Nonlinear Landau damping of a Langmuir wave packet

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Damping of a Langmuir wave packet due to its interaction with resonant electrons is considered. It is shown in the case of a packet with a fixed phase velocity that the front of the envelope becomes increasingly steep until a discontinuity is formed and "erodes" the packet. Damping of the packet in this case consists in a decrease in its wavelength and not in a reduction of its amplitude. The decay time is found as the ratio of the length of the packet to the velocity of the discontinuity. For a packet with a phase velocity which varies monotonically in space, the amplitude decreases simultaneously at all points of the packet and the decay time decreases. If the phase velocity is a random function of the coordinate, the damping again becomes "erosive."

1. INTRODUCTION

At the present time, the question as to whether phase mixing of resonant particles has an effect on the damping of packets of longitudinal waves in a plasma is under study. The importance of this problem is due to the fact that, both in astrophysics and under laboratory conditions, one has to deal with wave packets whose damping cannot be described within the framework of the theories of Mazitov[1] and O'Neil,[2] which refer to strictly spatially periodic waves.

In the papers of Mazitov and O'Neil, it was shown that for a spatially periodic wave, the damping process is characterized fully by the quantity γT, where γ is the linear damping decrement of the wave and T is the characteristic period of oscillation of the electrons trapped in the potential wells of the wave: $T = (m/ek^2\varphi)^{1/2}$ (in this formula, φ is the amplitude of the potential of the wave, k the wave vector, e and m the charge and mass, respectively, of the electron). If the amplitude of the wave is small, so that $\gamma T \gg 1$, then the linear approximation is valid and the wave is damped exponentially. But if the amplitude is large ($\gamma T \ll 1$), then phase mixing of the resonant particles occurs within a time of the order of T, after which the damping ceases. Here the energy of the wave in the final state becomes less than its initial value by only a small quantity (of the order of γT).

In contrast to[1,2], we consider the damping of a Langmuir wave packet of the form

$$\psi = \varphi(x, t) \cos (kx - \omega_p t - \theta(x, t)), \qquad (1)$$

where ω_p is the electron plasma frequency and $\varphi(x, t)$ and $\theta(x, t)$ are the amplitude and phase of the packet, both of which change slowly in space and in time. We shall show here that, in contrast to the spatially periodic problem, the packet is always damped to extinction.

2. DAMPING OF A PACKET WITH A FIXED PHASE VELOCITY

We first consider the particular case of a packet with a constant phase velocity ($\theta(x, t) = 0$). Since the frequency is independent of the wave vector for Langmuir oscillations of a cold plasma, the change in the envelope of the packet $\varphi(x, t)$ can be due only to damping by resonant electrons (dispersion spreading is lacking). These electrons enter into the packet from the left (we assume that the phase velocity of the wave is positive: $v_0 \equiv \omega_p/k > 0$), and traverse the region occupied by the packet in a time $t_0 \sim L/v_0$, where L is the characteristic width of the packet. If the width L is small, so that $t_0 \lesssim T$, then the distribution function of the resonant electrons does not change substantially during the time of their flight through the packet and the damping is determined by linear theory. It is of interest to note that in this case the quantity γT can be arbitrary and, in particular, can satisfy the relation $\gamma T < 1$, for satisfaction of which the damping of the spatially-periodic wave of comparable amplitude would be essentially nonlinear.

We shall consider the case of a long packet ($L \gg v_0 T$) under the assumption that $\gamma T \ll 1$ (if $\gamma T \gg 1$, the problem is linear, and the results are obvious beforehand). In order to distinguish more clearly between processes taking place at the leading and trailing edges of the packet, we shall assume that the envelope has the form shown in Fig. 1 at the initial instant.

Keeping in mind that to find the damping it suffices to know the distribution function in a narrow range of velocities around the point $v = v_0$, we shall use the linear approximation of the initial distribution function

$$f_0(v) = f_0(v_0) + (v - v_0) f_0'(v_0). \qquad (2)$$

The process of phase mixing is conveniently studied in a coordinate frame moving with the velocity v_0. In this system, the problem reduces in essence to the study of the motion of electrons in a spatially-periodic field (Fig. 2) with an amplitude that increases slowly (we speak of the leading edge of the packet) in time ($L/v_0 \gg T$!).

Slowness of the change in the amplitude means that the distribution function can always be regarded as "ergodic" (i.e., constant along the phase trajectories). In the region of untrapped particles, this ergodic distribution function can be found from the condition of invariance of the phase volume (the adiabatic invariant), which is satisfied for $L/v_0 \gg T$, and has the form

$$u_0 = \operatorname{sign} u \int_0^\lambda \left[\frac{2}{m}\left(\frac{mu^2}{2} - e\varphi \cos kx + e\varphi \cos kx'\right)\right]^{1/2} \frac{dx'}{\lambda},$$

where λ is the wavelength, $u = v - v_0$ the particle velocity in the system of the wave, and u_0 the initial value of this velocity (for $\varphi = 0$). Taking (2) into account, we use this relation to obtain the distribution function of the traversing particles at an arbitrary instant $t > 0$: $f(x, u, t) = f_0(v_0) + f_0'(v_0) u_0(x, u, t)$. In the region of trapped particles, the ergodic distribution

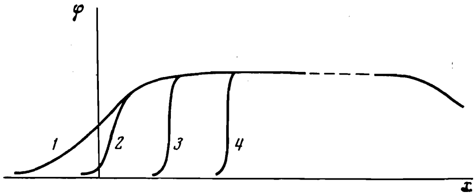

FIG. 1. Envelope of the packet. Curve 1 corresponds to the initial time and curves 2–4 to succeeding instants.

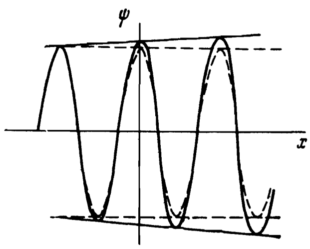

FIG. 2. Potential of the wave at a fixed instant of time. The dashed curves denote a purely harmonic approximation of this potential, used for the solution of the problem of phase mixing.

function is simply a constant, $f_0(v_0)$. This latter circumstance is connected with the fact that the difference $f_0(v) - f_0(v_0)$ is an odd function of u and the phase mixing of the particles trapped by the separatrix leads to vanishing of this difference.

Keeping in view the remarks we have made, we can calculate the energy U expended on the phase mixing of the particles (averaged over the length of the wave):

$$U=\left\langle\frac{m(v_0+u)^2}{2}\right\rangle\approx\frac{mv_0^2}{2}\langle 1\rangle+mv_0\langle u\rangle,$$

where

$$\langle A\rangle=\int_0^\lambda\frac{dx'}{\lambda}\int_{-\infty}^{+\infty}(f-f_0)A\,du.$$

In the spatially periodic problem now being considered, the number of particles per wavelength does not change. Therefore

$$\langle 1\rangle=0 \tag{3}$$

and the expression for U takes the form

$$U=mv_0\langle u\rangle=\frac{256}{9\pi^2}\gamma T\frac{k^2\varphi^2}{8\pi},$$
$$\gamma=\frac{\pi}{2}\frac{\omega_p}{n}|f_0'(v_0)|\,v_0^2, \tag{4}$$

where γ is the linear damping decrement. This result was obtained in[3] and independently for a packet of helicons in[4].

In order to obtain a closed equation for the envelope, we must make use of the law of energy conservation, which can be written in the form (see[5])

$$-\frac{\partial}{\partial t}\frac{k^2\varphi^2}{8\pi}=\frac{\partial}{\partial t}\left\langle\frac{m(v_0+u)^2}{2}\right\rangle+\frac{\partial}{\partial x}\left\langle\frac{m(v_0+u)^3}{2}\right\rangle$$

The first term on the right side can be neglected, since it is smaller than the second by a factor $(\gamma T)^{-1}$. As for the last term, it can be transformed by taking into account the smallness of u:

$$-\frac{\partial}{\partial t}\frac{k^2\varphi^2}{8\pi}=\frac{mv_0^3}{2}\frac{\partial}{\partial x}\langle 1\rangle+\frac{3mv_0^2}{2}\frac{\partial}{\partial x}\langle u\rangle. \tag{5}$$

At first glance, it appears that one ought to write the right side in the form $(3/2)\partial(Uv_0)/\partial x$ (see Eqs. (3), (4)). But since we are now considering a not completely periodic distribution of the potential (Fig. 2), relation (3) loses force, inasmuch as, generally speaking, the number of particles per wavelength in a nonperiodic problem can change. This change is easiest to calculate with the help of the equation of continuity, which, after averaging over the wavelength, takes the form

$$\frac{\partial}{\partial t}\langle 1\rangle+\frac{\partial}{\partial x}\langle v_0+u\rangle=0.$$

The first term is smaller than the second by a factor $(\gamma T)^{-1}$ and can be neglected. As a result, we get

$$\frac{\partial}{\partial x}\langle 1\rangle=-\frac{1}{v_0}\frac{\partial}{\partial x}\langle u\rangle.$$

After integration, we find the desired change in the density:[1)]

$$\Delta n=\langle 1\rangle=-\frac{1}{v_0}\langle u\rangle=-U/mv_0^2$$

and, with the aid of (5), we obtain the closed equation for the envelope:

$$\frac{\partial}{\partial t}\frac{k^2\varphi^2}{8\pi}=-v_0\frac{\partial U}{\partial x}.$$

Upon substitution of $\tilde{v}=(64/3\pi^2)v_0\gamma T$, this equation reduces to the form

$$\frac{\partial\tilde{v}}{\partial t}+\tilde{v}\frac{\partial\tilde{v}}{\partial x}=0.$$

It then follows that the envelope of the packet is so deformed that each of its points moves with the constant velocity

$$\tilde{v}=\frac{64}{3\pi^2}v_0\gamma T,$$

which is inversely proportional to $\sqrt{\varphi}$ (Fig. 1). Consequently, the leading edge of the packet becomes more and more steep, and ultimately a discontinuity appears on it.

Formally, the development of a discontinuity corresponds to having the derivative $\partial\varphi/\partial x$ become infinite, but this result is a consequence of the adiabatic approximation that we have used. In fact, as soon as the width of the discontinuity becomes of the order of v_0T, its further narrowing ceases. The velocity of the discontinuity will obviously be of the order of $v_0\gamma T$. Finding the numerical coefficient in this formula and determination of the form of the discontinuity are evidently possible only by numerical integration of the equations of motion of the electron in the region of the discontinuity.

We note that the evolution of very narrow (length $\sim 5\lambda$) packets of Langmuir waves has been analyzed numerically by Denavit and Sudan.[6] However, just because of the narrowness of the packets, these authors could not, in principle, observe the effects described above.

Events at the trailing edge of the packet have qualitatively the same character as those at the leading edge. But quantitative description of the corresponding processes is made more difficult by the fact that the distribution function of particles behind the discontinuity is not known after formation of the discontinuity on the leading edge, and it is precisely this that determines the quantitative characteristics of the processes taking place on the trailing edge.

Summing up the results of this section, we can state that the damping process of a long ($L \gg v_0T$) packet of Langmuir waves of high amplitude ($\gamma T \ll 1$) consists not in a lowering of the packet amplitude that is uniform over the length of the packet, but in a decrease in the length of the packet at a rate $\sim v_0\gamma T$. The time of vanishing of the packet will, accordingly, be of the order $L/v_0\gamma T$.

3. CONDITIONS OF APPLICABILITY OF RESULTS OBTAINED

Dispersion effects, which we did not consider important above, can have a definite effect on the structure of the discontinuity that arises at the front of the packet. This neglect is valid as long as the rate of dispersion spreading of the discontinuity, which is equal in order of magnitude to $(v_0T)^{-1}\gamma^2\omega/\partial k^2$, is small in comparison with the velocity of the discontinuity itself:

$$\frac{1}{v_0T}\left|\frac{\partial^2\omega}{\partial k^2}\right| \ll v_0\gamma T.$$

For Langmuir oscillations in an unbounded plasma, the derivative $\partial^2\omega/\partial k^2$ is determined by the thermal increment to the frequency, and is very small, so that this condition is not too restrictive.[2)]

The results given in the previous section generally also require modification in the case of a packet with variable phase $\theta(x, t)$ (see (1)), i.e., in the case in which not only the amplitude of the wave changes along the length of the packet, but also its phase velocity. It is clear that the corresponding corrections are insignificant if the change in the phase velocity Δv_0 is small in comparison with the width of the resonance region $\sqrt{e\varphi/m}$. If the condition $\Delta v_0 \gg \sqrt{e\varphi/m}$ is satisfied, then the presence of scatter in the phase velocities leads to a qualitative change in the damping process. Here one should distinguish the cases of regular and random changes in the phase velocity. In the first case, the entire packet can be divided up qualitatively into segments of length $l \sim (L/\Delta v_0)\cdot\sqrt{e\varphi/m} \ll L$, on each of which the phase velocity changes by an amount $\lesssim\sqrt{e\varphi/m}$ and which are damped independently of one another (there is no overlap of the resonance regions). Accordingly, damping time of the entire packet will be of the order

$$l/v_0\gamma T \sim (L/v_0\gamma T)\sqrt{e\varphi/m\Delta v_0^2}$$

(which is much smaller than for a packet of the same length but with fixed phase velocity). To obtain the last estimate, we have naturally assumed that $l > v_0T$ (if $l < v_0T$, then the damping is determined by the linear theory, see the beginning of Sec. 2).

In the second case, the damping of the packet should be described with the aid of the quasilinear theory.[3)][7,8] The corresponding problem is solved in the next section.

4. DAMPING OF A "QUASILINEAR" PACKET

The quasilinear equations have the form

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}=\frac{\partial}{\partial v}\mathscr{D}\frac{\partial f}{\partial v}, \tag{6}$$

$$\frac{\partial \mathscr{D}}{\partial t}=\mathscr{D}\frac{\pi\omega_p}{n}v^2\frac{\partial f}{\partial v}, \tag{7}$$

where $f(x, v, t)$ is the distribution function, $\mathscr{D}(x, v, t)$ the quasilinear diffusion coefficient, which is connected by the relation $\mathscr{D} = 4\pi^2e^2W/m^2$ with the spectral energy density W of the Langmuir oscillations, and n is the concentration of the plasma.

Depending on the initial energy density of the oscillations, the damping can take place in two essentially different ways. If the energy is small, so that the condition

$$\gamma\mathscr{D}^{-1}\Delta v_0^2 \gg 1, \tag{8}$$

is satisfied, where γ is the linear decrement of the oscillations, then the oscillations are damped to extinction, without producing any essential change in the electron distribution function. Here the spatial boundedness of the packet does not bring any essentially new effects into the problem (in comparison with the unbounded case). Therefore, we shall from the very beginning regard the energy of the oscillations as large (so that the inequality that is the reverse of (8) is satisfied). In the case of an infinitely long packet, rapid formation of a plateau in the distribution function takes place under such conditions, after which the damping ceases. Here the energy of the oscillations changes insignificantly. But if the packet has a finite length, then "fresh" particles enter continuously into the region occupied by it, and the packet is damped to extinction.

In accord with what has been said above, we shall make use of initial conditions of the form

$$\mathscr{D}|_{t=0}=\mathscr{D}_0(x)\begin{cases}1-(v-v_0)^2/\Delta v_0^2(x), & |v-v_0|<\Delta v_0(x)\\ 0, & |v-v_0|>\Delta v_0(x)\end{cases},$$
$$f|_{t=0}=f_0(v_0)+f_0'(v_0)\begin{cases}0, & |v-v_0|<\Delta v_0(x)\\ v-v_0, & |v-v_0|>\Delta v_0(x)\end{cases} \tag{9}$$

(it is understood that the scatter in the phase velocities $\Delta v_0(x)$ is small in comparison with the scale of change of the velocity distribution function). In order to distinguish more clearly between processes that take place on the leading and trailing edges of the packet, we shall assume that the functions $\mathscr{D}_0(x)$ and $\Delta v_0(x)$ have the forms shown in Fig. 3.

Introducing the new variable $\Delta v = v = v_0$ and taking into account that the important change in the functions f and $\mathscr{D}$ takes place in the interval $\Delta v \ll v_0$, we can simplify the system (6)—(7) somewhat:

$$\frac{\partial f}{\partial t}+v_0\frac{\partial f}{\partial x}=\frac{\partial}{\partial \Delta v}\mathscr{D}\frac{\partial f}{\partial \Delta v}, \tag{10}$$

$$\frac{\partial \mathscr{D}}{\partial t}=\mathscr{D}\frac{\pi\omega_{pe}}{n}v_0^2\frac{\partial f}{\partial \Delta v}. \tag{11}$$

It is of further convenience to introduce the dimensionless variables

$$u=\frac{\Delta v}{\Delta v_0^*},\qquad u_0=\frac{\Delta v_0}{\Delta v_0^*},$$
$$D=\frac{\mathscr{D}}{\mathscr{D}_0^*},\qquad D_0=\frac{\mathscr{D}_0}{\mathscr{D}_0^*},\qquad \tau=2\gamma t,\qquad \xi=\frac{2\gamma x}{v_0}, \tag{12}$$
$$g=\frac{f-f_0(v_0)}{\Delta v_0^*|\partial f_0/\partial v|_{v=v_0}},$$

where the meaning of the quantities Δv_0^* and $\mathscr{D}_0^*$ is clear from Fig. 3.

In these variables, Eqs. (10), (11) and the initial conditions (9) take the form

$$\varepsilon\left(\frac{\partial g}{\partial \tau}+\frac{\partial g}{\partial \xi}\right)=\frac{\partial}{\partial u}D\frac{\partial g}{\partial u}, \tag{13}$$

$$\partial D/\partial\tau = D\partial g/\partial u, \tag{14}$$

$$D|_{\tau=0}=D_0(\xi)\begin{cases}1-u^2/u_0^2(\xi), & |u|<u_0(\xi),\\ 0, & |u|>u_0(\xi),\end{cases} \tag{15}$$

$$g|_{\tau=0}=\begin{cases}0, & |u|<u_0(\xi),\\ -u, & |u|>u_0(\xi),\end{cases} \tag{16}$$

where the parameter ϵ, which is determined by the equality

$$\varepsilon=2\gamma\frac{\Delta v_0^{*2}}{\mathscr{D}_0^*},$$

is small in comparison with unity, in accord with what was said above.

It is evident that D and g are respectively even and

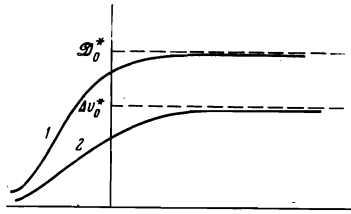

FIG. 3. Initial diffusion coefficient (curve 1) and initial scatter of phase velocities (curve 2) in the case of a "quasilinear" packet.

odd functions of u and therefore it is sufficient to study the system (13)–(14) in the region $u > 0$.

We denote by $\tilde{u}(\xi, \tau)$ the right boundary of this region in velocity space, where the quasilinear diffusion coefficient D is different from zero. It is evident that $\tilde{u}(\xi, \tau) \le u_0(\xi)$. Further, taking into account the smallness of the parameter ϵ, and neglecting the left side of Eq. (13) for that reason, we find that the function g has the form

$$g=\begin{cases}0, & u<\tilde{u}(\xi,\tau)\\ -u, & u>\tilde{u}(\xi,\tau)\end{cases}. \qquad (17)$$

To find the function D we use the exact expression

$$\varepsilon\left(\frac{\partial g}{\partial\tau}+\frac{\partial g}{\partial\xi}\right)=\frac{\partial}{\partial u}\frac{\partial D}{\partial\tau}, \qquad (18)$$

which is a consequence of (13) and (14). Substituting (17) here, we establish the fact that the function D satisfies the following simple equation in the region $u < \tilde{u}(\xi, \tau)$:[4)]

$$\frac{\partial}{\partial u}\frac{\partial D}{\partial\tau}=0.$$

Solving this equation with the initial condition (15), we find that

$$D=D_0(\xi)\left(1-\frac{u^2}{u_0^2(\xi)}\right)+A(\xi,\tau), \qquad (19)$$

where A is an arbitrary function. It can be found by noting that, in accord with the definition of the function $\tilde{u}(\xi, \tau)$, the condition $D|_{u\to\tilde{u}-0} = 0$ must be satisfied. As a result, we finally obtain for $u < \tilde{u}$

$$D=D_0\frac{\tilde{u}^2-u^2}{u_0^2}.$$

Thus, the form of the functions $D(u, \xi, \tau)$ and $g(u, \xi, \tau)$ is determined, and it remains only to find the function $\tilde{u}(\xi, \tau)$. For this purpose, we integrate Eq. (18) over the interval $(\tilde{u} - 0, \tilde{u} + 0)$. The result has the form

$$\varepsilon\tilde{u}\left(\frac{\partial\tilde{u}}{\partial\tau}+\frac{\partial\tilde{u}}{\partial\xi}\right)=-\frac{\partial D}{\partial\tau}\bigg|_{u=\tilde{u}-0}.$$

Calculating the derivative $\partial D/\partial\tau$ by means of the relation (19), we find an equation for $\tilde{u}$:

$$\frac{\partial\tilde{u}}{\partial\tau}+\frac{\varepsilon}{\varepsilon+2D_0/u_0^2}\frac{\partial\tilde{u}}{\partial\xi}=0. \qquad (20)$$

A similar method was used previously in[9]. The general solution of Eq. (20) has the form

$$\tilde{u}=h\left(\int_0^\xi\left[\varepsilon+2\frac{D_0(\xi')}{u_0^2(\xi')}\right]d\xi'-\varepsilon\tau\right).$$

The initial condition $u|_{\tau=0} = u_0(\xi)$ allows us in principle to find the function h. As an illustration, we set forth the solution for the very simple case in which the ratio $D_0(\xi)/u_0^2(\xi)$ does not depend on the coordinates: $D_0(\xi)/u_0^2(\xi) = D_0(\infty)/u_0^2(\infty) = 1$ (the last equality in this chain is connected with the fact that, in accord with the definition (12), $D_0(\infty) = u_0(\infty) = 1$). In this case, we have

$$\tilde{u}=u_0\left(\xi-\frac{\varepsilon}{2+\varepsilon}\tau\right).$$

Thus the packet is "eroded" at a constant rate. In dimensional variables, this rate can be estimated as

$$v_0(\gamma\Delta v_0^2/D_0)\ll v_0.$$

as far as the trailing edge of the packet is concerned, it remains immobile in the quasilinear case.

1)For reference, we also give the formula for change in the energy density U′ in the nonperiodic case: U′ = U/2. The difference between U′ and U is connected with the fact that the relation (3) is violated in the nonperiodic case.

2)If it is not satisfied, then dispersion effects limit the width of the discontinuity at the level $(v_0\gamma T)^{-1}|\partial^2\omega/\partial k^2|\gg v_0T$.

3)We have in mind that the scale of change in the phase velocity is small in comparison with v_0T. If the converse holds, then the problem reduces essentially to the first case mentioned above.

4)This equation is also valid for $u > \tilde{u}(\xi,\tau)$, but here D = 0, and it is satisfied identically. The question can be raised as to why it is impossible in the search for D to use the equation $\partial D/\partial\tau = 0$, which is obtained from (14) with (17). The fact is that the change of D with time is a quantity of order ϵ, while the solution (17) is valid only in the zeroth order in ϵ, so that the equation $\partial D/\partial\tau = 0$ is valid only in the zeroth order in ϵ.

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