A hydrogen-like atom in a superstrong magnetic field

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The relativistic problem of the terms of a hydrogen-like atom in a superstrong magnetic field is solved. The conditions under which the ground term approaches the lower boundary of the Dirac continuum are determined.

In the absence of a magnetic field, the ground term of a hydrogen-like atom reaches the lower boundary of the Dirac continuum for $Z \approx 170$.[1] If a nucleus with realistic charge $Z \ll 137$ is taken, the binding energy of the ground term is not large. However, if such an atom is placed in a superstrong magnetic field, the latter appreciably increases the binding energy. In the presence of a sufficiently strong field, this term may approach the boundary of the Dirac continuum.

The behavior of the ground term for a hydrogen-like atom with charge $Z \ll 137$ is investigated in this article in the presence of a constant, homogeneous, superstrong magnetic field. The analogous problem in the nonrelativistic limit[2] was solved earlier.

In the absence of the atomic nucleus, an electron placed in a magnetic field is found in Landau levels.[3] In the ground state an electron with spin along the magnetic field is centered in a cylindrical tube of small radius[1] $a \sim (eH)^{-1/2}$ and undergoes infinite motion along the field. Upon application of the electric field $-Ze^2/r$ of the charge of the nucleus, the motion along the field becomes finite. Thus, from a mathematical point of view the problem reduces to the one-dimensional Coulomb problem (see Fig. 1). The Coulomb potential is cutoff only at small distances of order a.

The binding energy of the ground term tends to $-\infty$ as $a \to 0$, and "collapse to the center" occurs.[2] If $a \neq 0$ the binding energy exhibits a logarithmic dependence on the cutoff parameter a. It is impossible to solve the problem exactly. In order to obtain analytic expressions, we shall everywhere confine our attention to logarithmic accuracy in a, i.e., to logarithmic accuracy in the magnetic field H.

Let us direct the x axis along H. In the x direction we have a system of two one-dimensional Dirac equations for the upper and lower components of a bispinor. In the direction transverse to H, the solution of the Dirac equation coincides with the solution obtained in the absence of an electric field.[3] It is convenient to choose the cutoff of the one-dimensional Coulomb potential in the following form:

$$V = -Ze^2 / (|x| + a).$$

Then, after replacing the independent variable by $x + a = t$ for $x > 0$ and by $-x + a = t$ for $x < 0$, the Dirac equations reduce to the usual relativistic Coulomb equations. First let us assume that $x > 0$. Let us make the customary (for the Coulomb problem) change to the components F and G of the Dirac bispinor:

$$F = \left(\frac{1+\varepsilon}{t}\right)^{1/2} (\chi_1 + \chi_2), \quad G = \left(\frac{1-\varepsilon}{t}\right)^{1/2} (\chi_1 - \chi_2);$$

here ε is the relativistic energy term. Then we obtain independent Whittaker equations for χ_1 and χ_2. Their solution, which vanishes as $x \to \infty$, has the form

$$\chi_1 = W_{\Lambda, iZe^2}(2\lambda t), \quad \chi_2 = \frac{\lambda}{Ze^2} W_{\Lambda+1, iZe^2}(2\lambda t),$$

where $\lambda = (1 - \varepsilon^2)^{1/2}$ and $\Lambda = Ze^2\varepsilon/\lambda - \frac{1}{2}$; here W denotes the Whittaker function. The characteristic region x of variation of the wave functions is of the order of an electron's Compton wavelength, $1/\lambda \sim 1$. The corresponding solution for $x < 0$ is found in a similar manner.

By matching F and G at $x = 0$, we find there are two classes of solutions which differ by their symmetry with respect to the substitution $x \to -x$: the first class of solutions corresponds to F being an even function, G an odd function, and the second class of solutions corresponds to the converse situation. States of the second class also occur in the usual three-dimensional relativistic Coulomb problem. In the nonrelativistic limit for such states, the radial wave function $rR(r)$, which is reduced to one-dimensional form, tends to zero as $r \to 0$. In the problem under consideration the energies of these states turn out to be close to the energies of the usual three-dimensional Coulomb problem. Therefore they are not of interest. On the other hand, the states of the first class are not present in the three-dimensional Coulomb problem. The quantization rule for these states has the form

$$W_{\Lambda, iZe^2}(2\lambda a) = \frac{\lambda}{Ze^2} W_{\Lambda+1, iZe^2}(2\lambda a).$$

We assume the magnetic field H to be so strong that $H \gg \lambda^2/e$. This means that $\lambda a \ll 1$, that is, the radius a of the small tube in which the electron effectively moves is small in comparison with the electron's Compton wavelength. Then the functions F and G are almost constant in the region $x \sim a$ near the origin of coordinates. Therefore, the fact that the exact solution in this region is unknown is unimportant.

Since $\lambda a \ll 1$, the Whittaker functions appearing in the last equation can be expanded in power series. Then we obtain the following approximate expression for the energy ε of the electron's ground state:

$$\varepsilon = \cos\left(Ze^2 \ln \frac{H}{H_0\lambda^2}\right). \tag{1}$$

Here H_0 is the relativistic unit of magnetic field strength, which is equal to $1/e = 4.42 \times 10^{13}$ Oe. The energies of all the excited levels of the first class turn out to be close to the energies of the corresponding second class levels, and therefore are not of interest.

In the nonrelativistic limit the cosine in expression (1) can be expanded in a power series. We obtain the well-known expression for the energy of the nonrelativistic ground state of a hydrogen-like atom in a strong magnetic field:[2,4]

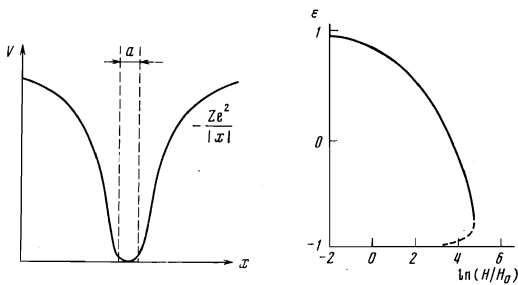$$\varepsilon \approx 1 - \frac{Z^2 e^4}{2} \ln^2 \frac{H}{Z^2 H_0'}. \tag{2}$$

FIG. 1   FIG. 2

FIG. 1. The effective potential of a hydrogen-like atom in a super-strong magnetic field. The quantity a is of the order of $(eH)^{-1/2}$.

FIG. 2. The energy ϵ for the ground state of a hydrogen-like atom in a superstrong magnetic field H as a function of the parameter ln (H/H_0). The charge Z = 60.

Here H_0' denotes the atomic unit of magnetic field strength, which is given by $H_0 e^4 = e^3 = 2.35 \times 10^9$ Oe.

The condition for the applicability of the relativistic solution (1) has the form $H \gg H_0$, whereas the condition for the applicability of the nonrelativistic solution (2) is given by

$$Z^2 H_0' \ll H \ll Z^2 H_0' \exp(1/Ze^2).$$

In addition, the solution (1) will not be applicable in the presence of a magnetic field H which is so strong that the decrease of the ground state energy becomes linear in H due to the anomalous correction $e^2/2\pi$ to the electron's magnetic moment. This phenomenon begins at fields H of the order of $2\pi H_0/e^2 = 4 \times 10^{16}$ Oe. Thus, the solution (1) is valid under the conditions

$$H_0 \ll H \ll H_0 2\pi/e^2.$$

A graph showing the dependence of ϵ on ln (H/H_0) for the case Z = 60 is shown in Fig. 2. One can easily show that the term has an infinite derivative when the energy $\epsilon \approx -1 + 2\,Z^2 e^4$ is close to the boundary of the lower continuum. This point is reached at the critical magnetic field H_{cr}, which is given by

$$H_{cr} = H_0 \exp\left(\frac{\pi}{Ze^2} - 2\ln\frac{1}{2Ze^2} - 2\right).$$

For example, we have $H_{cr} = 6 \times 10^{15}$ Oe for an atom with Z = 60. It is impossible to continue the term into the supercritical region since pair production occurs at the critical point, and it is impossible to consider the problem as a single-particle problem.

It is seen from the solution that, in a superstrong magnetic field the binding energy of the ground state of a hydrogen-like atom may become of the order of the electron's rest energy. If magnetic fields of the order of 10^{14} Oe exist in cosmic space, the spectra of atoms which fall into such fields will be strongly modified.

In conclusion let us estimate the effect due to the finite size of the atomic nucleus. It consists in the fact that, for sufficiently strong magnetic fields, the cutoff of the Coulomb potential will occur at nuclear dimensions. One can neglect the finite size of the nucleus if the radius a $\sim (eH_{cr})^{-1/2}$ of the cylindrical tube in which the electron moves will be larger than the nuclear radius R even for $H = H_{cr}$. Estimating $R \sim e^2$, we obtain the following condition for the applicability of the solution (1):

$$\frac{1}{e^2} \gg Z > \frac{1}{e^2}\ln^{-1}\frac{1}{e^2} \approx 30.$$

On the other hand, one must have $Z \ll 1/e^2 = 137$, since in the opposite case it would be impossible to regard the Coulomb field as weak in comparison with the magnetic field.

Let us note that the presence of the electron's spin is extremely important in the problem under consideration. In fact, for the Klein-Gordon equation describing the behavior of a spinless particle in a weak Coulomb field and a strong magnetic field, the logarithmic decrease of the ground term with increasing magnetic field is overwhelmed by the positive term $H/2$ which is linear in the field, and which exists even in the absence of the Coulomb field.

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[1]The system of units m = ħ = c = 1 is chosen.

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