$\alpha_2\eta e^{i\omega_2 t}$	$-\bar{\eta}e^{-i\omega_2 t}$
3	$\zeta e^{-i\omega_1 t}$	$\alpha_1\bar{\zeta}e^{i\omega_1 t}$	$\alpha_1\zeta e^{-i\omega_1 t}$	$\bar{\zeta}e^{i\omega_1 t}$
4	$\theta e^{-i\omega_2 t}$	$\alpha_2\bar{\theta}e^{i\omega_2 t}$	$\alpha_2\theta e^{-i\omega_2 t}$	$\bar{\theta}e^{i\omega_2 t}$

$0 < \varphi < \pi$. For example, we set $\gamma_+ = \xi e^{i\omega_1 t}$ and find the remaining γ. It follows from (7) that

$$\gamma_+' = -\frac{B_\varphi^2 - D_{2\pi-\varphi}^2 + \omega_1^2}{B_\varphi(D_\varphi + D_{2\pi-\varphi})}\gamma_+ = \alpha_1\gamma_+,$$

where

$$\alpha_{1,2} = \{-D_\varphi + D_{2\pi-\varphi} \pm \sqrt{(D_\varphi - D_{2\pi-\varphi})^2 - 4B_\varphi^2}\} / 2B_\varphi.$$

We note that $\alpha_1\alpha_2 = 1$. From (6), we get $\bar{\gamma}_-' = -\gamma_+$ and $\alpha_2\bar{\gamma}_- = -\gamma_+$. The remaining three solutions are found in similar fashion with

$$\gamma_+ = \eta e^{i\omega_2 t}, \quad \gamma_+ = \zeta e^{-i\omega_1 t}, \quad \gamma_+ = \theta e^{-i\omega_2 t}$$

(see the table).

To determine the shape of the normal oscillations, it is enough to use Eq. (3). For oscillations of type 1, we then obtain

$$\varepsilon_m = \xi e^{i(m\varphi+\omega_1 t)} - \alpha_1\bar{\xi}e^{-i(m\varphi+\omega_1 t)},$$

$$\varepsilon_m' = \alpha_1\xi e^{i[(m+1/2)\varphi+\omega_1 t]} - \bar{\xi}e^{-i[(m+1/2)\varphi+\omega_1 t]}.$$

When φ varies from 0 to π, the coefficient α falls from 1 to 0; therefore, the vortices describe ellipses elongated perpendicular to the street, and the direction of the motion along the ellipse is identical with the direction of the circulation of the vortex. For small φ, these will be mainly transverse oscillations of the street, propagating along the street in a direction counterparallel to its proper motion, with velocity $\omega_1/\varphi \to v_1 = 5.376$ as $\varphi \to 0$. In both chains, the vortices across the street move in the same phase—a bending of the street is obtained, in which its width remains almost unchanged.

Similar oscillations correspond to type 2, but the direction of motion of the vortices along the ellipse is counter to the circulation and the limiting velocity is $v_2 = 1.836$.

In oscillations of types 3 and 4, the vortices move along ellipses elongated along the street. In type 3, the motion of the vortex along the ellipse is counter to the direction of its circulation, in type 4, the two directions are the same. For small φ, the oscillations have the character of longitudinal waves, propagating along the street in the forward direction with velocities v_1 for type 3 and v_2 for type 4.

An arbitrary solution of the system (5) and (6) can be represented as the sum of oscillations of the four described types with certain complex ξ, η, ζ, θ, and the arbitrary small perturbation of the street can be represented by the corresponding integral with respect to φ.

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[1]H. Lamb, Hydrodynamics, Cambridge, 1933. Russian translation, Gostekhizdat, 1947.
[2]H. Villat, Leçons sur la thêorie des tourbillons, Gauthier-Villars, 1932. Russian translation, ONTI, 1937.

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