Asymptotics of the Green function and the charge renormalization constant in scalar electrodynamics in the $\alpha_0(\alpha_0 L)^n$-approximation

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(Submitted January 4, 1973)
Zh. Eksp. Teor. Fiz. **64**, 1935-1949 (June 1973)

Asymptotics of the nonrenormalized photon and meson Green's functions and vertex function are obtained in the high momentum range and also the charge renormalization constant are obtained in the so-called $\alpha_0(\alpha_0 L)^n$-approximation (the definition of which is given in the text). The analysis is performed in the Duffin-Kemmer formalism.

INTRODUCTION

The calculation of the asymptotic Green's functions and the renormalization constants by the method of summing the principal terms of the expansions in the coupling constants plays an important role in the investigation of the structure of renormalizable field theories[1–5]. Such calculations constitute a definite departure from standard perturbation theory and, besides being of independent significance, make it possible to investigate, to a certain degree, such questions as the problem of the self-consistency of the theory, the question of the true character (outside the scope of perturbation theory) of the divergences, the existence of solutions of the superconducting type, etc.

In spinor electrodynamics, such investigations were carried out[6, 7] up to the so-called "five-gamma" or $\alpha(\alpha L)^n$ approximation, which follows from the "three-gamma" or $(\alpha L)^n$ approximation of Landau, Abrikosov, Khalatnikov, and Fradkin[1, 2]. It is of interest to consider with the aid of analogous methods also the scalar electrodynamics. In this theory, calculation of the Green's functions in the "three-vertex" or $(\alpha_0 L)^n$ approximation was carried out by Gor'kov and Khalatnikov[5]. They have shown that in a specially chosen gauge of the photon Green's function (Feynman gauge), the equation for the vertex function contains in the $(\alpha_0 L)^n$ approximation only one three-vertex diagram, so that the system of the Dyson equations becomes closed.

In this paper we calculate the asymptotic forms of the photon and meson Green's functions D(k) and G(p) in the vertex function $\Gamma_\lambda(p, q)$ in the region $p^2, k^2 \gg q^2 >> m^2$, and also the renormalization constant of the charge Z_3 in the $\alpha_0(\alpha_0 L)^n$ approximation that follows $(\alpha_0 L)^n$. In this approximation, all the terms of the type $\alpha_0(\alpha_0 L)^n$ are summed in terms of the nonrenormalized coupling constant α_0, where n = 0, 1, 2, . . ., and L is the general symbol for large logarithmic parameters of the type $\ln(\Lambda^2/p^2)$, $\ln(\Lambda^2/m^2)$, etc. We neglect in this case only the terms $\alpha_0^2(\alpha_0 L)^n$ and smaller. The calculations are carried out in the Duffin-Kemmer formalism with the aid of the corresponding Dyson equations.

In the $\alpha_0(\alpha_0 L)^n$ approximation, the equation for the vertex function receives contributions not only from the three-vertex diagram, but also from the five- and seven-vertex diagrams (see Fig. 2). More complicated diagrams make no contribution, so that the system of Dyson's equations becomes closed and provides a basis for the investigation of the $\alpha_0(\alpha_0 L)^n$ approximation. The results of the calculations of the mesic Green's function G(p) (28) are represented by formulas (62), (69), and (70); those of the vertex function $\Gamma_\lambda(p, q)$ (49) are represented by formulas (50), (55), (57), (58), and (63); those of the photon Green's function and of the charge renormalization constant Z_3 by formulas (27) and (23).

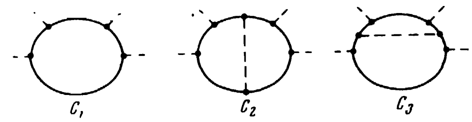

FIG. 1

THE CONSTANT Z_3

$Z_3(\alpha_0)$ is defined in terms of the polarization operator $\pi^*(k)$ by means of the formula[8]

$$Z_3^{-1}(\alpha_0) = 1 + i\frac{\delta_{\mu\nu}}{8}\frac{\partial^2}{\partial k_\mu \partial k_\nu}\Pi^*(k)\Big|_{k=0} = 1 + C. \quad (1)$$

It is easy to show that the $\alpha_0(\alpha_0 L)^n$ contribution to C is made by the following integrals[1)] (Fig. 1):

$$C = C_1 + C_2 + C_3; \quad (2)$$

$$C_1 = \frac{i}{12}\frac{\alpha_0}{4\pi^3}\mathrm{Sp}\int T_\mu(p)T_\nu(p)T_\nu(p)T_\mu(p)d^4p, \quad (3)$$

$$C_2 = \frac{i}{12}\left(\frac{\alpha_0}{4\pi^3}\right)^2 \mathrm{Sp}\int T_\mu(p)T_\nu(p)T_\sigma(p,k)T_\nu(k)T_\mu(k)T_\sigma(k,p)D(k-p)d^4p\,d^4k, \quad (4)$$

$$C_3 = \frac{i}{12}\left(\frac{\alpha_0}{4\pi^3}\right)^2 \mathrm{Sp}\int T_\mu(p)T_\sigma(p,k)T_\nu(k)T_\nu(k)T_\sigma(k,p)T_\mu(p)D(k-p)d^4p\,d^4k, \quad (5)$$

where we put for brevity

$$T_\mu(p,k) \equiv \Gamma_\mu(p,k)G(k), \quad T_\mu(p) \equiv T_\mu(p,p).$$

We consider first the integrals C_2 and C_3. In these integrals, to obtain the $\alpha_0(\alpha_0 L)^n$ terms it suffices to take all the functions in the $(\alpha_0 L)^n$ approximation, since one of the integration has a nonlogarithmic character. The functions G, $\Gamma\mu$, and D in the integrals (4) and (5) take in the $(\alpha_0 L)^n$ approximation in the form[5, 10] 2)]

$$G_0(k) = [-\hat{k}A_0(k) + imB_0(k)]^{-1} = \frac{-\hat{k}A_0 - imB_0 + iA_0^2B_0^{-1}(\hat{k}^2 - k^2)/m}{k^2A_0^2 + m^2B_0^2} \quad (6)$$

$$A_0(k) = \left(1 + \frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{k^2}\right)^{-3/2},$$

$$B_0(k) = \left(1 + \frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{k^2}\right)^3 \Big/ \left(1 + \frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{m^2}\right)^{9/2}, \quad \Lambda^2 \gg k^2 \gg m^2, \quad (7)$$

$$D_0(k-p) = \frac{1}{i(k-p)^2}d_0(k-p), \quad d_0(k-p) = \left(1 + \frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{(k-p)^2}\right)^{-1} \quad (8)$$

$$\Gamma_0^\mu(p,p) = -\frac{\partial}{\partial p_\mu}G^{-1}(p) = \beta_\mu A_0(p), \quad (9)$$

$$\Gamma_0^\mu(k,p) = \beta_\mu A_0(k) + \Big[\frac{i}{m}A_0^2(k)B_0^{-1}(k)(k_\mu k_\sigma - k^2\delta_{\mu\sigma})$$

$$-A_0(k)k_\mu\beta_\sigma\Big]\frac{1}{k^2}XJ_\sigma(k,p)+\ldots\qquad(k^2\gg p^2),\tag{10}$$

$$XJ_\sigma(k,p)=\frac{\alpha_0}{4\pi^3}X\int_p^k G(p-t)\Gamma_\sigma(p-t,p)D(t)d^4t=X[p_\sigma(A_0(p)A_0^{-1}(k)-1)+im\beta_\sigma A_0^{-1}(k)(B_0(k)-B_0(p))].\tag{11}$$

In the integrals (4) and (5), the functions (6) – (11) can be simplified when account is taken of the following remarks. Let the integration with respect to k in (4) and (5) be nonlogarithmic. In the integration of the terms containing $\ln k^2$ or $\ln(k-p)^2$, the $\alpha_0(\alpha_0 L)^n$ contribution which we need is made, as can be readily seen, by the regions $\Lambda^2 \gg k^2 \gg p^2$ and $k^2 \ll p^2$. The integrals that arise in the region $\Lambda^2 \gg k^2 \gg p^2$ are of the type

$$\alpha_0\int_p^\Lambda(\alpha_0\ln k^2)^n\left(\frac{p^2}{k^2}\right)^m\frac{d^4k}{k^4}=\alpha_0(\alpha_0\ln p^2)^n\int_p^\Lambda\left(\frac{p^2}{k^2}\right)^m\frac{d^4k}{k^4}+O(\alpha_0^2(\alpha_0L)^n),\quad m,n>0,\tag{12}$$

and in the region $p^2 \gg k^2 \gg m^2$ of the type

$$\alpha_0\int_m^p(\alpha_0\ln k^2)^n\left(\frac{k^2}{p^2}\right)^m\frac{d^4k}{p^2k^2}=\alpha_0(\alpha_0\ln p^2)^n\cdot\int_m^p\left(\frac{k^2}{p^2}\right)^m\frac{d^4k}{p^2k^2}+O(\alpha_0^2(\alpha_0L)^n),\quad m,n>0.\tag{13}$$

Relations (12) and (13) making it possible to make the substitution $\ln k^2 \to \ln p^2$ in the functions (6) - (11) without loss in the assumed accuracy; this leads to the following simplifications in the integrals (4) and (5):

$$A_0(k)\to A_0(p),\ B_0(k)\to B_0(p),\ d_0(k-p)\to d_0(p);\quad \Gamma_0^\mu(k,k),\ \Gamma_0^\mu(k,p)\to\beta_\mu A_0(p).\tag{14}$$

Substituting (14) and (9) in the integrals (4) and (5), we get

$$C_2=\frac{i}{12}\frac{\alpha_0}{4\pi^3}\mathrm{Sp}\int\beta_\mu G_0(p)\beta_\nu G_0(p)\Phi_{\nu\mu}(p)G_0(p)A_0^3(p)d^4p,$$
$$C_3=\frac{i}{12}\frac{\alpha_0}{4\pi^3}\mathrm{Sp}\int\beta_\mu G_0(p)\Phi_{\nu\nu}(p)G_0(p)\beta_\mu G_0(p)A_0^3(p)d^4p;\tag{15}$$

$$\Phi_{\nu\mu}(p)=\frac{\alpha_0}{4\pi^3}A_0^3(p)\int\beta_\sigma G_0(k)\beta_\nu G_0(k)\beta_\mu G_0(k)\beta_\sigma D_0(k-p)d^4k\tag{16}$$

$$=\frac{\alpha_0}{4\pi}d_0(p)\frac{1}{p^2}\left[-\frac{\hat pp_\mu p_\nu}{p^2}+\frac{1}{2}(\hat p\delta_{\mu\nu}-p_\mu\beta_\nu-p_\nu\beta_\mu)+2p_\mu X\beta_\nu+2p_\nu\beta_\mu X\right]\tag{17}$$

(in the last integral we have integrated with respect to k in the standard manner). Substituting (17) in (15) and (16) and carrying out the logarithmic integration, we obtain

$$C_2+C_3=-\frac{\alpha_0}{4\pi}\ln\left(1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{m^2}\right).\tag{18}$$

We proceed now to the integral C_1. To separate the $\alpha_0(\alpha_0L)^n$ contribution of interest to us, the functions G and Γ_μ in this integral must be taken both in the $(\alpha_0L)^n$ approximation (in which case the integration should be exact), and in the $\alpha_0(\alpha_0L)^n$ approximation (in which case logarithmic integration suffices). The functions G and Γ_μ, with allowance for the $\alpha_0(\alpha_0L)^n$ approximation terms, take the form

$$G(p)=\left[-\hat pA(p)+imB(p)-\frac{i}{m}Xp^2\varphi(p)\right]^{-1}=\frac{-\hat pA-imB+(i/m)A^2B^{-1}(\hat p^2-p^2)}{p^2A^2+m^2B^2}(1+\varphi A^{-2}B)+\frac{i}{m}Y\varphi A^{-2};\tag{19}$$

$$\Gamma_\mu(p,p)=-\frac{\partial}{\partial p_\mu}G^{-1}(p)=\beta_\mu A(p)+\hat p\frac{\partial}{\partial p_\mu}A(p)-im\frac{\partial}{\partial p_\mu}B(p)+\frac{i}{m}X2p_\mu\varphi(p)=\beta_\mu A+\frac{\alpha_0}{4\pi}\frac{\hat pp_\mu}{p^2}A_0d_0+\frac{\alpha_0}{2\pi}\frac{imp_\mu}{p^2}B_0d_0+\frac{i}{m}2p_\mu X\varphi,\tag{20}$$

where the argument p has been left out from all the functions for simplicity. In the last equation, in the differentiation of A(p) and B(p) we have replaced A and B by A_0 and B_0 without affecting the assumed accuracy.

Substituting (19) and (20) in (3) and carrying out the corresponding calculations under the integral sign, we obtain

$$C_1=\frac{i}{12}\frac{\alpha_0}{4\pi^3}\int\left\{\frac{-4p^4+20m^4B_0^4(p)A_0^{-4}(p)}{(p^2+m^2B_0^2(p)A_0^{-2}(p))^4}-13\frac{\alpha_0}{\pi}\frac{d_0(p)}{(p^2+m^2)^2}\right\}d^4p.\tag{21}$$

In the first term of the integrand of (21), the integration should be exact, and in the second logarithmic. The first term in (21) can be simplified by recognizing that the term $mB_0A_0^{-1}$ makes an $\alpha_0(\alpha_0L)^n$ contribution only in the region $p^2 \sim m^2$, and therefore, taking into account the explicit form of the functions A_0 and B_0 (7), it can be transformed into

$$mB_0(p)A_0^{-1}(p)\to mB_0(m)A_0^{-1}(m)=m.$$

Further integration of (21) entails no difficulty and leads to the result

$$C_1=\frac{\alpha_0}{12\pi}\left(\ln\frac{\Lambda^2}{m^2}-\frac{8}{3}\right)+\frac{13\alpha_0}{4\pi}\ln\left(1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{m^2}\right).\tag{22}$$

On the basis of the results (18) and (22), in accordance with formulas (1) and (2), we obtain the following value for Z_3 with allowance for the $(\alpha_0L)^n$ and $\alpha_0(\alpha_0L)^n$ approximations[3)]

$$Z_3^{-1}=1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{m^2}-\frac{2\alpha_0}{9\pi}+3\frac{\alpha_0}{\pi}\ln\left(1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{m^2}\right).\tag{23}$$

PHOTON GREEN'S FUNCTION

We proceed now to the photon Green's function D(k). The calculation of the asymptotic $\alpha_0(\alpha_0L)^n$ form of the function D(k) can be carried out in the usual manner with the aid of the Dyson's equations. If we know the value of Z_3, then the asymptotic form of D(k) can be obtained also by another method, using the connection between D and Z_3.

The nonrenormalized function D(k) satisfies the following relations[8]:

$$\lim_{k\to0}ik^2D(k)=Z_3,\tag{24}$$

$$\lim_{k\to\infty}ik^2D(k)=1.\tag{25}$$

The changeover in relations (24) and (25) from the exact value of D(k) to its asymptotic form, at the accuracy considered by us, correspond to replacing the limits $k\to0$ and $k\to\infty$ in the logarithmic terms by the limits $k\to m$ and $k\to\Lambda$. This change of the limits is reflected only in the value of the nonlogarithmic term, i.e., a constant, which we shall denote by c. Comparing relations (24) and (25) with the result (23), we find

$$d(k)=\left[1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{k^2}+3\frac{\alpha_0}{\pi}\ln\left(1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{k^2}\right)+\alpha_0c\right]^{-1}.\tag{26}$$

The constant c is determined from the agreement between (26) and the results of the calculations in the first order in α_0[9], and turns out to be equal to zero. Thus, the asymptotic form of the photon Green's function d(k), with allowance for the terms of the $(\alpha_0L)^n$ and $\alpha_0(\alpha_0L)^n$ approximations, takes the form

$$d(k)=\left[1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{k^2}+3\frac{\alpha_0}{\pi}\ln\left(1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{k^2}\right)\right]^{-1}.\tag{27}$$

MESON GREEN'S FUNCTION AND VERTEX FUNCTION

1. The meson Green's function G(p) has the following structure:

$$G^{-1}(p)=-\hat pA(p)+imB(p)-\frac{i}{m}Xp^2\varphi(p).\tag{28}$$

The $\alpha_0(\alpha_0 L)^n$ asymptotic forms of the functions A, B, and φ are easiest to find from the corresponding asymptotic vertex function $\Gamma_\lambda(p, q)$ in the region $p^2 \gg q^2 \gg m^2$, using the generalized Ward identity

$$(p-q)_\lambda \Gamma_\lambda(p,q) = G^{-1}(q) - G^{-1}(p) = \hat{p}A(p) - \hat{q}A(q) - im[B(p) - B(q)] + \frac{i}{m}X[p^2\varphi(p) - q^2\varphi(q)]. \quad (29)$$

In this case the asymptotic form of $\Gamma_\lambda(p, q)$ need be known only accurate to terms linear in m/p and q/p.

In the $\alpha_0(\alpha_0 L)^n$ approximation the equation for the vertex function can be shown to receive contributions, besides the three-vertex diagram $\Gamma_\lambda^{(3)}$, also from the five-vertex diagram $\Gamma_\lambda^{(5)}$ and the seven-vertex diagrams $\Gamma_\lambda^{(7)}$. The appearance of seven-vertex diagrams in this approximation is due to the fact that at large momenta the mesic Green's function assumes a constant value (with logarithmic accuracy), unlike in spinor electrodynamics, where the electron Green's function in the asymptotic region is inversely proportional to the momentum, and this is the reason why there are no seven-vertex diagrams in the $\alpha(\alpha L)^n$ approximation[6]. In the considered approximation, the equation for Γ_λ takes the form

$$\Gamma_\lambda(p,q) = \beta_\lambda + \Gamma_\lambda^{(3)}(p,q) + \Gamma_\lambda^{(5)}(p,q) + \Gamma_\lambda^{(7)}(p,q), \quad (30)$$

where $\Gamma_\lambda^{(3)}$, $\Gamma_\lambda^{(5)}$, and $\Gamma_\lambda^{(7)}$ are expressed in terms of Γ_ν, G, and D in accordance with the diagrams of Fig. 2.

To simplify the exposition that follows, we introduce the following notation. Any one of the functions f considered by us will be represented in the form

$$f = f_0 + \tilde{f}, \quad (31)$$

where f_0 is the contribution of the $(\alpha_0 L)^n$ approximation and $\tilde{f}$ is the contribution of the $\alpha_0(\alpha_0 L)^n$ approximation. An equation for $\tilde{\Gamma}_\lambda$ is obtained from (30) by separating the $\alpha_0(\alpha_0 L)^n$ parts in each term:

$$\tilde{\Gamma}_\lambda(p,q) = \tilde{\Gamma}_\lambda^{(3)}(p,q) + \tilde{\Gamma}_\lambda^{(5)}(p,q) + \tilde{\Gamma}_\lambda^{(7)}(p,q). \quad (32)$$

In accord with Fig. 2, the integral $\Gamma_\lambda^{(3)}(p, q)$ takes the form

$$\Gamma_\lambda^{(3)}(p,q) = \frac{\alpha_0}{4\pi^3}\int \Gamma_\mu(p, p-k)G(p-k)\Gamma_\lambda(p-k, q-k) \times G(q-k)\Gamma_\mu(q-k, q)D(k)d^4k. \quad (33)$$

To obtain $\tilde{\Gamma}_\lambda(p, q)$ from (33) it suffices, after representing each of the functions in the integrand in the form (31), to retain, first, the products of all f_0 (this integral will be designated I_λ) and, second, the product of five f_0 by one of the $\tilde{f}$ (the corresponding integrals will be designated $I_\lambda^{(1)}, \ldots I_\lambda^{(6)}$, where the superscript corresponds to the serial number of the function taken with the tilde, reading from left to right). In the integrals $\tilde{\Gamma}_\lambda^{(5)}$ and $\tilde{\Gamma}_\lambda^{(7)}$, which are not written out to save space, it suffices to take all the functions under the integral sign in the $(\alpha_0 L)^n$ approximation. As a result, Eq. (32) takes the form

$$\tilde{\Gamma}_\lambda(p,q) = I_\lambda(p,q) + \sum_{i=1}^{6} I_\lambda^{(i)}(p,q) + \tilde{\Gamma}_\lambda^{(5)}(p,q) + \tilde{\Gamma}_\lambda^{(7)}(p,q). \quad (34)$$

This equation for the vertex function $\tilde{\Gamma}_\lambda(p, q)$, in accordance with the problem posed above, will be solved in the region $p^2 \gg q^2 \gg m^2$ with accuracy up to terms linear in m/p and q/p.

2. Let us consider the integral $I_\lambda(p, q)$

$$I_\lambda(p,q) = \frac{\alpha_0}{4\pi^3}\int \Gamma_0^\mu(p, p-k)G_0(p-k)\Gamma_0^\lambda(p-k, q-k) \times G_0(q-k)\Gamma_0^\mu(q-k, q)D_0(k)d^4k. \quad (35)$$

It gives both an $(\alpha_0 L)^n$ contribution, which results from

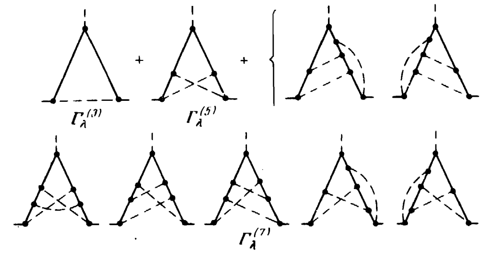

FIG. 2

logarithmic integration, and the sought $\alpha_0(\alpha_0 L)^n$ contribution, the determination of which calls for a more accurate integration. The integrands are given by formulas (6) - (11). In the integral (35), the functions G and Γ_μ, which contain the momentum p, can be simplified (cf. the analogous simplification (14) in the integrals (4) and (5)), and reduce effectively to

$$\Gamma_0^\mu(p, p-k) \to \beta_\mu A_0(p), \; \Gamma_0^\lambda(p-k, q-k) \to \beta_\lambda A_0(p), \; A_0(p-k) \to A_0(p), \; B_0(p-k) \to B_0(p). \quad (36)$$

We note that since we are calculating $\tilde{\Gamma}_\lambda(p, q)$ accurate to terms linear in m/p and q/p, it suffices to calculate the function $\Gamma_0^\mu(q - p, q)$, which enters in (35), accurate to terms linear in m/k and q/k.

As a result of the simplifications (36), the integral (35) takes the form

$$I_\lambda(p,q) = \frac{\alpha_0}{4\pi^3}A_0^2(p)\int \beta_\mu G_0(p-k)\beta_\lambda G_0(q-k)\Gamma_0^\mu(q-k, q)D_0(k)d^4k. \quad (37)$$

Substituting in (35) the expressions (6) - (11) and integrating with the aid of the formulas of the Appendix of [6], we obtain

$$I_\lambda(p,q) = \frac{\alpha_0}{4\pi}\Big\{-\frac{3}{4}\beta_\lambda A_0 d_0 + \frac{1}{2}A_0 d_0 \frac{1}{p^2}(p_\sigma\beta_\lambda - \hat{p}\delta_{\sigma\lambda} - p_\lambda\beta_\sigma)[J_\sigma(p,q) + q_\sigma] + \frac{1}{p^2}\Big[\frac{i}{m}A_0 B_0^{-1}(p_\lambda p_\sigma - p^2\delta_{\lambda\sigma}) - p_\lambda\beta_\sigma\Big]X\Big[\frac{1}{4}A_0 d_0(J_\sigma(p,q) + q_\sigma) - \frac{3}{4}q_\sigma A_0 d_0(q) + im\beta_\sigma(B_0 d_0 - A_0 B_0(q)A_0^{-1}(q)d_0(q))\Big]\Big\}. \quad (38)$$

3. We now consider the integral $\tilde{\Gamma}_\lambda^{(5)}(p, q)$. This integral takes the following form (see Fig. 2) (here and below, where there is no danger of misunderstanding, we shall not write out the corresponding index in the functions of the $(\alpha_0 L)^n$ approximation):

$$\Gamma_\lambda^{(5)}(p,q) = \left(\frac{\alpha_0}{4\pi^3}\right)^2\int \Gamma_\mu(p, p-t)G(p-t)\Gamma_\sigma(p-t, p-t-k) \times G(p-t-k)\Gamma_\lambda(p-t-k, q-t-k)G(q-t-k)\Gamma_\mu(q-t-k, q-k) \times G(q-k)\Gamma_\sigma(q-k, q)D(t)D(k)d^4k\,d^4t. \quad (39)$$

The functions G, Γ_μ, and D in (39) are given by formulas (6) - (11). To separate the $\alpha_0(\alpha_0 L)^n$ contribution, one of the integrations in (39) (with respect to d^4k) should be logarithmic, and the other (with respect to d^4t) must be exact. The logarithmic integration occurs in the regions $\Lambda^2 \gg k^2 \gg p^2$ and $p^2 \gg k^2 \gg q^2$. In the region $p^2 \gg k^2 \gg q^2$ the integrand of (39) can be simplified (cf. the simplification of the logarithmic functions in the integrals (4) and (5)), and $\tilde{\Gamma}_\lambda^{(5)}$ reduces to the form

$$\tilde{\Gamma}_\lambda^{(5)}(p,q) \to \left(\frac{\alpha_0}{4\pi^3}\right)^2\int \beta_\mu G(p-t)\beta_\sigma G(p-t)\beta_\lambda G(q-t-k) \times \Gamma_\mu(q-t-k, q-k)G(q-k)\Gamma_\sigma(q-k, q)D(k)D(t)A^3(t)d^4k\,d^4t = \Big\{\frac{\partial}{\partial p_\sigma}[\Gamma_0^\lambda(p, q-k) + I_\lambda(p, q-k)]\Big\}J_\sigma(p,q) = \frac{\alpha_0}{4\pi}A(p)d(p)\frac{\beta_\lambda p_\sigma}{p^2}J_\sigma(p,q). \quad (40)$$

In the region of logarithmic integration $\Lambda^2 \gg k^2 \gg p^2$, carrying out the corresponding simplifications of the integrands, we obtain

$$\tilde{\Gamma}_\lambda^{(5)}(p,q)\to\left(\frac{\alpha_0}{4\pi^3}\right)^2\int\Gamma_\mu(p,p-t)G(p-t)\beta_\sigma G(p-t-k)\beta_\lambda G(q-t-k)\times\beta_\mu G(q-k)\Gamma_\sigma(q-k,q)D(k)D(t)A^3(k)\,d^4k\,d^4t. \tag{41}$$

In the calculation of this integral it is convenient to separate the contributions of the term $\sim q - p$, so that this makes it possible to simplify significantly the remaining integral. The calculations are standard and the result is

$$\tilde{\Gamma}_\lambda^{(5)}(p,q)\to\frac{3}{4}\frac{i}{m}X\left(\frac{\alpha_0}{4\pi}\right)^2\int_p^\Lambda(p-q)_\lambda A^2(k)B^{-1}(k)d^2(k)\frac{dk^2}{k^2}+\frac{\alpha_0}{4\pi^3}\int_p^\Lambda\left\{-\frac{\partial}{\partial k_\lambda}[\Gamma_\sigma^\sigma(p,q-k)+I_\sigma(p,q-k)]\right\}G(q-k)\times\Gamma_\sigma(q-k,q)D(k)d^4k=\left(\frac{\alpha_0}{4\pi}\right)^2\int_p^\Lambda\left[-\frac{1}{4}\beta_\lambda-\frac{3}{4}\frac{i}{m}X(p+q)_\lambda A(k)B^{-1}(k)\right]A(k)d^2(k)\frac{dk^2}{k^2}. \tag{42}$$

Gathering together the results (40) and (42), we obtain for $\tilde{\Gamma}_\lambda^{(5)}(p,q)$ the final expression:

$$\tilde{\Gamma}_\lambda^{(5)}(p,q)=\frac{\alpha_0}{4\pi}\frac{\beta_\lambda p_\sigma}{p^2}A(p)d(p)J_\sigma(p,q)+\left(\frac{\alpha_0}{4\pi}\right)^2\int_p^\Lambda\left[-\frac{1}{4}\beta_\lambda-\frac{3}{4}\frac{i}{m}X(p+q)_\lambda A(k)B^{-1}(k)\right]A(k)d^2(k)\frac{dk^2}{k^2}. \tag{43}$$

4. We proceed now to the integrals $\Gamma_\lambda^{(7)}(p, q)$. To obtain the $\alpha_0(\alpha_0 L)^n$ contribution to $\Gamma_\lambda^{(7)}$ it suffices to take all the integrand functions in the $(\alpha_0 L)^n$ approximation, since one of the integration has a nonlogarithmic character. These integrals, just as the integrals $\Gamma_\lambda^{(5)}$ considered above, are calculated in accord with the following scheme: first we separate the regions of exact and logarithmic integrations; we then simplify the logarithmic functions in the integrals (for an analogous simplification see the integrals (4) and (5)); this simplification makes it possible to carry out exact integration, using standard integrals, and the final result is obtained after performing the two remaining logarithmic integrations.

We omit the calculations, which are quite cumbersome, and present only the final answer for $\tilde{\Gamma}_\lambda^{(7)}$:

$$\tilde{\Gamma}_\lambda^{(7)}(p,q)=-\frac{21}{4}\frac{i}{m}\left(\frac{\alpha_0}{4\pi}\right)^2\int_p^\Lambda[XJ_\lambda(k,q)+J_\lambda(k,p)X]A^2(k)B^{-1}(k)d^2(k)\frac{dk^2}{k^2}. \tag{44}$$

5. We consider now the integrals $I_\lambda^{(i)}(p, q)$. We write out explicitly only the expression for $I_\lambda^{(1)}$:

$$I_\lambda^{(1)}(p,q)=\frac{\alpha_0}{4\pi^3}\int\tilde{\Gamma}_\mu(p,p-k)G_0(p-k)\Gamma_0^\lambda(p-k,q-k)\times G_0(q-k)\Gamma_0^\mu(q-k,q)D_0(k)\,d^4k. \tag{45}$$

The remaining $I_\lambda^{(i)}$ differ from (45) only that for the functions with the tilde we take not Γ_μ, but functions whose positions correspond to the value of the index (i). The $I_\lambda^{(i)}$ contain one $\alpha_0(\alpha_0 L)^n$-approximation function, and therefore the $\alpha_0(\alpha_0 L)^n$ contribution to them is obtained from the logarithmic integration in the regions $\Lambda^2 \gg k^2 \gg p^2$ and $p^2 \gg k^2 \gg q^2$.

We consider first the region $p^2 \gg k^2 \gg q^2$. Expanding the integrand in this region in powers of the parameter k/p, it is easy to estimate that the required contribution is made to the functions $\Gamma_\mu(p, p - k)$, $G(p - k)$, and $\Gamma_\lambda(p - k, q - k)$ only by the first term of the expansion. This allows us to write

$$I_\lambda^{(1)}(p,q)+I_\lambda^{(2)}(p,q)+I_\lambda^{(3)}(p,q)\to\{\tilde{\Gamma}_\mu(p,p)G_0(p)\beta_\lambda+\beta_\mu\tilde{G}(p)\beta_\lambda A_0(p)+\beta_\mu G_0(p)\tilde{\Gamma}_\lambda(p,q)\}A_0(p)J_\mu(p,q), \tag{46}$$

$$I_\lambda^{(4)}(p,q)+I_\lambda^{(5)}(p,q)+I_\lambda^{(6)}(p,q)\to\beta_\mu G_0(p)\beta_\lambda A_0^2(p)\times\frac{\alpha_0}{4\pi^3}\int_q^p\{\tilde{G}(q-k)\Gamma_0^\mu(q-k,q)D_0(k)+G_0(q-k)\tilde{\Gamma}_\mu(q-k,q)D_0(k)+G_0(q-k)\Gamma_0^\mu(q-k,q)\tilde{D}(k)\}d^4k. \tag{47}$$

The functions A_0, G_0, Γ_0^μ, D_0, and $\tilde{\Gamma}_\mu(p, p)$ that enter in in (46) and (47) are given by formulas (6) - (11) and (20), while the function $G(p)$ is determined from (19) and takes the form

$$\tilde{G}(p)=\frac{1}{p^2A_0^2}\left[\hat{p}\tilde{A}-im(\tilde{B}-2\tilde{A}B_0A_0^{-1})-\frac{i}{m}\tilde{B}A_0^2B_0^{-2}(\hat{p}^2-p^2)\right]+\frac{i}{m}Y\varphi A_0^{-2}+G_0(p)\varphi B_0A_0^{-2}. \tag{48}$$

The asymptotic form of the function $\tilde{\Gamma}_\lambda(p, q)$ in the region $p^2 \gg q^2 \gg m^2$ can be represented, accurate to terms linear in m/p and q/p, in the form

$$\tilde{\Gamma}_\lambda(p,q)=\beta_\lambda A_1+\frac{i}{m}Xp_\lambda\varphi_1+\tilde{\gamma}_\lambda\left(\frac{p}{p},q\right)+\tilde{\gamma}_\lambda\left(\frac{1}{p},q\right), \tag{49}$$

where $\tilde{\gamma}_\lambda(p/p, q)$ denotes terms of zeroth order, and $\tilde{\gamma}_\lambda(1/p, q)$ denotes terms of first order in the parameters m/p and q/p.[4] The contribution of the integrals I_λ, $\tilde{\Gamma}_\lambda^{(5)}$, and $\tilde{\Gamma}_\lambda^{(7)}$ to the function $\tilde{\gamma}_\lambda(p/p, q)$ is proportional to the matrix X. It is easy to show that the integrals $I_\lambda^{(i)}$ also make a contribution having the same structure to $\gamma_\lambda(p/p, q)$.

Substituting (49) in the generalized Ward identity (29) and separating terms having the same matrix structure, we obtain the following relations:

$$A_1=A(p),\qquad \varphi_1=\varphi(p), \tag{50}$$

$$p_\lambda\tilde{\gamma}_\lambda\left(\frac{p}{p},q\right)=\frac{i}{m}Xpq\,\varphi(p), \tag{51}$$

$$p_\lambda\tilde{\gamma}_\lambda\left(\frac{1}{p},q\right)=im[\tilde{B}(q)-\tilde{B}(p)]+\hat{q}[\tilde{A}(p)-\tilde{A}(q)]. \tag{52}$$

Substituting in (46) and (47) all the necessary functions and performing the matrix operations, we obtain

$$I_\lambda^{(1)}(p,q)+I_\lambda^{(2)}(p,q)+I_\lambda^{(3)}(p,q)=\frac{1}{p^2}\left\{\frac{i}{m}AB^{-1}(2\tilde{A}-\tilde{B}AB^{-1})\times(p_\mu p_\lambda-p^2\delta_{\mu\lambda})-2\frac{i}{m}p_\mu p_\lambda\varphi-p_\lambda\beta_\mu\tilde{A}-\frac{3\alpha_0}{4\pi}\hat{p}\frac{p_\lambda p_\mu}{p^2}Ad+\frac{\alpha_0}{2\pi}p_\mu\beta_\lambda Ad\right\}XJ_\mu(p,q), \tag{53}$$

$$I_\lambda^{(4)}(p,q)+I_\lambda^{(5)}(p,q)+I_\lambda^{(6)}(p,q)=\frac{1}{p^2}\left[\frac{i}{m}A^2B^{-1}(p_\mu p_\lambda-\delta_{\mu\lambda}p^2)-Ap_\lambda\beta_\mu\right]\times\frac{\alpha_0}{4\pi^3}\int_q^p\left\{-im\beta_\mu\left(\frac{\tilde{B}}{A}-\frac{\tilde{A}B}{A^2}+\frac{\varphi B^2}{A^3}+\frac{B\tilde{d}}{Ad}\right)-\frac{1}{2}q_\mu\frac{\tilde{d}}{d}+\left[\frac{5}{4}\frac{\tilde{A}}{A}-\frac{3}{4}\frac{\tilde{B}}{B}-\frac{1}{2}\frac{\varphi B}{A^2}-\frac{1}{2}\frac{\tilde{d}}{d}\right]J_\mu(k,q)+\frac{1}{A}\hat{k}\tilde{\gamma}_\mu\left(\frac{1}{-k},q\right)-im\frac{B}{A^2}\tilde{\gamma}_\mu\left(\frac{k}{k},q\right)\right\}d(k)\frac{d^4k}{ik^4}. \tag{54}$$

We now proceed to calculate the integrals $I_\lambda^{(i)}$ in the logarithmic integration region $\Lambda^2 \gg k^2 \gg p^2$. The functions $\tilde{\Gamma}_\mu(p, p - k)$ and $\tilde{G}(p - k)$ which enter in the integrals are taken in the form (49) and (48), while the function $\tilde{\Gamma}_\lambda(p - k, q - k)$ is given in (A.6) of the Appendix. We note that the considered region of logarithmic integration in the integrals $I_\lambda^{(i)}$ makes no contribution to the function $\tilde{\gamma}(1/p, q)$ from (49), and therefore $\tilde{\gamma}_\mu(1/p, q)$ is determined completely by the results (38), (43), (44), (53), and (54).

For the calculations that follow, it is convenient to represent the functions $\tilde{\gamma}_\mu(p/p, q)$ and $\tilde{\gamma}_\mu(1/p, q)$ in the form

$$\tilde{\gamma}_\mu\left(\frac{p}{p},q\right)+\tilde{\gamma}_\mu\left(\frac{1}{p},q\right)=\left[\frac{i}{m}AB^{-1}(p_\mu p_\sigma-p^2\delta_{\mu\sigma})-p_\mu\beta_\sigma\right]\times\frac{1}{n^2}\tilde{\gamma}_\sigma+\tilde{\gamma}_\mu^*\left(\frac{p}{p},q\right)+\tilde{\gamma}_\mu^*\left(\frac{1}{p},q\right), \quad (55)$$

where on the basis of (38), (43), (44), (53), and (54) we have

$$\tilde{\gamma}_\sigma=\tilde{A}J_\sigma(p,q)+\frac{\alpha_0}{4\pi}\left[\frac{1}{4}Ad(J_\sigma(p,q)+q_\sigma)-\frac{3}{4}q_\sigma Ad(q)\right.$$
$$\left.+im\beta_\sigma(Bd-AB(q)A^{-1}(q)d(q))\right]+A\frac{\alpha_0}{4\pi^3}\int_q^p\left\{-im\beta_\mu\left(\frac{\tilde{B}}{A}\right.\right.$$
$$\left.-\frac{\tilde{A}B}{A^2}+\frac{\varphi B^2}{A^3}+\frac{B\tilde{d}}{Ad}\right)-\frac{1}{2}q_\mu\frac{\tilde{d}}{d}+\left(\frac{5}{4}\frac{\tilde{A}}{A}-\frac{3}{4}\frac{\tilde{B}}{B}-\frac{1}{2}\frac{\varphi B}{A^2}-\frac{1}{2}\frac{\tilde{d}}{d}\right)$$
$$\left.\cdot J_\mu(k,q)+\frac{1}{A}\hat{k}\tilde{\gamma}_\mu^*\left(\frac{1}{-k},q\right)-im\frac{B}{A^2}\tilde{\gamma}_\mu^*\left(\frac{k}{k},q\right)-\frac{1}{2}\frac{1}{A}\tilde{\gamma}_\mu\right\}d(k)\frac{d^4k}{ik^4}, \quad (56)$$

$$\tilde{\gamma}_\mu^*\left(\frac{1}{p},q\right)=\frac{\alpha_0}{4\pi}Ad\frac{1}{p^2}\left[\frac{1}{2}(p_\sigma\beta_\mu-\hat{p}\delta_{\sigma\mu}-p_\mu\beta_\sigma)(J_\sigma(p,q)+q_\sigma)\right.$$
$$\left.+3\left(p_\sigma\beta_\mu-\hat{p}\frac{p_\mu p_\sigma}{p^2}\right)J_\sigma(p,q)\right]. \quad (57)$$

In the foregoing expressions, the functions A, B, d, and φ under the integral sign have the argument k, and those outside the integral have the argument p. We note that the function $\tilde{\gamma}_\sigma$ can be obtained by starting from relations (52) and (55) - (57) in the form

$$\tilde{\gamma}_\sigma=im\beta_\sigma\left[\tilde{B}(p)-\tilde{B}(q)-\frac{\alpha_0}{8\pi}(B(p)-B(q))d(p)\right]+q_\sigma\left[\tilde{A}(q)-\tilde{A}(p)-\frac{\alpha_0}{8\pi}A(q)d(p)\right]. \quad (58)$$

Substituting the corresponding $(\alpha_0 L)^n$- and $\alpha_0(\alpha_0 L)^n$-approximation functions in the integrals $I_\lambda^{(i)}$ and carrying out the logarithmic integration in the region $\Lambda^2 \gg k^2 \gg p^2$, we obtain

$$\sum_{i=1}^{6}I_\lambda^{(i)}(p,q)\to\frac{\alpha_0}{4\pi^3}\int_p^\Lambda\left\{\beta_\lambda\left[-\frac{1}{2}\left(\tilde{A}+A\frac{\tilde{d}}{d}\right)-\frac{B}{A}\varphi+\frac{\alpha_0}{16\pi}Ad\right]\right.$$
$$+\frac{i}{m}X(p+q)_\lambda\left(\frac{1}{2}\varphi-\frac{9\alpha_0}{16\pi}\frac{A^2}{B}d\right)+\frac{i}{m}[XJ_\lambda(k,q)+J_\lambda(k,p)X]$$
$$\times\left[\frac{1}{2}\varphi-\frac{3\alpha_0}{16\pi}\frac{A^2}{B}d-\frac{3}{4}\frac{A}{B}\left(\tilde{A}-\tilde{B}\frac{A}{B}\right)\right]+\left[X\tilde{\gamma}_\mu^*\left(\frac{k}{k},q\right)\right.$$
$$\left.\left.+\tilde{\gamma}_\mu^*\left(\frac{k}{k},p\right)X\right]\left(2\frac{k_\lambda k_\mu}{k^2}-\delta_{\lambda\mu}\right)\right\}d(k)\frac{d^4k}{ik^4}. \quad (59)$$

6. Gathering together the calculation results (38), (43), (44), and (53) - (59), we obtain the following expression for $\Gamma_\lambda(p, q)$:

$$\tilde{\Gamma}_\lambda(p,q)=\beta_\lambda\tilde{A}+\frac{i}{m}Xp_\lambda\varphi+\frac{1}{p^2}\left[\frac{i}{m}\frac{A}{B}(p_\sigma p_\lambda-p^2\delta_{\lambda\sigma})-p_\lambda\beta_\sigma\right]\tilde{\gamma}_\sigma$$
$$+\tilde{\gamma}_\lambda^*\left(\frac{p}{p},q\right)+\tilde{\gamma}_\lambda^*\left(\frac{1}{p},q\right)=-\frac{3\alpha_0}{16\pi}Ad\beta_\lambda+\frac{1}{p^2}\left[\frac{i}{m}\frac{A}{B}(p_\sigma p_\lambda-p^2\delta_{\lambda\sigma})\right.$$
$$\left.-p_\lambda\beta_\sigma\right]\left[\tilde{\gamma}_\sigma+\tilde{\gamma}_\lambda^*\left(\frac{1}{p},q\right)\right]+\frac{1}{p^2}\frac{i}{m}\left[\frac{A}{B}\left(\tilde{A}-\tilde{B}\frac{A}{B}\right)(p_\mu p_\lambda-p^2\delta_{\lambda\mu})-2p_\lambda p_\mu\varphi\right]$$
$$\times XJ_\mu(p,q)+\frac{\alpha_0}{4\pi^3}\int_p^\Lambda\left\{\beta_\lambda\left[-\frac{1}{2}\left(\tilde{A}+A\frac{\tilde{d}}{d}\right)-\frac{B}{A}\varphi\right]\right. \quad (60)$$
$$+\frac{i}{m}X(p+q)_\lambda\left(\frac{1}{2}\varphi-\frac{3\alpha_0}{4\pi}\frac{A^2}{B}d\right)+\frac{i}{m}[XJ_\lambda(k,q)+J_\lambda(k,p)X]$$
$$\times\left[\frac{1}{2}\varphi-\frac{3\alpha_0}{2\pi}\frac{A^2}{B}d-\frac{3}{4}\frac{A}{B}\left(\tilde{A}-\tilde{B}\frac{A}{B}\right)\right]+\left[X\tilde{\gamma}_\mu^*\left(\frac{k}{k},q\right)\right.$$
$$\left.\left.+\tilde{\gamma}_\mu^*\left(\frac{k}{k},p\right)X\right]\left(2\frac{k_\lambda k_\mu}{k^2}-\delta_{\lambda\mu}\right)\right\}d(k)\frac{d^4k}{ik^4}.$$

In the foregoing expression we did not write out explicitly the functions $\gamma_\lambda^*(1/p, q)$ and $\tilde{\gamma}_\sigma$, since they are given by (57) and (58).

Separating in (60) the terms having the same matrix structure, we obtain three integral equations for the functions $\tilde{A}$, φ, and $\tilde{\gamma}_\lambda^*$, the solutions of which are

$$\tilde{A}(p)=A_0(p)\frac{3\alpha_0}{2\pi}\left(-3\xi^{-1}\ln\xi-\frac{3}{8}\xi^3-\frac{71}{16}\xi^{-1}+\frac{75}{16}\right),$$
$$\varphi(p)=\frac{9\alpha_0}{16\pi}(\xi^{-4}-1)\xi^{-3}\eta^{9/2}, \quad (61)$$

$$\tilde{\gamma}_\lambda^*\left(\frac{p}{p},q\right)=\frac{i}{m}Xq_\lambda\varphi+\frac{1}{p^2}\frac{i}{m}X\left[\frac{A}{B}\left(\tilde{A}-\tilde{B}\frac{A}{B}\right)-2\varphi\right]\times(p_\mu p_\lambda-p^2\delta_{\lambda\mu})J_\mu(p,q); \quad (62)$$

$$\xi=1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{p^2},\quad \eta=1+\frac{\alpha_0}{12\pi}\ln\frac{\Lambda^2}{m^2}. \quad (63)$$

We proceed now to the function $\tilde{B}(p)$. We have determined this function by starting from relation (52), which is the consequence of the generalized Ward identity. Substituting the expressions (55) - (57) obtained above for $\tilde{\gamma}_\lambda(1/p, q)$ in (52) and separating the terms linear in m, we obtain the following equation:

$$\tilde{B}(p)-\tilde{B}(q)\frac{A}{A(q)}=\frac{\tilde{A}}{A}(B-B(q))+\frac{\alpha_0}{4\pi}\left(\frac{7}{4}Bd-\frac{3}{4}B(q)d\right.$$
$$\left.-A\frac{B(q)}{A(q)}d(q)\right)+A\frac{\alpha_0}{4\pi}\int_q^p\left\{-\frac{3}{2}\left(\frac{\tilde{B}}{A}-\frac{\tilde{A}B}{A^2}+\frac{B\tilde{d}}{Ad}\right)+\frac{3\alpha_0}{16\pi}\frac{B}{A}d\right. \quad (64)$$
$$\left.-\frac{B(q)}{A}\left(\frac{B}{A^2}\varphi-\frac{1}{2}\frac{\tilde{d}}{d}+\frac{1}{2}\frac{\tilde{A}}{A}+\frac{3\alpha_0}{16\pi}d\right)\right\}d(k)\frac{dk^2}{k^2},$$

where the functions A, B, d, and φ under the integral sign have the argument k, and those outside the integral sign have the argument p.

When solving this equation in the region $\Lambda^2 \gg p^2 \gg q^2 \gg m^2$, it is convenient to go over to the region $p^2 \approx \Lambda^2$. This changes the value of the nonlogarithmic term in (64), i.e., the constant, which we denote by $\alpha_0 g/\pi$, and the equation takes the form

$$\tilde{B}(\Lambda)-\frac{\tilde{B}(q)}{A(q)}=\frac{\alpha_0}{\pi}g+\frac{\alpha_0}{4\pi}\left(B(\Lambda)-\frac{B(q)}{A(q)}d(q)\right)$$
$$+\frac{\alpha_0}{4\pi}\int_q^\Lambda\left\{-\frac{3}{2}\left(\frac{\tilde{B}}{A}-\frac{\tilde{A}B}{A^2}+\frac{B\tilde{d}}{Ad}\right)+\frac{3\alpha_0}{16\pi}\frac{B}{A}d\right. \quad (65)$$
$$\left.-\frac{B(q)}{A}\left(\frac{B}{A^2}\varphi-\frac{1}{2}\frac{\tilde{d}}{d}+\frac{1}{2}\frac{\tilde{A}}{A}+\frac{3}{16}\frac{\alpha_0}{\pi}d\right)\right\}d(k)\frac{dk^2}{k^2}.$$

We have taken into account here the fact that, in accord with (7), (8), and (61),

$$A_0(\Lambda)=d_0(\Lambda)=1,\quad \tilde{A}(\Lambda)=-3\alpha_0/16\pi.$$

The functions A, B, and d in (65) are given by formulas (7) and (8), while the functions $\tilde{d}$, $\tilde{A}$, and φ are given by (27) (61), and (62).

The solution of Eq. (65) is

$$\tilde{B}(p)=B_0(p)\frac{\alpha_0}{\pi}\left(9\xi^{-1}\ln\xi-\frac{9}{16}\xi^3+\frac{47}{8}\xi^{-1}+g-\frac{85}{16}\right). \quad (66)$$

The constant g is determined from the agreement between formula (65) and the result of the calculations of $\tilde{B}$ in first order is α_0(A.9), and is equal to

$$g=-5/16. \quad (67)$$

Substituting (67) in (66), we obtain ultimately

$$\tilde{B}(p)=B_0(p)\frac{\alpha_0}{\pi}\left(9\xi^{-1}\ln\xi-\frac{9}{16}\xi^3+\frac{47}{8}\xi^{-1}-\frac{45}{8}\right). \quad (68)$$

Thus, the asymptotic forms of the functions A and B, with the $(\alpha_0 L)^n$ and $\alpha_0(\alpha_0 L)^n$ approximations taken into account, are

$$A(p)=\xi^{-3/2}\left[1+\frac{3\alpha_0}{2\pi}\left(-3\xi^{-1}\ln\xi-\frac{3}{8}\xi^3-\frac{71}{16}\xi^{-1}+\frac{75}{16}\right)\right] \quad (69)$$

$$B(p)=\xi^3\eta^{-3/2}\left[1+\frac{\alpha_0}{\pi}\left(9\xi^{-1}\ln\xi-\frac{9}{16}\xi^3+\frac{47}{8}\xi^{-1}-\frac{45}{8}\right)\right]. \quad (70)$$

The author is deeply grateful to P. I. Fomin for suggesting the topic and help with the work and to D. V. Volkov for useful discussions.

APPENDIX 1

In this Appendix we find the $\alpha_0(\alpha_0 L)^n$ asymptotic form of the function $\Gamma_\lambda(p, p+q)$ in the momentum region $p^2 \gg q^2 \gg m^2$. In this region, the function $\Gamma_\lambda(p, p+q)$ can be expanded in powers of the parameter q/p. We confine ourselves in the expression to terms linear in q, and in this case $\Gamma_\lambda(p, p+q)$ can be represented in the form

$$\tilde{\Gamma}_\lambda(p,p+q)=\tilde{\Gamma}_\lambda(p,p)+\frac{i}{m}Xq_\lambda\varphi_2+\delta_\lambda\left(\frac{q}{p}\right)+\delta_\lambda\left(\frac{q}{p^2}\right), \quad (A.1)$$

Where $\tilde{\delta}_\lambda(q/p)$ corresponds to the terms $\sim q/p$, and $\tilde{\delta}_\lambda(q/p^2)$ to the terms $\sim qm/p^2$. We note that the terms $\tilde{\delta}_\lambda$, which contain an odd number of β matrices or matrices proportional to the matrix X, make no contribution to the integrals $I_\lambda^{(i)}$ considered by us, and will therefore be omitted. Substituting (A.1) in the generalized Ward identity (29) and taking into account the structure of $\tilde{\Gamma}_\lambda(p, p)$ (20), we obtain

$$\varphi_2=\varphi(p). \quad (A.2)$$

We now proceed to the functions $\tilde{\delta}_\lambda$. In the logarithmic integration, there are no contributions proportional to q/p and qm/p^2 in the $(\alpha_0 L)^n$ approximation [5]. It is easy to verify that in the $\alpha_0(\alpha_0 L)^n$ approximation the integral containing logarithmic integration make no contribution to the functions $\tilde{\delta}_\lambda$. The terms of interest to us can appear only in the exact integration in the integral I_λ. This integral takes the form

$$I_\lambda(p,p+q)=\frac{\alpha_0}{4\pi^3}\int\Gamma_0^\mu(p,p-k)G_0(p-k)\Gamma_0^\lambda(p-k,p+q-k) \times G_0(p+q-k)\Gamma_0^\mu(p+q-k,p+q)D(k)d^4k, \quad (A.3)$$

where the functions G, Γ, and D are represented by the formulas (6) - (11). In the calculation of the terms $\tilde{\delta}_\lambda$ these functions can be simplified. This simplification is connected with the fact that upon integration of the logarithmic terms (containing $\ln k^2$) the $\alpha_0(\alpha_0 L)^n$ contribution proportional to q/p and qm/p^2 arises in the regions $k^2 \gg p^2$ and $p^2 \gg k^2 \gg q^2$, the integration in which leads effectively to the substitution $\ln k^2 \to \ln p^2$ uder the integral sign (see relations (12) and (13)). We thus obtain

$$\ln(p-k)^2,\ \ln(p+q-k)^2,\ \ln(p+q)^2 \to \ln p^2,$$
$$\Gamma_0^\mu(p,\ p-k),\ \Gamma_0^\mu(p-k,\ p+q-k),\ \Gamma_0^\mu(p+q-k,\ p+q) \to \beta_\mu A_0(p) \quad (A.4)$$

and as a result of these simplifications, Eq. (A.3) takes the form

$$I_\lambda(p,p+q)=\frac{\alpha_0}{4\pi^3}A_0^3(p)\int\beta_\mu G(p-k)\beta_\lambda G(p+q-k)\beta_\mu D(k)d^4k, \quad (A.5)$$

where all the logarithmic functions have the argument p.

The integration of (A.5) with the required accuracy yields

$$\delta_\lambda(q/p)=0, \quad \delta_\lambda\left(\frac{q}{p^2}\right)=\frac{\alpha_0}{8\pi}\frac{imq_\lambda}{p^2}B(p)d(p). \quad (A.6)$$

APPENDIX 2

We obtain here $\tilde{B}(p)$ in first order in α_0. By definition, we have

$$G^{-1}(p)=-\hat{p}A+imB=-\hat{p}+im-(\Sigma(p)-\Sigma(p_0)), \quad (A.7)$$

where $\Sigma(p)$ is the mass operator, $\hat{p}_0 = im$, and $p_0^2 = -m^2$. Calculations of $\Sigma(p)$ and $\Sigma(p_0)$ in the first order in α_0 are carried out in standard manner, and the result is

$$\Sigma(p)-\Sigma(p_0)=-\frac{3\alpha_0}{16\pi}\hat{p}+\frac{5\alpha_0}{16\pi}im. \quad (A.8)$$

From this we get

$$\tilde{B}^{(1)}(p)=-5\alpha_0/16\pi. \quad (A.9)$$

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[1] We adhere in the main the notation of the book by Akhiezer and Berestetskiĭ [9]. The metric and the β matrices are the same as in [9].

[2] Here and throughout, the $(\alpha_0 L)^n$-approximation functions will be marked by a zero subscript. For convenience, the vector indices of the vector quantities will be in the form of superscripts.

[3] We note that the value of Z_3^{-1} in the second order in α_0 coincides with the result of Sinclair, Hagen and Kim[11].

[4] The terms $\tilde{\gamma}_\lambda(1/p, q)$, which are proportional to the matrix X, make no contribution to the integrals considered by us, and will therefore be disregarded.

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Translated by J. G. Adashko
206