Theory of the anomalous skin effect in a plasma with a diffuse boundary

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A theory is developed of the anomalous skin effect in a diffuse-boundary plasma located in a magnetic field. Penetration of an electromagnetic wave into a finite size plasma layer is also investigated. Final solutions are obtained for cases when the electron concentration outside the plasma drops exponentially or according to a power law.

1. INTRODUCTION

The question of the anomalous skin effect in a plasma with a diffuse boundary was raised by Kapitza in connection with a study of a microwave pinch discharge in a high-pressure gas[1]. Since the plasma boundary in the high-frequency discharge is not abrupt, the existing theory of the anomalous skin effect (see, e.g.[2]), in Secs. 33 and 34 of[3], Secs. 17 and 18 of[4], etc.), which was developed for metals with sharp boundaries, cannot be applied directly to a plasma. In the absence of a magnetic field, a theory for the anomalous skin effect in a plasma with a diffuse boundary was constructed by Liberman, Pitaevskiĭ, and one of the authors[5] under the assumption that the concentration of the electrons depends monotonically on the coordinate. The final formulas were obtained in[5] for the case when the electron density decreases exponentially with distance outside the plasma.

In Sec. 2 we develop the theory of the anomalous skin effect in a plasma having a diffuse boundary and situated in a constant and uniform magnetic field. In Sec. 3 we consider the anomalous skin effect in a plasma layer with diffuse boundaries. This question is important, since the characteristic dimensions of the plasma in the experiments can be of the same order as, and even smaller than the electron mean free path. Since there are no grounds for assuming that the electron concentration always decreases exponentially with increasing distance outside the plasma, considerable interest attaches to the solution, in Sec. 4, of the integral equation for the case of the extremely anomalous skin effect in a plasma with a power-law dependence of the electron concentration on the coordinate.

We shall assume that in the absence of the high-frequency field the plasma is at equilibrium and that the dependence of the electron density $n_e(x)$ on the coordinate is specified and is maintained by the field of external forces $e\mathbf{E}_0(x)$ that act on the electrons (including the forces exerted by the ions). The potential $\varphi(x)$ of this field is connected with $n_e(x)$ by the Boltzmann formula

$$n_e(x) = n_0 \exp(-e\varphi(x) / kT_e). \tag{1.1}$$

In Secs. 2 and 4 we shall assume that the electron density $n_e(x)$ is a monotonic function and depends only on one coordinate x. The singularities resulting from the non-monotonicity of the function $n_e(x)$ are investigated in Sec. 3. Since the plasma is assumed to be in equilibrium, the presence of the electron concentration gradient does not affect the distribution of the constant magnetic field **H** (which is perpendicular to the density gradient); this field penetrates freely into the plasma and will henceforth be assumed uniform.

The magnetic field bends the electron trajectories. The electron motion in the direction of the concentration gradient is finite. The motion of the electrons in a plasma layer is likewise finite (see Sec. 3), for in this case $\varphi(x) \to \infty$ both as $x \to -\infty$ and as $x \to +\infty$, leading to the presence of two classical turning points (see Fig. 4 below). Moving along a finite trajectory, the electron can return many times to the skin layer, where it interacts actively with the field of the electromagnetic wave. In the general case, the connection between the current density $\mathbf{j}$ in the plasma and the electromagnetic field $\mathbf{E}$ is integral. For a wave propagating in the direction of the gradient electron concentration x we have

$$j_\mu(x) = \int_{-\infty}^{+\infty} \Sigma_{\mu\nu}(x, x') E_\nu(x')\, dx'; \quad \mu, \nu = y, z. \tag{1.2}$$

In the present paper we obtain expressions for the conductivity kernel $\Sigma_{\mu\nu}(x, x')$ at arbitrary relations between the electron mean free path, the dimensions of their trajectories, and the depth of penetration of the field into the plasma, and also at an arbitrary $n_e(x)$.

In the presence of a magnetic field, just as in the case of a sharp boundary, the interaction of the electrons with the high-frequency field has a resonant character if the period of the electron motion is close to an integer number of periods of the field (the Azbel'-Kaner cyclotron resonance[6]).

Besides the cyclotron resonance in a plasma with a continuous boundary, a new resonant phenomenon takes place, the gist of which consists of the following. If one period of motion of the electron is close to an integer odd number of half-periods of the high-frequency field, then after each revolution the field exerts on the electron a force which is equal in magnitude but opposite in sign. Therefore, if the electron acquires energy in the given revolution, then in the next revolution it will return this energy. The electrons thus do not interact resonantly with the electromagnetic wave, as a result of which the wave attenuates weakly. This phenomenon can be naturally called cyclotron antiresonance.

The investigation in Sec. 4 shows that the problem of the strongly anomalous skin effect in a plasma with a power-law $n_e(x)$ dependence is meaningful only if the exponent is $p > 4$. It should be noted that the effective depth of field penetration into the plasma has a characteristic dependence on the frequency in the case of a power-law function $n_e(x)$. The shapes of the cyclotron-resonance and cyclotron-antiresonance lines depend significantly on the exponent in the case of a power-law function $n_e(x)$, and differ strongly from the corresponding lines in a plasma with an exponentially decreasing electron concentration.

2. ANOMALOUS SKIN EFFECT IN A PLASMA WITH A DIFFUSE BOUNDARY IN A MAGNETIC FIELD. CYCLOTRON RESONANCE AND CYCLOTRON ANTIRESONANCE

We choose the y axis parallel to the constant and uniform magnetic field **H**. The kinetic equation linearized with respect to the small electromagnetic field $\mathbf{E}e^{i\omega t}$ takes the form

$$(i\omega+\nu_{\text{eff}})f_1+v_x\frac{\partial f_1}{\partial x}+\frac{eE_0}{m}\frac{\partial f_1}{\partial v_x}+\Omega\left(v_z\frac{\partial f_1}{\partial v_x}-v_x\frac{\partial f_1}{\partial v_z}\right)=\frac{eE_\nu}{m}\frac{\partial f_0}{\partial v_\nu},\quad \nu=y,z. \tag{2.1}$$

Here f_1 is the increment to the equilibrium distribution function

$$f_0=\left(\frac{m}{2\pi kT_e}\right)^{3/2}n_e(x)\exp\left(-\frac{mv^2}{2kT_e}\right),$$

and is proportional to the external electromagnetic field in whose penetration into the plasma we are interested; $E_0(x)$ is the electric field that contains the electrons and is constant in time. Its potential $\varphi(x)$ is connected with the electron concentration $n_e(x)$ by the Boltzmann formula (1.1); ω is the frequency of the electromagnetic wave, ν_{eff} is the effective number of collisions, and $\Omega = eH/mc$ is the Larmor frequency of the electrons.

Equation (2.1) is a first-order linear differential equation. Its solution reduces to integration of ordinary differential equations (the so-called characteristic equations)

$$\frac{dx}{v_x}=\frac{dv_x}{eE_0/m+\Omega v_z}=-\frac{dv_z}{\Omega v_x}(=dt),$$

which, when rewritten in the form

$$\frac{dx}{dt}=v_x,\quad \frac{dv_x}{dt}=\frac{e}{m}E_0+\Omega v_z,\quad \frac{dv_z}{dt}=-\Omega v_x, \tag{2.2}$$

are the equations of motion of the electron in the constant and homogeneous magnetic field H and in the electric field $E_0(x)$. Multiplying the second equation of (2.2) by v_x, the third by v_z, and adding, we obtain an integral of the motion, namely the law of conservation of the energy ϵ:

$$\varepsilon={}^1\!/_2 m(v_x^2+v_z^2)+e\varphi(x). \tag{2.3}$$

From the first and third equations of (2.2) we obtain one more integral of the motion

$$x_0=x+v_z/\Omega. \tag{2.4}$$

Eliminating v_z from (2.3) and (2.4), we get

$$\varepsilon=\frac{mv_x^2}{2}+e\tilde{\varphi}(x),\quad \tilde{\varphi}(x)=\varphi(x)+\frac{m\Omega^2}{2e}(x-x_0)^2,$$

i.e., the electron moves in the direction of the x axis in a field having an effective potential $\tilde{\varphi}(x)$ that increases without limit as $x\to\pm\infty$ (see Fig. 1). The electrons move along finite trajectories, and the classical turning points x_1^* and x_2^* are determined as functions of ϵ and x_0 from the equation

$$e\tilde{\varphi}(x^*)=\varepsilon.$$

The electrons reverse direction at the classical turning point. The corresponding boundary conditions on the distribution function are

$$f(v_x)=f_1(-v_x),\quad x=x_{1,2}^*. \tag{2.5}$$

They define uniquely an electron distribution function that satisfies Eq. (2.1). As a result we obtain for the density of the electric current in the plasma

$$j_\mu(x)=-2e^2\int_0^\infty dv_x\int_{-\infty}^{+\infty}dv_y\int_{-\infty}^{+\infty}dv_z\frac{v_\mu}{\operatorname{sh}\Phi(x_1^*,x_2^*)}\left[\int_{x_1^*}^{x}\operatorname{ch}\Phi(x_1^*,x')\operatorname{ch}\Phi(x_2^*,x)+\int_x^{x_2^*}\operatorname{ch}\Phi(x_1^*\ x)\operatorname{ch}\Phi(x_2^*,x')\right]\frac{v_\nu(x')}{v_x(x')}\frac{\partial f_0}{\partial\varepsilon}E_\nu(x')dx';\quad \mu,\nu=y,z. \tag{2.6}$$

Here $v_\mu(x')$ is the velocity, at the point x′, of an electron having a velocity v at the point x. It is determined by eliminating ϵ, x_0, and $v_z(x')$ from the relations

$$\varepsilon=\frac{m}{2}(v_x^2+v_z^2)+e\varphi(x)=\frac{m}{2}[v_x^2(x')+v_z^2(x')]+e\varphi(x'),$$

$$x_0=x+v_z/\Omega=x'+v_z(x')/\Omega.$$

We have

$$\left.\begin{aligned}&v_z(x')=v_z+\Omega(x-x'),\quad v_y(x')=v_y=\text{const},\\ &v_x(x')=\pm\left\{v_x^2+v_z^2-[v_z+\Omega(x-x')]^2+\frac{2e}{m}[\varphi(x)-\varphi(x')]\right\}^{1/2}.\end{aligned}\right\} \tag{2.7}$$

The arguments of the hyperbolic functions in (2.6) are

$$\Phi(x_1,x_2)=(i\omega+\nu_{\text{eff}})\int_{x_1}^{x_2}\frac{dx''}{v_x(x'')}=(i\omega+\nu_{\text{эфф}})t(x_1,x_2), \tag{2.8}$$

where

$$t(x_1,x_2)=\int_{x_1}^{x_2}\frac{dx''}{v_x(x'')}$$

is the time during which the electron covers the distance from x_1 to x_2.

In the region of integration in (2.6), the radicand of (2.7) is not negative, with $v_x^2(x')=0$ only at the classical turning points $x'=x_{1,2}^*$. The region of integration is shown schematically in Fig. 2 in terms of the coordinates x′ and v_z. At specified x and x′, the region of integration with respect to v_z takes the form

$$-\infty<v_z<v_z^*(x,x'),\quad x>x',$$
$$v_z^*(x,x')<v_z<+\infty,\quad x'>x,$$

where $v_z^*(x, x')$ is determined from the condition $v_x^2(x') = 0$:

$$v_z^*(x,x')=\frac{v_x^2}{2\Omega(x-x')}-\frac{1}{2}\Omega(x-x')+\frac{e}{m\Omega}\frac{\varphi(x)-\varphi(x')}{x-x'}.$$

Reversing the order of integration with respect to v_z and x_1, we obtain for the electric-current density in the plasma the expression (1.2) with a conductivity kernel

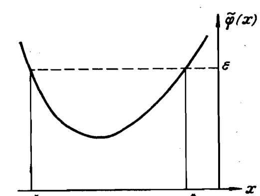

FIG. 1. Dependence of the effective potential $\tilde{\varphi}$ on the coordinate.

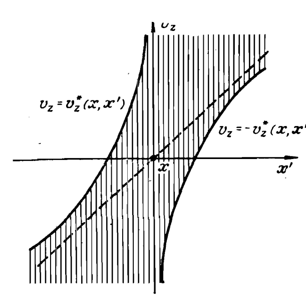

FIG. 2. Integration region in formula (2.6).

$$\Sigma_{\mu\nu}(x,x') = \begin{cases} -2e^2\int_0^\infty dv_x \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{v_z^*} dv_z \frac{v_\mu v_\nu(x')}{v_x(x')} \frac{\partial f_0}{\partial \varepsilon} \frac{\operatorname{ch}\Phi(x_1^*,x')\operatorname{ch}\Phi(x_2^*,x)}{\operatorname{sh}\Phi(x_1^*,x_2^*)}, & x'<x; \\ -2e^2\int_0^\infty dv_x \int_{-\infty}^{+\infty} dv_y \int_{v_z^*}^{+\infty} dv_z \frac{v_\mu v_\nu(x')}{v_x(x')} \frac{\partial f_0}{\partial \varepsilon} \frac{\operatorname{ch}\Phi(x_1^*,x)\operatorname{ch}\Phi(x_2^*,x')}{\operatorname{sh}\Phi(x_1^*,x_2^*)}, & x'>x; \end{cases} \quad (2.9)$$

this expression is valid for any relation between the depth of penetration of the field into the plasma, the electron mean free path, the Larmor radius, and the dimensions of the transition region at the plasma boundaries.

We confine ourselves below to the most interesting case, of the extremely anomalous skin effect, when the depth δ of penetration into the plasma is small in comparison with the characteristic dimensions of the electron trajectories as well as in comparison with the mean free path:

$$\delta \ll \bar{v}/\Omega, l; \quad l=\bar{v}/|i\omega+\nu_{\text{eff}}|, \quad (2.10)$$

$\bar{v} = (2kT_e/m)^{1/2}$ is the average thermal velocity of the electrons. In this case, the most significant values of x and x′ are those that differ from the classical turning point by an amount on the order of δ, inasmuch as the electromagnetic field and current density in the plasma attenuate rapidly at larger distances. The arguments of the hyperbolic functions of (2.9) vary significantly over distances on the order of the dimensions of electron trajectories that are much larger than the penetration depth. We can therefore put in (2.9)

$$\frac{\operatorname{ch}\Phi(x_1^*,x')\operatorname{ch}\Phi(x_2^*,x)}{\operatorname{sh}\Phi(x_1^*,x_2^*)} = \frac{\operatorname{ch}\Phi(x_1^*,x_1^*)\operatorname{ch}\Phi(x_2^*,x_1^*)}{\operatorname{sh}\Phi(x_1^*,x_2^*)} = \operatorname{cth}\Phi(x_1^*,x_2^*), \quad (2.11)$$

$$\frac{\operatorname{ch}\Phi(x_1^*,x)\operatorname{ch}\Phi(x_2^*,x')}{\operatorname{sh}\Phi(x_1^*\ x_2^*)} = \operatorname{cth}\Phi(x_1^*,x_2^*). \quad (2.12)$$

In the limiting case of a strongly anomalous skin effect, only a small fraction of the electron trajectory (of the order of δ) near the turning point is of importance. In this case the bending of the electron trajectory by the magnetic field in the skin layer can be neglected. As a result, Eq. (2.9) reduces to the form ($\partial f_0/\partial\epsilon = -f_0/kT_e$); we reverse the order of integration with respect to v_x and v_z, after which the dependence on the magnetic field remains significant only in the factor $\coth\Phi(x_1^*, x_2^*)$):

$$\Sigma_{\mu\nu}(x,x') = \begin{cases} \frac{2e^2}{kT_e}\int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z \int_{v_x^*}^{+\infty} dv_x \frac{v_\nu v_\mu}{(v_x^2-v_x^{*2})^{1/2}} f_0 \operatorname{cth}\Phi(x_1^*,x_2^*), & x'<x, \\ \frac{2e^2}{kT_e}\int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z \int_{0}^{+\infty} dv_x \frac{v_\nu v_\mu}{(v_x^2-v_x^{*2})^{1/2}} f_0 \operatorname{cth}\Phi(x_1^*,x_2^*), & x'>x, \end{cases}$$

where

$$v_x^{*2} = 2e[\varphi(x')-\varphi(x)]/m.$$

We have used here also the fact that the potential $\varphi(x)$ is monotonic: $\varphi(x') > \varphi(x)$ at $x' < x$. The character of the interaction of the electrons with the electromagnetic field in the skin layer on a small segment of the trajectory does not depend on the magnetic field. The magnetic field enters only in the factor $\coth\Phi(x_1^*, x_2^*)$, which takes into account the return of the electrons to the skin layer after each period of electron motion on the finite trajectory along the x axis. We confine ourselves to the case when the period of the finite motion of the electron along the x axis does not depend on the electron energy. This occurs, for example, when the electron concentration depends exponentially on the coordinate in a region exceeding the dimensions of the electron trajectories along the x axis, and also when the dimensions of the transition region at the plasma boundary are small in comparison with the average Larmor radius of the electrons. In both cases we have

$$\Phi(x_1^*,x_2^*) \approx \pi\beta = \pi(i\omega+\nu_{\text{eff}})/\Omega$$

(Ω is the frequency of revolution of the electrons and coincides in this case with the Larmor frequency), and does not depend on the electron velocity. The remaining integrals do not differ from the corresponding integrals without the magnetic field (see, e.g., [5]), and we obtain

$$\Sigma_{\mu\nu} = \frac{e^2}{\sqrt{\pi}\,m\bar{v}}\operatorname{cth}(\pi,\beta)[n_e(x)n_e(x')]^{1/2}K_0\left(\frac{e}{2kT_e}|\varphi(x)-\varphi(x')|\right)\delta_{\mu\nu},$$

where $K_0(x)$ is a Macdonald function. The conductivity kernel is thus isotropic in the region (2.10).

The integral equation for the electromagnetic field in a plasma (wavelength $\lambda = c/\omega \gg \delta$) is given by

$$\frac{d^2E_\mu}{dx^2} = \frac{i\operatorname{cth}\pi\beta}{\delta_0^3}\int_{-\infty}^{+\infty} E_\mu(x')\exp\left\{\frac{-e[\varphi(x)+\varphi(x')]}{2kT_e}\right\} \times K_0\left(\frac{e}{2kT_e}|\varphi(x)-\varphi(x')|\right)dx', \quad \mu=y,z \quad (2.13)$$

$\delta_0 = (c^2\bar{v}m/4\pi^{1/2}e^2n_0\omega)^{1/3}$ is of the same order of magnitude as the anomalous skin depth of the field penetration into a plasma with a sharp boundary and with a constant electron density n_0. Equation (2.13) differs from the equation without the magnetic field only by the factor $\coth(\pi\beta)$. This equation cannot be solved for an arbitrary $\varphi(x)$. Let us consider the case when the electron density decreases exponentially as $x \to -\infty$ and tends to a certain constant limit n_0 as $x \to +\infty$:[1)]

$$n_e(x) = \begin{cases} n_0e^{x/a}, & x\to-\infty, \quad a=kT_e/eE_0. \\ n_0, & x\to+\infty \end{cases}$$

Just as in[5], we consider the case when the deviation of the electron density from exponential can be neglected in a region in which the high frequency field is still appreciable. This means that the electromagnetic wave attenuates strongly even before it reaches the place where the electron density in the plasma begins to deviate strongly from exponential. The corresponding condition takes the form

$$L \equiv \ln(a^3|\operatorname{cth}\pi\beta|/\delta_0^3) \gg 1. \quad (2.14)$$

The region in which the electric-current density is maximal is separated from the region in which the electron density is close to n_0 by a distance on the order of aL. Thus, under the condition (2.14), it suffices to obtain the solution of the integral equation (2.13) in the region where the electron density does not differ from exponential:

$$n_e(x) = n_0e^{x/a}, \quad a=kT_e/eE_0.$$

The potential $\varphi(x)$ is then a linear function of the coordinate, $\varphi(x) = -E_0x$. In terms of the dimensionless variables

$$x=a(\xi-L), \quad f(\xi)=E_\mu(x(\xi)) \quad (2.15)$$

Eq. (2.13) takes the form

$$d^2f(\xi)/d\xi^2 = e^{i(\pi/2+g)}\int_{-\infty}^{+\infty} f(\xi)e^{(\xi+\xi')/2}K_0(|\xi-\xi'|/2)d\xi', \quad (2.16)$$

where

$$g = \operatorname{Im}\ln\operatorname{cth}\pi\beta = -\operatorname{arctg}\frac{\sin(2\pi\omega/\Omega)}{\operatorname{sh}(2\pi\nu_{\text{eff}}/\Omega)}.$$

Equation (2.16) differs from the corresponding equation without the magnetic field (see[5], formula (22)) only by a complex factor, and can be solved by the same method. Leaving out the details, which are given in[5], we obtain

$$f(\xi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(k)e^{k\xi}\,dk,\quad -1<c<0,$$
$$F(k)=\alpha\frac{\pi^k e^{ikg}\sqrt{k}}{\Gamma^3(1+k)}\left(\frac{2\pi i}{1-e^{-2\pi ik}}\right)^{1/2}e^{-2\pi ik}. \qquad (2.17)$$

It is easy to verify that this function is regular in the same region as function (30) of[5], inasmuch as $|g| < \pi/2$. By virtue of the linearity and homogeneity of (2.16), the constant α in (2.17) is arbitrary.

To calculate the reflection coefficient r and the active part of the surface impedance Re Z (see formulas (16), (18), and (19) of[5]), we calculate the asymptotic form of f(x) at $\lambda/a \gg |\xi| \gg 1$, $\xi < 0$:

$$f(\xi) = \alpha[\xi+\ln\pi+3C+i(\pi/2+g)]$$

(C = 0.577 ... is the Euler constant), from which we obtain in accordance with[5]

$$r(H)=1-2\frac{\omega a}{c}\left[g+\frac{\pi}{2}-i(3C+\ln\pi+L)\right],$$
$$\mathrm{Re}\,Z(H)=\frac{4\pi a\omega}{c^2}\left(\frac{\pi}{2}+g\right).$$

Just as without the magnetic field, the depth of penetration of the wave into the plasma is of the order of a, which is the reciprocal of the argument of the exponential.

Let us trace the variation of the surface impedance of the plasma with increasing magnetic field. In the absence of a magnetic field we have

$$\mathrm{Re}\,Z(0) = 2\pi^2 a\omega/c^2,\quad H=0. \qquad (2.18)$$

In a weak magnetic field, when the Larmor frequency is small in comparison with the effective number of collisions, the surface impedance differs from its value without the magnetic field by an exponentially small oscillating increment

$$\mathrm{Re}\,Z(H)=\mathrm{Re}\,Z(0)\left[1-\frac{4}{\pi}e^{-2\pi\nu_{\mathrm{eff}}/\Omega}\sin\frac{2\pi\omega}{\Omega}\right],\quad \Omega\ll\nu_{\mathrm{eff}}.$$

With increasing magnetic field, in the region

$$\Omega\sim\omega\gg\nu_{\mathrm{eff}}$$

the surface resistance of the plasma is a periodic function of the frequency ω, and the function Re $Z(H)/\omega$ is closer to rectangular in shape the better the inequality $\Omega \gg \nu_{\mathrm{eff}}$ is satisfied. The abrupt jumps of the impedance at

$$\omega=n\Omega,\quad n=1,2,3,\ldots, \qquad (2.19)$$

are connected with the Azbel'-Kaner cyclotron resonance (see[6] and Secs. 35 and 36 of[3]). Under the condition (2.19), one period of motion of the electron corresponds to an integer number of periods of the electromagnetic field. On entering the skin layer, the electron has after each revolution the same phase as the field, and the interaction between them is resonant.

If

$$\omega=(n+1/2)\Omega,\quad n=0,1,2,\ldots, \qquad (2.20)$$

then one revolution of the electron along a finite trajectory corresponds to an odd number of half-periods of the electromagnetic field. After each revolution, the phase difference between the electron and the field in the skin layer is exactly reversed, i.e., the electron does not interact resonantly with the field under the condition (2.20). This is manifest in the Re Z(H) dependence as jumps of the surface resistance, under the condition (2.20), in a direction opposite to that for (2.19). It is natural to call the resonant non-interaction of the electron with electromagnetic under the condition (2.20) "cyclotron antiresonance." The dependence of the surface impedance on ω/Ω at certain values of $\nu_{\mathrm{eff}}/\Omega$ is shown in Fig. 3.

In the region of stronger magnetic fields

$$\bar{v}/a\gg\Omega\gg\omega$$

the surface resistance does not depend on the magnetic field and is given by

$$\mathrm{Re}\,Z=\frac{4\pi a\omega}{c^2}\mathrm{arctg}\frac{\nu_{\mathrm{eff}}}{\omega},\quad \frac{\bar{v}}{a}\gg\Omega\gg\omega.$$

With further increase of the magnetic field, the dimensions of the electron orbits in the x-axis direction become smaller than the depth of penetration of the field into the plasma, and the condition under which the skin effect is strongly anomalous is violated.

A few words concerning the possibility of experimentally observing cyclotron antiresonance in a plasma with a diffuse boundary. The point is that the current in the plasma has a minimum under the condition (2.20). In metals with an abrupt boundary, this minimum of the current is screened by the strong nonresonant surface current of the electrons that glide in the skin layer[7-9]. On the other hand, in a plasma with exponentially growing electron concentration, all the electrons in the region of the effective interaction of the electromagnetic field with electrons are under the influence of the same uniform field E_0. Thus, in a plasma with a diffuse boundary all the electrons have the same period of revolution, and the resonance conditions (2.20) are satisfied simultaneously for all the electrons. We note that the period of revolution for all the electrons can be regarded as the same in both limiting cases, $\bar{v}/\Omega \gg aL$ as well as $\bar{v}/\Omega \ll aL$. In the first case, the electron moves in a constant and homogeneous magnetic field H, and in the second case in crossed mutually perpendicular fields E_0 and H.

3. ANOMALOUS SKIN EFFECT IN A PLASMA LAYER WITH DIFFUSE BOUNDARIES. INFLUENCE OF THE LAYER THICKNESS

In this section we investigate the penetration of an electromagnetic wave into a plasma layer with diffuse boundaries. In the absence of a magnetic field, the kinetic equation (2.1) takes the form

$$(i\omega+\nu_{\mathrm{eff}})f_1+v_x\frac{\partial f_1}{\partial x}+\frac{eE_0}{m}\frac{\partial f_1}{\partial v_x}=\frac{eE_y}{m}\frac{\partial f_0}{\partial v_y}. \qquad (3.1)$$

This equation was solved in[5] by the method of characteristics for monotonic $n_e(x)$. For a plasma layer, $n_e(x)$ tends to zero as $x\to\pm\infty$. We assume that $n_e(x)$ is monotonic in the sections $(-\infty, 0)$ and $(0, +\infty)$. Accordingly, $\varphi(x)$ increases without limit as $x\to\pm\infty$. Just as in a

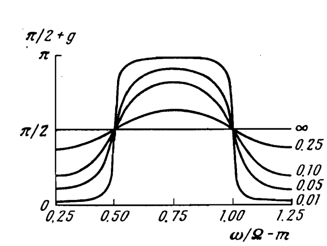

FIG. 3. Dependence of the quantity $\pi/2 + g$ on the magnetic field. The numbers at the curves denote the values of $\nu_{\mathrm{eff}}/\Omega$.

magnetic field, the motion of the electrons in the x-axis direction is finite. The classical turning points $x^*_{1,2}$ are obtained from the equation

$$e\varphi(x^*) = \varepsilon. \tag{3.2}$$

The boundary conditions on the distribution function are given as before by relations (2.5). From the condition (2.5) we obtain a unique solution of (3.1) and then, calculating the current density $j_i(x)$, we obtain an integral equation for the electromagnetic field in the plasma

$$\frac{d^2E_y(x)}{dx^2} = \frac{i}{\delta_0^3}\left[\int_{-\infty}^{x} E_y(x')G(x,x')dx' + \int_{x}^{+\infty} E_y(x')G(x',x)dx'\right], \quad \lambda \gg \delta, \tag{3.3}$$

with a kernel

$$G(x',x) = 2\int_0^\infty \frac{\operatorname{ch}\Phi(x_1^*,x)\operatorname{ch}\Phi(x_2^*,x')}{\operatorname{sh}\Phi(x_1^*,x_2^*)}\frac{\exp(-\varepsilon/kT_e)\,dv_x}{(v_x^2+2e|\varphi(x)-\varphi(x')|/m)^{1/2}}. \tag{3.4}$$

A kernel of symmetric form is obtained in (3.3) by putting

$$\varepsilon = \frac{mv_x^2}{2} + \frac{e}{2}[\varphi(x)+\varphi(x')+|\varphi(x)-\varphi(x')|].$$

The arguments of the hyperbolic functions are given by (2.8), with

$$v_x(x'') = (2[\varepsilon - e\varphi(x'')]/m)^{1/2}.$$

A kernel of the type (3.4) goes over into a kernel of the type (14) of[5] if $\operatorname{Re}\Phi(x_1^*, x_2^*) \gg 1$. This condition can be transformed into

$$\bar{v}/\nu_{\text{eff}} \ll \Delta, \tag{3.5}$$

where Δ is the effective thickness of the plasma layer and depends on the frequency of the electromagnetic wave, on the temperature, and on the concrete function of the function $n_e(x)$. In the general case it is natural to assume that the plasma layer is of the form $n_e(x) \approx n_e$ at $0 \lesssim x \lesssim d$ and that the electron concentration decreases at $x < 0$ and $x > d$, tending to zero as $|x| \to \infty$. Figure 4 shows a plot of the potential $\varphi(x)$ against the coordinate. We assume that the alternating electromagnetic field practically does not penetrate into the region $x > 0$ (with constant electron concentration), and is essentially attenuated already at

$$|x| \sim aL \gg a, \tag{3.6}$$

where a is the characteristic distance over which the electron density decreases appreciably, i.e., $n_e(x)$ is a function of the dimensionless quantity x/a. In addition, we shall consider the extremely anomalous skin effect:

$$l \gg \delta,\ a. \tag{3.7}$$

From Boltzmann's formula (1.1) and from the condition (3.6) it follows that

$$e\varphi(-aL) \gg kT_e. \tag{3.8}$$

Just as in Sec. 2, the conditions (3.6) and (3.7) enable us to use relations (2.11) and (2.12). Then

$$G(x',x) = 2\int_0^\infty \operatorname{cth}\Phi(x_1^*,x_2^*)\frac{\exp(-\varepsilon/kT_e)}{(v_x^2+2e|\varphi(x)-\varphi(x')|/m)^{1/2}}dv_x. \tag{3.9}$$

If

$$d \gg aL$$

the motion of the electron from one turning point to the other occurs mainly in the region of constant concentration. In this case we have

$$\Phi(x_1^*,x_2^*) \approx (i\omega+\nu_{\text{eff}})\frac{d}{(2\varepsilon/m)^{1/2}}.$$

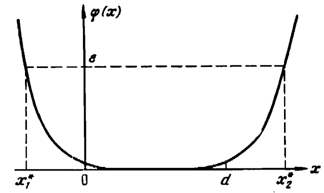

FIG. 4. Dependence of the potential φ on the coordinate.

By virtue of (3.8), we can put

$$\Phi(x_1^*,x_2^*) = \frac{(i\omega+\nu_{\text{eff}})d}{(e[\varphi(x)+\varphi(x')+|\varphi(x)-\varphi(x')|]/m)^{1/2}}. \tag{3.10}$$

Substituting (3.10) in (3.9) and integrating, we get

$$G(x',x) = \operatorname{cth}\left\{\frac{(i\omega+\nu_{\text{eff}})d}{(e[\varphi(x)+\varphi(x')+|\varphi(x)-\varphi(x')|]/m)^{1/2}}\right\}$$

$$\times\exp\left\{-\frac{e}{2kT_e}[\varphi(x)+\varphi(x')]\right\}K_0\left(\frac{e}{2kT_e}|\varphi(x)-\varphi(x')|\right)$$

Eq. (3.3) now takes the form

$$\frac{d^2E_y(x)}{dx^2} = \frac{i}{\delta_0^3}\int_{-\infty}^{+\infty} G(x',x)E_y(x')dx'. \tag{3.11}$$

As follows from (3.6), it suffices to know the potential $\varphi(x)$ only on the "tail" of $n_e(x)$, i.e., at $-x \sim aL$. Assume that the electron concentration decreases exponentially as $x \to -\infty$,

$$n_e(x) = n_0 e^{x/a}, \quad |x| \gtrsim aL, \quad x<0, \quad a = kT_e/eE_0.$$

If we make the substitution $x = a(\xi - L)$ in (3.11), then it is easy to note that we can put $x = x' = -aL$ in the argument of the hyperbolic cotangent, since $\xi \sim 1$ and $L \gg 1$. In the region of low electron concentrations, it is mainly the fast particles that penetrate. Therefore the distances $x \sim -aL$ are reached by electrons of velocity $v \sim (2e\varphi(-aL)/m)^{1/2} \approx \bar{v}\sqrt{L}$, which is much larger than the thermal velocity. Equation (3.11) takes the form

$$\frac{d^2E_y(x)}{dx^2} = \frac{i}{\delta_0^3}\operatorname{cth}\left[(i\omega+\nu_{\text{eff}})\frac{\Delta}{\bar{v}}\right]\int_{-\infty}^{+\infty} E_y(x')\exp\left(\frac{x+x''}{2a}\right)K_0\left(\frac{|x-x'|}{2a}\right)dx',$$

$\Delta = d/\sqrt{L}$ is the effective thickness of the plasma layer, with $L \gg \Delta/l$.

In terms of the dimensionless variables (2.15) we have

$$\frac{d^2f(\xi)}{d\xi^2} = ie^{ig}\int_{-\infty}^{+\infty} f(\xi')e^{(\xi+\xi')/2}K_0\left(\frac{|\xi-\xi'|}{2}\right)d\xi', \tag{3.12}$$

where now

$$g = -\operatorname{arctg}\frac{\sin(2\omega\Delta/\bar{v})}{\operatorname{sh}(2\nu_{\text{eff}}\Delta/\bar{v})}.$$

The quantity L is determined from the transcendental equation

$$L = \ln\left|\frac{a^3}{\delta_0^3}\operatorname{cth}\left[(i\omega+\nu_{\text{eff}})\frac{d}{\bar{v}\sqrt{L}}\right]\right|, \quad \frac{d}{a} \gg L \gg 1, \tag{3.13}$$

which is obtained when the dimensionless quantities are introduced. Equation (3.12) differs from (3.16) in that Ω is replaced by $\pi\bar{v}/\Delta$. The reflection coefficient and the surface resistance are given by

$$r(d) = 1 - 2\frac{\omega a}{c}\left[g + \frac{\pi}{2} - i(3C + \ln\pi + L)\right], \tag{3.14}$$

$$\operatorname{Re} Z(d) = \frac{4\pi\omega a}{c^2}\left(\frac{\pi}{2} + g\right). \tag{3.15}$$

In the limiting case of a small mean free path (3.5), we obtain from (3.13)

$$L = \ln(a^3/\delta_0^3),$$

so that

$$\operatorname{Re} Z(d)=\operatorname{Re} Z(\infty)\left[1-\frac{4}{\pi}\exp\left(-\frac{2\nu_{\text{eff}} d}{\bar{v}(\ln(a^3/\delta_0^3))^{1/2}}\right)\right.$$
$$\left.\times\sin\left(\frac{2\omega d}{\bar{v}(\ln(a^3/\delta_0^3))^{1/2}}\right)\right],\quad \frac{\bar{v}}{\nu_{\text{eff}}}\ll\frac{d}{(\ln(a^3/\delta_0^3))^{1/2}}$$

In the opposite case of small effective thickness of the plasma layer,

$$\bar{v}/\nu_{\text{eff}}\gg\Delta \tag{3.16}$$

the surface resistance is given by the formula

$$\operatorname{Re} Z(d)=\frac{4\pi\omega a}{c^2}\operatorname{arctg}\left[\frac{2\nu_{\text{eff}}\Delta}{\bar{v}}\Big/\sin\frac{2\omega\Delta}{\bar{v}}\right], \tag{3.17}$$

from which we see that the plasma layer is, generally speaking, a higher conductivity than a plasma occupying a half-space (see formula (2.18)). Indeed, in a plasma bounded on one side, the electron reflected from the turning point will move in the interior of the plasma until it collides with another particle. In the presence of two boundaries, the electron will "travel" many times, by virtue of the condition (3.16), between the turning point x_1^* and x_2^*, and during the time between two collisions it visits the skin layer approximately $\bar{v}/2\nu_{\text{eff}}\Delta$ times. As seen from (3.17), this is precisely the factor by which the surface resistance is decreased if it is assumed that $\sin(2\omega\Delta/\bar{v})\sim 1$. It is seen from Fig. (3.17) that the size effect becomes manifest also in that Re Z(d) is not a monotonic function of the frequency in the region $\omega\sim\bar{v}/(d)\gg\nu_{\text{eff}}$.

Let us assume that the mean free path is large:

$$l\gg\Delta. \tag{3.18}$$

Then the surface resistance does not depend on Δ:

$$\operatorname{Re} Z(d)=\frac{4\pi\omega a}{c^2}\operatorname{arctg}\frac{\nu_{\text{eff}}}{\omega}, \tag{3.19}$$

and the reflection coefficient is equal to

$$r=1+\frac{2\omega a}{c}\left[-\operatorname{arctg}\frac{\nu_{\text{eff}}}{\omega}+i\left(3C+\ln\frac{a^3 l}{\delta_0^3 d}\right)\right]. \tag{3.20}$$

The condition for the applicability of these formulas is

$$\frac{d}{a}\gg\ln\frac{a^3 l}{d\delta_0^3}\gg\max\left(\frac{d^2}{l^2},1\right).$$

The quantity L depends in this case on the electron mean free path. From this dependence it follows, in principle, that at a sufficiently large mean free path ($\ln(l/d)\gg 1$) the high-frequency field attenuates in the region of the exponential "tail," even if $a\sim\delta_0$.

We consider now a thin plasma layer, such that

$$d/a\ll L. \tag{3.21}$$

The motion of the electrons between the classical turning points takes place in the region where an appreciable change takes place in the electron density. In this case, to calculate the phase $\Phi(x_1^*, x_2^*)$ it is necessary to know the potential $\varphi(x)$ in the entire region of electron motion. Neglecting the small region of order d near $x\approx 0$, we assume that the potential depends on x linearly:

$$\varphi(x)=\frac{kT_e}{e}\frac{|x|}{a},\quad d\ll|x|\leqslant La.$$

Taking (3.6), (3.8), (3.15), and (3.31) into account, we obtain

$$\Phi(x_1^*, x_2^*)=4(i\omega+\nu_{\text{eff}})a\sqrt{L}/\bar{v}.$$

The effective layer thickness Δ is now equal to $\Delta = 4a\sqrt{L}$. The equation for the field takes the form (3.12), and L is the root of the equation

$$L=\ln\left|\frac{a^3}{\delta^3}\operatorname{cth}\left[\frac{4(i\omega+\nu_{\text{eff}})}{\bar{v}}a\sqrt{L}\right]\right|,\quad L\gg 1,\quad \Delta/l.$$

The reflection coefficient in the surface resistance is obtained from formulas (3.14), and (3.15). In particular, under the condition (3.18), the surface resistance is given by formula (3.19), and in expression (3.20) for the reflection coefficients it is necessary to replace $\ln(a^3 l/\delta_0^3 d)$ by $\ln(la^2/4\delta_0^3)$. The region of applicability of these formulas now takes the form

$$\frac{l^2}{a^2}\gg\ln\frac{la^2}{\delta_0^3}\gg\max\left(\frac{d}{a},1\right).$$

4. ANOMALOUS SKIN EFFECT IN A PLASMA IN WHICH THE ELECTRON DENSITY IS A POWER-LAW FUNCTION OF THE COORDINATE

In this section we consider the anomalous skin effect in a plasma in which the electron density decreases in power-law fashion as $x\to-\infty$.

Let the electron density tend to a constant value n_0 as $x\to+\infty$ (inside the plasma) and decreases like $|x|^{-p}$ as $x\to-\infty$. We assume that the damping of the electromagnetic wave in the region with the maximum concentration is so large that it suffices to consider the penetration of the field only in the region where the electron density decreases in power-law fashion. This means that the penetration depth δ of the field into the plasma should be much larger than the anomalous penetration depth δ_0 at the density n_0:

$$\delta\gg\delta_0. \tag{4.1}$$

If $n_e(x)$ becomes of the order of n_0 at $|x|\sim b$ (see Fig. 5), then as $x\to\infty$ we can express the electron density in the form

$$n_e(x)=n_0(b/|x|)^p,\quad |x|\gg b,\quad x<0. \tag{4.2}$$

The potential $\varphi(x)$ of the field leading to the concentration (4.2) is given by

$$\varphi(x)=p\frac{kT_e}{e}\ln\frac{|x|}{b}.$$

In the case of the extremely anomalous skin effect, when the electron mean free path is large

$$l\gg\delta, \tag{4.3}$$

the expression given in[5] for the kernel (see (20)) takes the form

$$G(x,x')=\left(\frac{|xx'|}{b^2}\right)^{-p/2}K_0\left(\frac{p}{2}\left|\ln\frac{x}{x'}\right|\right). \tag{4.4}$$

Substituting (4.4) in the integral equation and changing over to the dimensionless variables

$$\xi=-x/\delta,\quad f(\xi)=E_y(-\delta\xi),\quad \delta=b(b/\delta_0)^{3/(p-3)},$$

we arrive at the equation

$$\frac{d^2f(\xi)}{d\xi^2}=i\int_0^\infty\frac{f(\xi')K_0(p|\ln(\xi'/\xi)|/2)}{(\xi\xi')^{p/2}}d\xi'. \tag{4.5}$$

We assume that $p\neq 3$, for otherwise Eq. (4.5) has a unique trivial solution $f(\xi)\equiv 0$.

Assume that, just as in[5], the current density outside the plasma decreases to zero rapidly enough to be able to neglect the integral term in (4.5) as $\xi\to+\infty$. Then $f''(\xi)=0$ and $f(\xi)\sim\alpha\xi$ as $\xi\to+\infty$. We shall show that this asymptotic relation holds true at $p>4$. Let the mixed higher term take the form ξ^{1-n} as $\xi\to+\infty$, with $n>0$, i.e.,

$$f(\xi)=\alpha(\xi+\beta\xi^{1-n}),\quad n>0,\quad \xi\to+\infty.$$

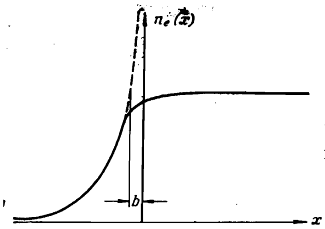

FIG. 5. Dependence of the electron concentration of the coordinates in the case of a power-law fall-off.

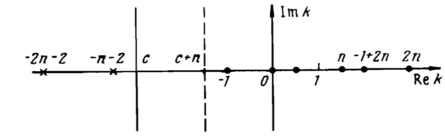

FIG. 6. Positions of the integration contour before and after the shift: ● – first-order poles, X – second-order branch points.

We substitute this asymptotic relation in (4.5). We then obtain on the left $\alpha\beta(1-n)(-n)\xi^{-n-1}$, and in the right the highest-order term in the form

$$i\alpha\xi^{2-p}\int_0^\infty u^{1-p/2}K_0(p|\ln u|/2)\,du$$

(we have put $\xi' = u\xi$). This integral converges at $p > 2$. Comparing the exponents of ξ on the left and on the right, we obtain $p - 3 = n > 0$. Thus, at $4 > p > 3$ the asymptotic form of $f(\xi)$ is

$$f(\xi) = \alpha(\xi + \beta\xi^{4-p}), \quad 3 < p < 4,$$

from which we see that in this region we cannot neglect the current as $\xi \to +\infty$, since this asymptotic form does not follow from the equation $f''(\xi) = 0$. At $p > 4$ we have

$$f(\xi) = \alpha(\xi + \beta). \qquad (4.6)$$

Thus, the current density outside the plasma (as $\xi \to +\infty$) can be neglected only if $p > 4$.

To solve (4.5) we use the Mellin transformation[10]

$$F(k) = \int_0^\infty f(\xi)\xi^{k-1}d\xi, \quad f(\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(k)\xi^{-k}\,dk, \quad c = \operatorname{Re} k. \qquad (4.7)$$

We choose the real number c inside the region where the function F(k) is regular. Just as in the case of an exponential dependence of the electron density on the coordinate, we note that the integral

$$\int_0^\infty \xi'^{-k-p/2}K_0\left(\frac{p}{2}\left|\ln\frac{\xi'}{\xi}\right|\right)d\xi' = \xi^{-(k+p/2-1)}\frac{\pi}{\sqrt{1-k}\sqrt{k-1+p}} \qquad (4.8)$$

converges in the band $1 - p < \operatorname{Re} k < 1$. When the root is extracted, the principal value is always chosen here as the regular branch. Substituting (4.7) in (4.5) and taking (4.8) into account, we obtain

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} k(k+1)F(k)\xi^{-k-2}\,dk = \frac{i}{2\pi i}\int_{c-i\infty}^{c+i\infty}\xi^{-(k+p-1)}\frac{\pi F(k)\,dk}{\sqrt{1-k}\sqrt{k-1+p}}. \qquad (4.9)$$

From this we get a functional equation for F(x), by shifting the integration contour in the left-hand side of (4.9) by an amount

$$n = p - 3 \qquad (4.10)$$

(see Fig. 6). The sought function F(k) is such that $k(k+1)F(k)$ is regular on the shift strip. The functional equation takes the form

$$(k+n)(k+n+1)F(k+n) = i\pi F(k)\,/\,\sqrt{1-k}\sqrt{k+n+2}, \qquad (4.11)$$

and its general solution is

$$F(k) = \left(\frac{\pi}{n^3}\right)^{k/n}\varphi(k)\Gamma^{-1}\left(\frac{k+n}{n}\right)\Gamma^{-1}\left(\frac{k+n+1}{n}\right)\Gamma^{-1/2}\left(\frac{k-1}{n}\right)\times\Gamma^{-1/2}\left(\frac{k+n+2}{n}\right). \qquad (4.12)$$

Here $\varphi(k)$ is an arbitrary analytic periodic function with period n.

Let us examine the analytic properties of the function (4.12). It follows from (4.6) that F(k) should have first-order poles at the points $k = -1$ and $k = 0$. This is obtained by a suitable choice of the function $\varphi(k)$. It is necessary here to get rid also of the branch points $k = 1, 1-n, 1-2n, \ldots$, which is essential to be able to shift the contour through a distance n (Fig. 6). In addition to the poles at the points $k = -1$ and 0, the function (4.12) will have first-order poles at all the points $k = n, 2n, 3n, \ldots$ and $k = -1+n, -1+2n, \ldots$. This follows from the periodicity of the function $\varphi(k)$. (The poles at the points $k = -n, -2n, -3n, \ldots$ and $k = -1-n, -1-2n, \ldots$ are "erased" by the zeroes of the functions $\Gamma^{-1}((k+n)/n)$, $\Gamma^{-1}((k+n+1)/n)$).

Figure 6 shows all the singularities of the function F(k), and also the positions of the integration contour before and after the shift by n. The value of c must be chosen here in the interval $-n-2 < c < -1$. Using also the condition that the function (4.12) be regular at infinity, we easily obtain the function $\varphi(k)$, apart from a constant factor

$$\varphi(k) = e^{-2\pi ik/n}\left(\frac{2\pi i}{1-e^{-2\pi i(k-1)/n}}\right)^{1/2}(1-e^{-2\pi ik/n})^{-1}(1-e^{-2\pi i(k+1)/n})^{-1}.$$

As a result we get

$$F(k) = \left(\frac{\pi}{n^3}\right)^{k/n}e^{-2\pi ik/n}\left(\frac{2\pi i}{1-e^{-2\pi i(k-1)/n}}\right)^{1/2}\Gamma^{-1}\left(\frac{k+n}{n}\right)\Gamma^{-1}\left(\frac{k+n+1}{n}\right)\times\Gamma^{-1/2}\left(\frac{k-1}{n}\right)\Gamma^{-1/2}\left(\frac{k+n+2}{n}\right)(1-e^{-2\pi ik/n})^{-1}(1-e^{-2\pi i(k+1)/n})^{-1}. \qquad (4.13)$$

The function $f(\xi)$ is obtained from (4.7). We have already proved that the function $f(\xi)$ obtained in this manner has the asymptotic form (4.6) only if $n > 1$. As $n \to 1$, the poles at zero and at the point $-1 + n$ coalesce to form a second-order pole. This gives rise to the logarithmic asymptotic form

$$f(\xi) = \alpha(\xi + \beta_1\ln\xi + \beta_0), \quad n = 1, \quad \xi \to +\infty. \qquad (4.14)$$

At $0 < n < 1$ the asymptotic form of $f(\xi)$ as $\xi \to +\infty$ receives contributions not only from the poles at the points 0 and -1, but also from the poles $-1 + n, -1 + 2n, \ldots$, which lie between 0 and -1. At $n > 1$ we have

$$f(\xi) = \alpha\left[\xi - \frac{n}{2}\left(\frac{\pi}{n^3}\right)^{1/n}\frac{\pi e^{\pi i/2n}}{\Gamma(1/n)\Gamma(2/n)\sin(\pi/n)}\right], \quad \xi \to +\infty \qquad (4.15)$$

hence

$$\beta = -\frac{n}{2}\left(\frac{\pi}{n^3}\right)^{1/n}\frac{\pi}{\Gamma(1/n)\Gamma(2/n)\sin(\pi/n)}e^{\pi i/2n}.$$

A detailed investigation shows that the asymptotic form of (4.15) holds true at $n - 1 \gg (\ln(l/\delta))^{-1}$. In addition, it must be remembered that the plasma parameters l, $n = p - 3$, b, and δ_0 are connected by conditions (4.1) and (4.3), which can be expressed in the form of the inequality

$$l \gg b(b/\delta_0)^{3/n} \gg \delta_0. \qquad (4.16)$$

From formulas (18) and (19) of[5], recognizing that in our case $B = -\delta\beta$, we obtain the reflection coefficient

$$r = 1 + \frac{b\omega}{2c}\frac{n\pi}{\Gamma(1/n)\Gamma(2/n)}\left(\frac{b^3\pi}{\delta_0^3n^3}\right)^{1/n}\left[i\left(\sin\frac{\pi}{2n}\right)^{-1} - \left(\cos\frac{\pi}{2n}\right)^{-1}\right],$$

$$n - 1 \gg \left(\ln\frac{l}{\delta}\right)^{-1},$$

and the surface resistance

$$\operatorname{Re} Z=\frac{\pi^2 n}{\Gamma(1/n)\Gamma(2/n)\cos(\pi/2n)}\frac{\omega}{c^2}b\left(\frac{\pi b^3}{n^3\delta_0^3}\right)^{1/n},\quad n-1\gg\left(\ln\frac{l}{\delta}\right)^{-1} \tag{4.17}$$

We now consider specially the case n = 1. It is easily seen that in this case if a first-order pole exists at the point k = −1, then a second-order pole must exist at the point k = 0. The function (4.13) is continuous at n = 1. We obtain the function $f(\xi)$ from formula (4.7). As $\xi\to+\infty$ we have

$$f(\xi)=\alpha\left[\xi-\frac{\pi^2}{4}+\frac{i\pi}{2}\left(\ln\frac{\pi}{\xi}+\frac{5}{2}C-\frac{7}{4}\right)\right]$$

in agreement with (4.14).

We have obtained a logarithmically distorted asymptotic form (4.6). "Cutting off" the logarithm at values $\xi\sim l/\delta$, we obtain formulas that are valid at $|n-1|\ll(\ln(l/\delta))^{-1}$:

$$r=1+\pi\frac{\omega}{c}\delta\left(\frac{i\pi}{2}-\ln\frac{l}{\delta}\right),$$

$$\operatorname{Re} Z=2\pi^2\frac{\omega}{c^2}\delta\ln\frac{l}{\delta},\quad \delta=b\left(\frac{b}{\delta_0}\right)^3,\quad \ln\frac{l}{\delta}\gg 1.$$

We now find an asymptotic expression for the function $f(\xi)$ as $\xi\to 0$. The main contribution to the integral (4.7) is made in this case by a small vicinity of the branch point k = −n −2. The presence of this branch point follows from the recurrence formula (4.11) and is not connected with the behavior of the function $f(\xi)$ as $\xi\to+\infty$. Deforming the contour in the manner shown in Fig. 7, we obtain

$$f(\xi)=\alpha\frac{e^{i\pi(1-1/n)/2}}{\pi n}\left(\frac{n^3}{\pi}\right)^{1/n}\left[\frac{3n+9}{2\pi n}\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\right]^{1/2}\frac{\xi^{n+2}}{|\ln\xi|^{1/2}}. \tag{4.18}$$

at $b/\delta\ll\xi\ll 1$. For this formula to be valid, it suffices to satisfy only the condition (4.16), since $f(\xi)$ is continuous at n = 1. The effective depth of penetration of the field into the plasma, as follows from (4.17), is of the order of $b(b/\delta_0)^{3/n}$. It follows therefore that the effective depth of penetration increases like $\omega^{1/n}$ when the frequency of the field is increased. This peculiar dependence on the frequency is directly connected with the power-law dependence of the electron density on the coordinate[2)]. The electromagnetic field likewise depends in this case on the coordinate mainly in power-law fashion (and not exponentially). It is clear from this that the quantity $\delta=b(b/\delta_0)^{3/n}$ does not have the same simple meaning as the quantity δ = a for a plasma with an electron density having an exponential dependence on x. In the case of the power-law dependence, δ is the distance from the origin to the place in the plasma where the electromagnetic wave differs already substantially from the wave in free space. It is natural to write the condition that the plasma attenuate in the region of the plasma tail in the form

$$\delta\gg b, \tag{4.19}$$

where b characterizes the scale of variation of the potential (just as the quantity a does in Sec. 3). Now let the plasma, with a power-law decrease of $n_e(x)$, be situated in a magnetic field. If the Larmor radius is large in comparison with the depth of penetration and with the dimensions of the transition layer, then the

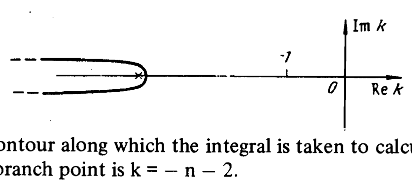

FIG. 7. Contour along which the integral is taken to calculate $f(\xi)$ as $\xi\to 0$; the branch point is k = − n − 2.

quantity $\Phi(x_1^*, x_2^*)$ is practically independent of the electron velocity. In a magnetic field, the condition (4.19) for the fast damping of the wave takes the form

$$(b/\delta_0)^3|\operatorname{cth}\pi\beta|\gg 1.$$

The equation for the electromagnetic-wave wave differs in this case from (4.5) by a factor $\coth(\pi\beta)$ in front of the integral. From this we obtain directly the reflection coefficient and the surface resistance:

$$r=1+i\frac{\omega}{c}\frac{n\pi}{\Gamma(1/n)\Gamma(2/n)\sin(\pi/n)}b\left(i\pi\frac{b^3}{n^3\delta_0^3}\operatorname{cth}\pi\beta\right)^{1/n}, \tag{4.20}$$

$$\operatorname{Re} Z=\frac{2\pi^2 n}{\Gamma(1/n)\Gamma(2/n)\sin(\pi/n)}\frac{\omega}{c^2}b\left(\frac{\pi b^3}{n^3\delta_0^3}\right)^{1/n}\operatorname{Im}(i\operatorname{cth}\pi\beta)^{1/n},$$
$$n=p-3,\quad n-1\gg(\ln(l/\delta))^{-1}. \tag{4.21}$$

These formulas are valid if

$$l,\ \bar v/\Omega\gg b(b/\delta_0)^{3/n}|\operatorname{cth}\pi\beta|^{1/n}\gg b.$$

We write out the solutions for n = 1:

$$r=1+i\pi\frac{\omega}{c}\delta\operatorname{cth}\pi\beta\left[\frac{\pi}{2}+g+i\ln\left(\frac{l}{\delta}|\operatorname{th}\pi\beta|\right)\right], \tag{4.22}$$

$$\operatorname{Re} Z=\frac{2\pi^2\omega}{c^2}\delta\left[\left(\frac{\pi}{2}+g\right)\operatorname{Im}(\operatorname{cth}\pi\beta)+\operatorname{Re}(\operatorname{cth}\pi\beta)\ln\left(\frac{l}{\delta}|\operatorname{th}\pi\beta|\right)\right],$$
$$\delta=b(b/\delta_0)^3,\quad g=\operatorname{Im}\ln(\operatorname{cth}\pi\beta),\quad \ln(l|\operatorname{th}\pi\beta|/\delta)\gg 1. \tag{4.23}$$

Cyclotron resonance may be observed in the frequency region $\omega\sim\Omega\gg\nu_{\text{eff}}$. The cyclotron-resonance line shapes for exponents n and for different values of the parameter ν_{eff}/Ω are shown in Fig. 8. At n = 1 the line shape is almost symmetrical and differs significantly from the case of an exponential $n_e(x)$ dependence. The dependence of Re Z on ω/Ω is contained in the factor $n|\coth\pi\beta|^{1/n}\sin[(\pi/2)+g)/n]$, which goes over into $\pi/2+g$ as $n\to\infty$, as is the case for an exponential $n_e(x)$ dependence (see Fig. 3). At small n, the cyclotron antiresonances produce no singularities in Re Z. With increasing n, the antiresonance become manifest more and more clearly (see Fig. 8c), and in the limit of large n (just as for the exponential behavior) it leads to jumps of Re Z of the same magnitude as in the case of resonance.

In the case of a plasma layer with a power-law de-

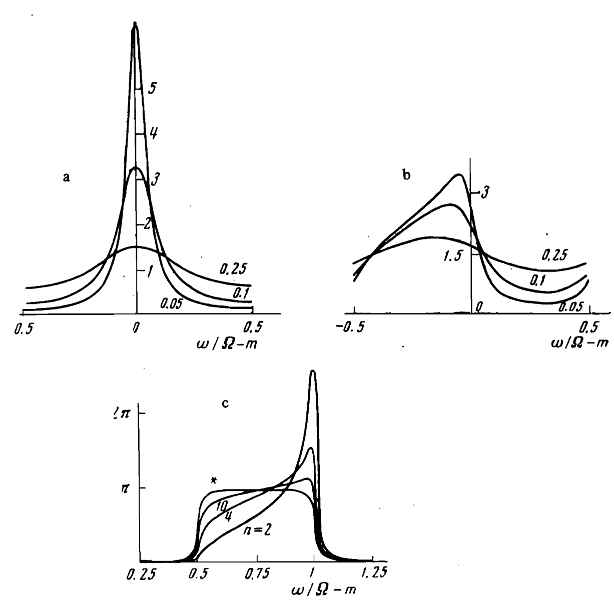

FIG. 8. Dependence of the quantity Im (i coth $\pi\beta$)$^{1/n}$ on the magnetic field: a) n = 1, b) n = 3 (the numbers at the curves indicate the values of ν_{eff}/Ω); c) the values of n are indicated at the curves and the astefisk marks the limiting curve as $n\to\infty$, $\nu_{\text{eff}}/\Omega=0.01$.

crease of $n_e(x)$, we obtain at $x \to \pm\infty$ and $d \gg \delta$ the same formulas (4.20)–(4.23), in which now $\pi\beta = (i\omega + \nu_{eff})\Delta/\bar{v}$, $\Delta = d/(p \ln(\delta/b))^{1/2}$, and δ is the root of the transcendental equation

$$\delta = b\left(\frac{b}{\delta_0}\right)^{3/n}\left|\operatorname{cth}\frac{(i\omega+\nu_{\text{эфф}})d}{\bar{v}(p\ln(\delta/b))^{1/2}}\right|^{1/n}.$$

The final formulas are valid if

$$b \ll \delta \ll d,\ l.$$

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[1)]The case when $n_e(x)$ decreases in power-law fashion as $x \to \infty$ is considered in Sec. 4.

[2)]This, of course, does not mean that the wave field penetrates better into the plasma with increasing frequency. The thickness of the skin layer increases, and the wave field decreases at each point in the plasma, as is seen, for example, from (4.18) by substituting $\xi = |x|/\delta$.

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