Turning region of the Poynting vector of a wave incident on an inhomogeneous plasma towards the direction of an external magnetic field

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Results are presented of a theoretical investigation of the turning region of the Poynting vector $\bar{S}$ of a wave incident on a plane boundary (xy) of a magnetoactive plasma which is inhomogeneous along the z coordinate, toward the direction of the vector of a stationary magnetic field $\mathbf{H}_0$. Two dependences of the electron density N(z) are considered. The frequency of the waves ω is much greater than that of the lower-hybrid frequency ω_L and also the gyro- and Langmuir frequencies (Ω_H and Ω_0) of the ions. This permits one to solve the corresponding problem in the quasi-longitudinal approximation valid for an electronic (extraordinary) wave excited in a plasma. It is shown that the thickness of the turning region of $\bar{S}$ assumes the maximal value ($z \to \infty$) at distances z from the plasma boundary which vary between 0.1 and 7 wavelengths. The thickness of the region decreases rapidly with increase of the gradient of the electron density.

1. INTRODUCTION

It is known that at frequencies ω lower than the gyrofrequency ω_H of the electrons, only one of the modes of the electromagnetic waves excited in a magnetoactive plasma can propagate with little attenuation, becoming channeled about the vector of the external magnetic field $\mathbf{H}_0$. At $\omega < \Omega_H$ (Ω_H is the ion gyrofrequency), the corresponding wave is the ionic (ordinary) wave, and at $\Omega_H < \omega < \omega_H$, the weakly damped mode is the electronic (extraordinary) wave. The angle between the wave energy flux and the vector $\mathbf{H}_0$ is determined by the direction of the Poynting vector $\mathbf{S}$, which in a nonabsorbing medium is collinear with the group-velocity vector $\mathbf{u} = d\omega/d\mathbf{k}$ ($\mathbf{k}$ is the wave vector, $|\mathbf{k}| = \omega n/c$, and n is the refractive index). The corresponding theorem for a nonabsorbing magnetoactive homogeneous plasma (collision number $\nu = 0$) was proved in[1], where the following formulas were derived:

$$\operatorname{tg}\psi = \operatorname{tg}(\mathbf{k},\mathbf{u})\frac{\sin\theta}{n}\frac{\partial n}{\partial(\cos\theta)},$$
$$|\mathbf{u}| = \frac{c}{\partial(\omega n)/\partial\omega}\frac{1}{\cos\psi} = \frac{u_0}{\cos\psi}. \quad (1)$$

Here u_0 is the group velocity of the wave in an isotropic medium ($\mathbf{H}_0 = 0$), and ψ is the angle between the direction of the wave vector $\mathbf{k}$ and the Poynting vector $\mathbf{S}$ or the group-velocity vector $\mathbf{u}$.

To study a number of processes in a plasma, it is important, however, to determine as rigorously as possible how the Poynting vector gradually "presses" towards the direction of the field $\mathbf{H}_0$ or, as is now customarily stated, the wave becomes "trapped" in "magnetic-force channels." A rigorous solution of this problem in the general case, which is essential for a complete investigation, is hardly possible at the present time, owing to the mathematical difficulty of the differential equations that are obtained in this case, and particularly because of the complicated equation for the refractive index n when account is taken of the motion of the ions, especially in the case of an inhomogeneous plasma. It is therefore necessary to proceed along the way of a rigorous solution of a particular, limiting type of problem, which can ultimately yield the general picture of the phenomena of interest to us.

We report in this article the first results of a rigorous solution of one such problem for waves incident normally on the flat boundary of a one-dimensionally inhomogeneous plasma; the frequency of the waves is much higher than the lower hybrid frequency ω_L, but is lower than the gyrofrequency of the electrons ω_H in the entire plasma region considered. In this case one can neglect with confidence the influence of the ion motion.

2. MATHEMATICAL FORMULATION OF THE PROBLEM AND ITS SOLUTION

Assume that a plane wave is normally incident on a plasma that is inhomogeneous with respect to the coordinate z, so that on the plasma boundary, which is assumed to be flat (the xy plane), the wave vector $\mathbf{k}_0$ of the incident wave ($|\mathbf{k}_0| = \omega/c$) is directed upward and coincides with the positive direction of the z axis. The magnetic field vector $\mathbf{H}_0$ is assumed to lie in the (xz) plane, and we direct it downward and denote by θ the angle between the vector $-\mathbf{H}_0$ and the z axis. We assume that the frequency ω of the incident wave, the electron density $N = N(z)$, and the number of collisions ν have values throughout the plasma such that the conditions of the so-called quasilongitudinal propagation of the waves passing through it[2,3] are satisfied, i.e., that the angle θ satisfies the inequality

$$\frac{\sin^2\theta}{\cos\theta} \ll 2\frac{\omega}{\omega_H}\left|1 - \frac{\omega_0^2}{\omega^2} - i\frac{\nu}{\omega}\right|, \quad (2)$$

where $\omega_0 = (4\pi Ne^2/m)^{1/2}$ is the Langmuir frequency of the electrons (e and m are respectively the charge and mass of the electrons). Even from the very form of the condition (2) one can see that we are dealing with frequencies at which we can neglect the influence of the ion motion, something permissible if $\omega \gg \Omega_H, \omega_L, \Omega_0$, where Ω_0 is the Langmuir frequency of the ions, and the lower hybrid frequency

$$\omega_L = \sqrt{\omega_H\Omega_H}(1 + \omega_H^2/\omega_0^2)^{-1}.$$

In the assumed approximation, the complex refractive indices for the ordinary and extraordinary waves and the corresponding values of the polarization $\rho = E_y/E_x$, where E_x and E_y are the horizontal components of the electric field, are adequately described by the formulas

$$n^2 = 1 - \omega_0^2 / \omega(\omega \pm \omega_H \cos\theta - i\nu), \quad \rho = \mp i, \tag{3}$$

and the ratio of the normal component of the electric field E_z to its horizontal component E_x is equal to

$$\frac{E_z}{E_x} = \pm \frac{\omega_0^2 \omega_H \sin\theta}{(\omega \pm \omega_H \cos\theta - i\nu)(\omega^2 - \omega_0^2 - i\nu\omega)}. \tag{4}$$

In the case of normal incidence for the components E_x and E_y we obtain from Maxwell's equations two linear differential equations of second order, the coefficients of which are expressed in terms of the values of the refractive indices and the polarization of the ordinary and extraordinary waves (see[2], Sec. 18). By defining n^2 and ρ with the aid of formulas (3), we obtain from these equations the wave equation for the quantities $\epsilon_\pm = E_x \pm iE_y$:

$$d^2\varepsilon_\pm / dz^2 + k_0^2 n^2 \varepsilon_\pm = 0. \tag{5}$$

In all the foregoing formulas, the plus and minus signs correspond to the ordinary and extraordinary waves, respectively. We have already noted that only the extraordinary electron wave (minus sign in (3)—(5)) is weakly damped. Therefore all the calculation results that follow pertain only to this case, and we leave out the indices plus or minus and the signs corresponding to them.

We assume the electron density to be variable only with respect to the coordinate z, i.e., $N = N(z)$, and assume that the number of collisions $\nu = \nu_0$ is constant. Such a limitation means that the plasma is weakly ionized, namely that $N_n/N \gg 1$ (N_n is the concentration of the neutral particles). In this case, the $\nu(z)$ dependence should have little influence on the investigation of the angle of rotation of the incident-wave energy-flux vector towards the direction of the external magnetic field H_0.

We consider two relations of the type $N = N(z)$, which reduce Eq. (5) to a Whittaker equation. Namely, we assume

$$\text{model I: } N(z) = N_{\infty 1}(1 + be^{-\alpha_1 z})(1 - e^{-\alpha_1 z}), \tag{6}$$

$$\text{model II: } N(z) = N_{\infty 2}(1 - e^{-\alpha_2 z})^2. \tag{7}$$

For both these relations we have $N(0) = 0$, and $N_{\infty 1}$ and $N_{\infty 2}$ are the corresponding limiting values at infinity. The difference between (6) and (7) lies in the fact that the relation (6) has a maximum at

$$z = z_m = \alpha_1^{-1} \ln \frac{2b}{b-1},$$

and (7) is a monotonically increasing function. For the numerical values of the parameters chosen below, the dependences (6) and (7) take the form shown in Fig. 1.

We choose the variables $\eta = -\alpha z$ and $\xi = 2ae^{\eta}$. Then Eq. (5) for the functions (6) and (7) reduces to

$$d^2\varepsilon / d\eta^2 = (a^2 e^{2\eta} - 2ase^{\eta} + p^2)\varepsilon$$

and has a solution (see[4])

$$\varepsilon = e^{-\eta/2} Y_{s,p}(\xi),$$

where the function $Y_{s,p}(\xi)$ is a solution of the Whittaker equation:

$$y'' = \left(\frac{1}{2} - \frac{s}{\xi} + \frac{p^2 - 1/4}{\xi^2}\right) y.$$

Thus, in the previously assumed notation,

$$\varepsilon(z) = e^{\alpha z/2} Y_{s,p}(2ae^{-\alpha z}), \tag{8}$$

where we have for both functions (6) and (7)

$$p = p_{1,2} = i \frac{1}{\alpha_{1,2}} \frac{\omega}{c} \sqrt{1 - QN_{\infty 1,2}},$$

for the function (6)

$$a = a_1 = \frac{1}{\alpha_1} \frac{\omega}{c} \sqrt{bQN_{\infty 1}},$$

$$s = s_1 = i \frac{1}{\alpha_1} \frac{b-1}{2\sqrt{b}} \frac{\omega}{c} \sqrt{QN_{\infty 1}}$$

and for the function (7)

$$a = s = a_2 = s_2 = \frac{1}{\alpha_2} \frac{\omega}{c} \sqrt{QN_{\infty 2}},$$

$$Q = \frac{4\pi e^2/m}{\omega - \omega_H \cos\theta - i\nu}.$$

We now choose for the Whittaker equation a solution corresponding to a damped wave that goes off to infinity. Assuming the waves to have a time dependence $e^{i\omega t}$, we have in the notation of ξ and η (see[5])

$$Y_{s,p}(\xi) = \xi^{1/2 - p} e^{-1/2\xi} \Phi(\beta, \gamma, \xi), \tag{9}$$

where $\Phi(\beta, \gamma, \xi)$ is the confluent hypergeometric function, $\beta = 1/2 - p - s$, and $\gamma = 1 - 2p$. Ultimately, leaving out the constant factor, we obtain the sought solutions for the relations (6) and (7) in the form

$$\varepsilon(z) = \exp\left\{i \frac{\omega}{c} \left[1 - \frac{(4\pi e^2/m) N_\infty}{\omega(\omega - \omega_H \cos\theta - i\nu)}\right]^{1/2} z\right\} f(z) = \exp\left\{i \frac{\omega}{c} \mu z\right\} f(z), \tag{10}$$

$$f(z) = \exp\{-ae^{-\alpha z}\} \Phi(\beta, \gamma, 2ae^{-\alpha z}), \tag{11}$$

and to abbreviate the notation we introduce the symbol

$$\mu = \sqrt{1 - QN_\infty} = \left[1 - \frac{(4\pi e^2/m) N_\infty}{\omega(\omega - \omega_H \cos\theta - i\nu)}\right]^{1/2}. \tag{12}$$

as $z \to \infty$ we have $f(z) \to 1$, and $\epsilon(z)$, as expected, describes plane waves defined by the refractive index (3) for a homogeneous medium with electron density $N = N_\infty$. The extraordinary wave considered by us corresponds to the following solution of Eq. (5) for the electric-field components E_x and E_y: $E_x = \epsilon(z)$, $E_y = i\epsilon(z)$, while E_z is determined by formula (4), where the minus sign should be taken.

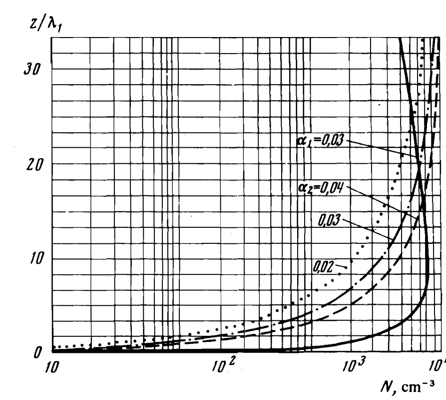

FIG. 1. Dependence of height above the plasma the electron density on the boundary z/λ_1 (λ_1 = 3 km).

Using Maxwell's equations, we obtain the following expressions for the components of the magnetic field of the wave:

$$H_x = e^{i\omega\mu z/c}\left[i\mu f(z) + \frac{c}{\omega} f'(z)\right], \quad H_y = iH_x, \quad H_z = 0.$$

The time-averaged Poynting vector $\overline{S}$ is calculated from the formula*

$$\overline{S} = {}^1/_2 \mathrm{Re}\,[\mathbf{EH}^*].$$

Its components are equal to

$$\bar{S}_x = -\frac{\omega_H \omega_c^2}{2\omega\omega^2}\sin\theta\cdot\exp\left\{-2\frac{\omega}{c}(\operatorname{Im}\mu)z\right\}\operatorname{Re}K(z), \tag{13}$$

$$\bar{S}_y = -\frac{\omega_H \omega_0^2}{2\omega\omega^2}\sin\theta\cdot\exp\left\{-2\frac{\omega}{c}(\operatorname{Im}\mu)z\right\}\operatorname{Im}K(z),$$

$$\bar{S}_z = -\exp\left\{-2\frac{\omega}{c}(\operatorname{Im}\mu)z\right\}\operatorname{Re}\left\{|f(z)|^2\mu^* + i\frac{c}{\omega}f(z)f'^*(z)\right\},$$

where

$$K(z) = \frac{|f(z)|^2\mu^* + ic\omega^{-1}f(z)f'^*(z)}{[1-(\omega_H/\omega)\cos\theta - i\nu/\omega][1-\omega_0^2/\omega^2 - i\nu/\omega]},$$

f(z) and μ are determined by formulas (11) and (12) (the asterisk is the sign of complex conjugation).

Naturally, the Poynting vector emerges from the xz plane, in which the vector of the magnetic field $\mathbf{H}_0$ lies; this is due to the fact that the plasma is magnetoactive (see[1]). The tangent of the angle that this vector forms with the z axis (i.e., with the direction $\mathbf{k}_0$ of the incident wave) is equal to

$$\operatorname{tg}\psi = \frac{(\bar{S}_x^2+\bar{S}_y^2)^{1/2}}{\bar{S}_z} = -\frac{1}{2}\frac{|F(z)|}{\operatorname{Re}F(z)}\frac{\omega_0^2\omega_H\sin\theta}{|\omega-\omega_H\cos\theta - i\nu||\omega^2-\omega_0^2-i\nu\omega|}, \tag{14}$$

$$F(z) = \mu^* + i\frac{c}{\omega}\ln' f^*(z), \tag{15}$$

and the logarithmic derivative f(z) that enters in (15) and is determined from (11) with allowance for the relation (see[6])

$$\frac{d\Phi(\beta,\gamma,\xi)}{d\xi} = \frac{\beta}{\gamma}\Phi(\beta+1,\gamma+1,\xi),$$

is equal to

$$\ln' f(z) = \alpha a e^{-\alpha z}\left[1 - 2\frac{\beta}{\gamma}\frac{\Phi(\beta+1,\gamma+1,\xi)}{\Phi(\beta,\gamma,\xi)}\right].$$

We note here that ψ_{xz}, i.e., the projection of the angle ψ on the xz plane, in which the vector $\mathbf{H}_0$ lies, is much larger, as is physically clear, than ψ_{yz}, i.e., than the projection of the angle ψ on the yz plane. The calculated results that follow do indeed confirm this assumption. In concluding this section, we point out that in the limit as $z \to \infty$, when the position of the Poynting vector $\bar{\mathbf{S}}$ should correspond to wave propagation in a homogeneous medium, we obtain from (14) a simple formula for tan ψ, namely

$$\operatorname{tg}\psi_\infty = -\frac{\sin\theta}{2}\frac{|\mu|}{\operatorname{Re}\mu}\frac{\omega_H\omega_0^2}{|\omega_H\cos\theta-\omega+i\nu||\omega_0^2-\omega^2+i\nu\omega|}. \tag{16}$$

When $\nu = 0$, formula (16) naturally coincides exactly with the expression obtained from (1) for tan ψ after first differentiating with respect to cos θ the complete formula for the refractive index of the electron wave (see[3,4]), and then using condition (2) for quasi-longitudinal propagation.

3. ANALYSIS OF THE RESULTS OF THE NUMERICAL CALCULATIONS

To find the dependence of the angle ψ on the height z, in accordance with formulas (14) and (15), we calculated the values of the confluent hypergeometric function $\Phi(\beta, \gamma, \xi)$ with a BESM-6 computer as the sum of the corresponding series, since none of the known asymptotic expressions for $\Phi(\beta, \gamma, \xi)$ (see[7]) turned out to be suitable for the considered values of the argument ξ and the parameters β and γ. The calculations were performed for the following values of the parameters ($f = \omega/2\pi$):

$$\begin{array}{ll} \theta = 10^\circ & \theta = 15^\circ; \\ f_1 = 5\cdot 10^4\ \text{sec}^{-1} & f_2 = 10^5\ \text{sec}^{-1} \\ \lambda_1 = 6\ \text{km} & \lambda_2 = 3\ \text{km} \\ N_{\infty 1} = 4\cdot 10^3\ \text{cm}^{-3}, & N_{\infty 2} = 10^4\ \text{cm}^{-3}; \\ \omega_H = 8.8\cdot 10^6\ \text{sec}^{-1} & \nu = 10^6\ \text{sec}^{-1} \\ \alpha_1 = 0.03\ \text{km}^{-1}, & \alpha_2 = 0{,}01;\ 0{,}03;\ 0{,}04\ \text{km}^{-1}; \\ b = 5.8. & \end{array} \tag{17}$$

We note that the quasilongitudinality conditions (2) improve with decreasing frequency ω and angle θ. At the chosen values (17), the inequality (2) in our calculations was in best agreement with the relation (0.3 < 1), and in general varied in the range (~0.13 < 1) and (~2×10^{-3} < 1). On the other hand, in the range $z/\lambda \approx 0.15-7$, where, as seen below, tan ψ = tan ψ_∞ throughout, the range of the inequality (2) was from (0.13 < 1) to (~10^{-2} < 1).

The main feature of the behavior of the angle ψ is that it reaches the limiting value ψ_∞ quite rapidly (see Figs. 2 and 3). In the considered models (6) and (7), for different values of α characterizing the gradient of the electron density dN/dz, the rotation of the Poynting vector occurs at values $z/\lambda \approx 0.15-7$. The limiting values of the angles ψ_∞ and of their components ψ_{xz} and ψ_{yz} for different values of ω, θ, and $\alpha = 0.03\ \text{km}^{-1}$ are listed in the table.

It is seen from the table that the Poynting vector indeed deviates little from the xz plane, in which the magnetic-field vector $\mathbf{H}_0$ is located, a fact already noted

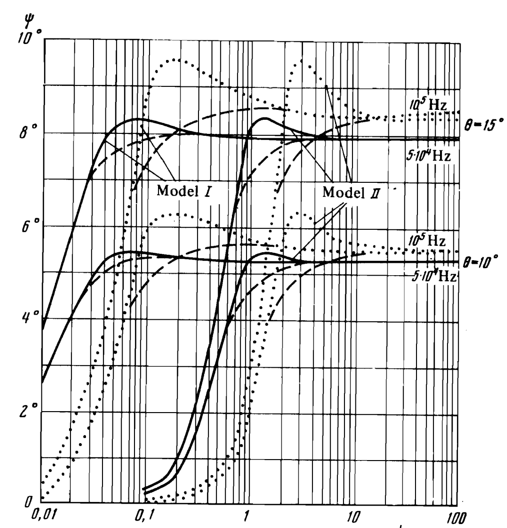

FIG. 2. Dependence of the angle of rotation ψ of the Poynting vector on the height z/λ.

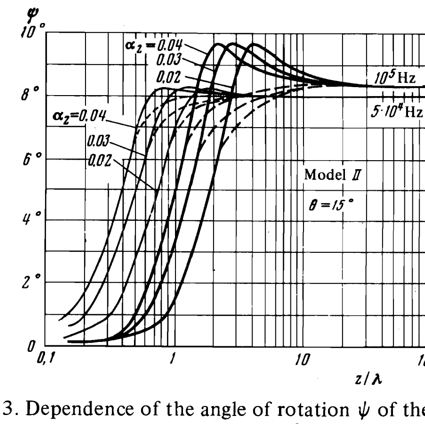

FIG. 3. Dependence of the angle of rotation ψ of the Poynting vector on the height z/λ; model II, $\theta = 15°$.

	Frequency f, sec^{-1}	$\theta = 10°$			$\theta = 15°$		
		ψ_{xz}	ψ_{yz}	ψ	ψ_{xz}	ψ_{yz}	ψ
Model I	$5 \cdot 10^4$	5°13′	—0°31′	5°14′	7°53′	—0°47′	7°56′
	10^5	5°31′	—0°45′	5°34′	8°22′	—1°09′	8°26′
Model II	$5 \cdot 10^4$	5°12′	—0°24′	5°13′	7°52′	—0°36′	7°54′
	10^5	5°27′	—0°30′	5°28′	8°15′	—0°47′	8°17′

Note: The minus sign preceding the values of ψ_{yz} indicate that the Poynting vector is inclined towards the negative y axis.

above. For both models of N(z), the functions $\psi(z)$ have maxima at $z = z_m$, where $\omega_0^2(z_m)/\omega^2 = 1 + (\nu/\omega)^2$.

We assume, however, that in fact, under real conditions, $\psi(z)$ increases monotonically (see the dashed curves in Figs. 2 and 3), and that the appearance of the maxima is due to the fact that we are considering the solution of the problem in a quasilongitudinal approximation. Favoring this assumption are a number of circumstances. Thus, with decreasing θ and ω, when the conditions for the applicability of the quasilongitudinal approximation improve, the quantity

$$\delta\psi = (\psi_{max} - \psi_\infty) / \psi_\infty$$

decreases, as can be seen directly from an examination of Figs. 2 and 3. Further, an analysis of the formulas for $\tan \psi(z)$ shows that in the limit as $\nu \to \infty$ or $\omega \to 0$, when the conditions for applicability of the quasilongitudinal approximation also improve, we have $\delta\psi \to 0$, i.e., the maxima of $\psi(z)$ vanish. We notice also that the maxima are attained in the vicinity of points where $\omega_0^2(z)/\omega^2 = 1$, when the conditions of the quasilongitudinal approximation are satisfied worst of all (see (2)).

Another important feature of the height dependence of $\psi(z)$ is that when the gradient of the electron density dN/dz increases, the region of rotation of the Poynting vector $\bar{S}$ towards the limiting value of the angle ψ_∞ decreases rapidly. This circumstance is particularly noticeable in Fig. 2, where the results of calculations are compared for the two models I and II. Calculations of the ratio of $(dN/dz)_I$ to $(dN/dz)_{II}$ as a function of z show that in the effective region of values of z, where the turning of the vector S takes place, the angle decreases 2—3 times faster than the ratio of the electron-density gradients.

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[EH] = E × H*.

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