High density bose gas with strong attraction between the particles

V. S. Babichenko
Moscow State University

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An exact solution is considered for a model of a Bose particle system with a distinct interaction potential which permits the self-consistent-field approximation to be applied. The dependence of the chemical potential on the particle number density is determined and the principal thermodynamic functions of the system and the spectrum of elementary excitations are calculated. It is shown that the presence of a bound state of a particle pair does not affect the results.

The model of a Bose gas with attraction between the particles at large distances has been considered previously by a number of authors.[1-3] In these papers, the pair interaction potential was chosen in the form of a shortrange "hard" repulsive core and a longrange shallow potential well. A general property of systems with attraction is the presence of a certain critical density n_{cr} for which the states with densities $n < n_{cr}$ are unstable. Systems with weak attraction, i.e., those in which the pair interaction potential has a shallow well, have n_{cr} such that $n_{cr}a^3 \ll 1$, where a is the radius of the shortrange hard core. In such systems, therefore, the gas approximation is applicable relative to the repulsive core for small pressures, while ordinary perturbation theory is applicable so far as the shallow well is concerned. But in a model with a hard repulsive core, it has not been possible to consider the case of strong pair attraction, i.e., cases in which the depth of the well cannot be considered to be small. This is because in this case $n_{cr}a^3$ is of the order of unity.

In order to consider the case of strong attraction, the repulsive core has been assumed to be "soft" in the present study, so that it was possible to solve the problem exactly. The repulsive part of the pair interaction potential is chosen in the form of a Yukawa potential and the attractive part in the form of a potential well, the depth of which satisfies the condition $U_0R^2 \gtrsim \hbar^2/m$ (R is the radius of the well, U_0 its depth, m the mass of the particle). As a result of the fact that the repulsive part of the pair interaction potential is a soft core, the system has a high critical density, relative to both the longrange part of the pair interaction potential and the shortrange part. This makes it possible to apply the self-consistent-field approximation, which is expressed in the selection of the principal sequence of diagrams, which is analogous to the principal sequence of diagrams for high-density Bose systems with a Coulomb pair interaction.[4] The choice of a soft repulsive core does not correspond to the real situation.

However, the exact solution of such a model of a system of Bose particles is of interest, inasmuch as this model leads to a strongly compressed state of the system and in this sense is a complement to the well-studied system of a weakly non-indeal Bose gas.

FORMULATION OF THE MODEL AND THERMODYNAMIC PROPERTIES

We chose the model of the pair interaction potential in the form

$$U(r)=\frac{\gamma}{r}\exp\left(-\frac{r}{R}\right)-U_0\exp\left[-\left(\frac{r}{R}\right)^2\right],$$

where $\gamma > 0$, $U_0 > 0$, $R > R_0$. As will be seen from what follows, Gaussian form of the second term is not essential. It can be chosen, for example, in the form of a term that is exponentially damped with the distance. The sole condition for the second term is a sufficiently smooth change with distance, such that its Fourier transform is localized in a small region of momentum space $p \lesssim \hbar/R$. The form of the pair interaction potential is shown in Fig. 1.

The condition of strong attraction or the deep-well condition means that $U_0R^2 \gtrsim \hbar/m$. For what follows, it is convenient to choose the system of units $\hbar = 1$, $m = 1$, $R_0 = 1$. The Fourier transform of the pair interaction potential will have the form

$$U(p)=\frac{\gamma}{1+p^2}-\pi^{3/2}U_0\exp\left(-\left(\frac{pR}{2}\right)^2\right).$$

The basic condition that is superimposed on the pair interaction potential parameters will be smallness of the Fourier transform at small momenta, i.e.,

$$U(p)_{p=0}=\gamma-\pi^{3/2}U_0R^3=\beta\pi^{3/2}U_0R^3,\quad 0<\beta\ll 1.$$

Therefore the fundamental relation between the parameters of the potential takes the form

$$\gamma=(1+\beta)\pi^{3/2}U_0R^3.$$

Using the condition of strong attraction, we easily find that $\gamma \gtrsim 1$. A graph of the Fourier transform of the potential is shown in Fig. 2 (curve a).

To calculate the chemical potential of the system, we pick out the principal sequence of condensate diagrams, i.e., those diagrams which do not have entering and exiting terminals other than the condensate lines. As will be shown in what follows, the particle number density of the condensate n_0 and the chemical potential of the system μ satisfy the condition

$$(n_0\gamma)^{3/2} \gg \mu \gg \gamma \gtrsim 1.$$

We now estimate some very simple diagrams of the series of perturbation theory. We shall consider the

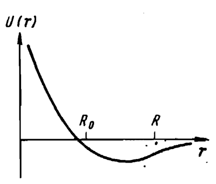

FIG. 1.

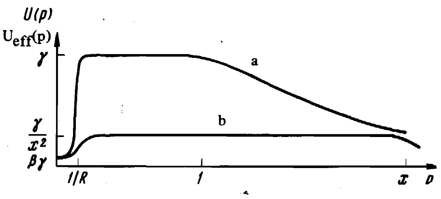

FIG. 1. Pair interaction potential.
FIG. 2. Fourier transform of pair interaction, a—U(p), b—$U_{eff}(p)$.

diagrams shown in Fig. 3. The quantity corresponding to the diagram of Fig. 3a has the form

$$n_0^2\int\frac{U^2(q)}{(\omega-q^2/2+\mu+i\delta)(-\omega-q^2/2+\mu+i\delta)}\frac{d^4q}{(2\pi)^4}$$
$$=in_0\int\frac{U^2(q)}{(-q^2+2\mu)}\frac{d^3q}{(2\pi)^3}.$$

It is easy to establish the fact that the principal contribution to this integral is made by momenta $q\sim 1$. Therefore the quantity corresponding to the diagram of Fig. 3a is proportional to

$$n_0^2\int\frac{dq\,q^2\gamma^2}{(1+q^2)^2(q^2-2\mu)}\sim n_0\gamma\frac{n_0\gamma}{\mu}.$$

The quantity corresponding to the diagram of Fig. 3b can be represented in the form

$$n_0^3\int\frac{U^3(q)}{(\omega-q^2/2+\mu+i\delta)^2(-\omega-q^2/2+\mu+i\delta)}\frac{d^4q}{(2\pi)^4}$$
$$=in_0^3\int\frac{U^3(q)}{(-q^2+2\mu)^2}\frac{d^3q}{(2\pi)^3}.$$

The principal contribution to this integral is made by momenta $q\sim 1$. Therefore the quantity corresponding to the diagram of Fig. 3b is proportional to

$$n_0^3\int\frac{\gamma^3q^2\,dq}{(1+q^2)^3(q^2-2\mu)^2}\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)^2.$$

For the diagram of Fig. 3c, we can write

$$n_0^2\int\frac{d^4q_1}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}U(q_1)U(q_2)U(q_1-q_2)$$
$$\times\left[\left(\omega_1-\frac{q_1^2}{2}+\mu+i\delta\right)\left(-\omega_1-\frac{q_1^2}{2}+\mu+i\delta\right)\left(\omega_2-\frac{q_2^2}{2}+\mu+i\delta\right)\right.$$
$$\left.\times\left(-\omega_2-\frac{q_2^2}{2}+\mu+i\delta\right)\right]^{-1}\approx n_0^2\int\frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}$$
$$\times\frac{U(q_1)U(q_2)U(q_1-q_2)}{(-q_1^2+2\mu)(-q_2^2+2\mu)}\sim n_0\gamma\frac{n_0\gamma}{\mu}\frac{\gamma}{\mu}.$$

The principal contribution here is made by momenta $q_1\sim 1$, $q_2\sim 1$.

The expression for the single-loop diagrams, examples of which are shown in Fig. 4a., will have the form

$$\int\frac{d^3q}{(2\pi)^3}\frac{U^m(q)}{(-q^2+2\mu)^{m-1}}$$

where m is the order of the diagram. In this expression, the characteristic momenta are

$$q\leqslant 1/R,\quad q\sim 1,\quad q\sim\mu^{1/2}.$$

For momenta $q\lesssim 1/R$, the quantity $U(q)\sim\beta\gamma$, where $\beta\ll 1$. The contribution of the momenta $q\lesssim 1/R$ to the integral will be of the order

$$\frac{1}{R^3}\frac{\beta^m\gamma^m n_0^m}{\mu^{m-1}}.$$

The momenta $q\sim 1$ give the contribution

$$\gamma^m n_0^m/\mu^{m-1}.$$

The momenta $q\sim\mu^{1/2}$ made a contribution of order

$$\gamma^m n_0^m\mu^{5/2-2m}.$$

As will be seen from what follows, $\mu\gg 1$; therefore the principal contribution will be made by momenta $q\sim 1$.

In the expressions for the two-loop diagrams (Fig. 4b), which contain the same number of condensate lines as the single-loop diagrams, there will be the additional small factor γ/μ, i.e., the single-loop diagrams containing m entering and m exiting condensate lines will be of the order $n_0^m\gamma^m/\mu^{m-1}$, while the two-loop diagrams will be of order $n_0\gamma^{m+1}/\mu^m$. Similarly, the three-loop diagrams (Fig. 4c) will contain the additional small factor $(\gamma/\mu)^2$.

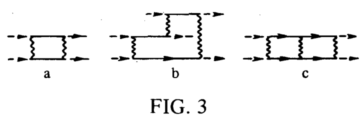

FIG. 3

We now calculate the chemical potential of the system. The value of the chemical potential μ is calculated as the sum of the diagrams shown in Fig. 5a. The quantity $G_1(p)$ denotes the sum of diagrams of type Fig. 6a, $\hat{G}_1(p)$ those of the type of Fig. 6b, and $\check{G}_1(p)$ those of the type of Fig. 6c. The diagrams of Fig. 6a include all diagrams that do not have a single loop and contain one entering and one exiting line. The diagrams of Fig. 6b are diagrams that do not have a single loop and contain two entering external lines. Figure 6c includes diagrams containing two exiting external lines.

As is known,[5]

$$G_1'(q)=\frac{\omega+q^2/2+\Sigma_{11}^{(0)}(q)-\mu}{\omega^2-\varepsilon_1^2(q)+i\delta},$$
$$\hat{G}_1(q)=\frac{\Sigma_{02}^{(0)}(q)}{\omega^2-\varepsilon_1^2(q)+i\delta},$$

where

$$\varepsilon_1(q)=\{[q^2/2+\Sigma_{11}^{(0)}(q)-\mu]^2-[\Sigma_{02}^{(0)}(q)]^2\}^{1/2},$$
$$\Sigma_{11}^{(0)}(q)=\Sigma_{02}^{(0)}(q)=n_0U(q).$$

Then the expression for μ takes the form

$$\mu=n_0U(0)+i\int[G_1'(q)+\hat{G}_1(q)]U(q)\frac{d^4q}{(2\pi)^4}.$$

After integration over the frequency, we get

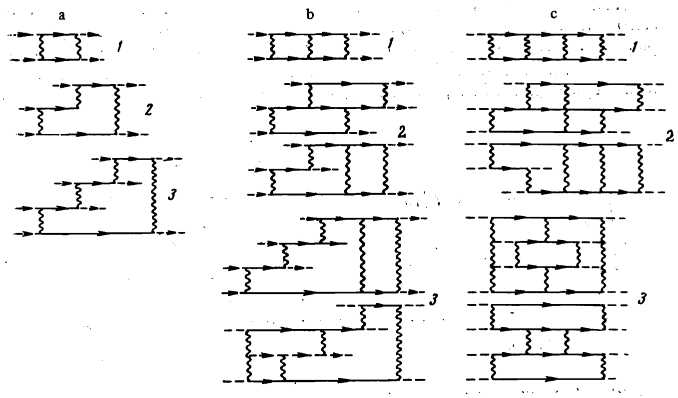

FIG. 4. Table of estimates of certain diagrams.

$$a_1\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right),\quad a_2\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)^2,\quad a_3\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)^3,$$
$$b_1\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)\left(\frac{\gamma}{\mu}\right),\quad b_2\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)^2\left(\frac{\gamma}{\mu}\right),\quad b_3\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)^3\left(\frac{\gamma}{\mu}\right),$$
$$c_1\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)\left(\frac{\gamma}{\mu}\right)^2,\quad c_2\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)^2\left(\frac{\gamma}{\mu}\right)^2,\quad c_3\sim n_0\gamma\left(\frac{n_0\gamma}{\mu}\right)^3\left(\frac{\gamma}{\mu}\right)^2.$$

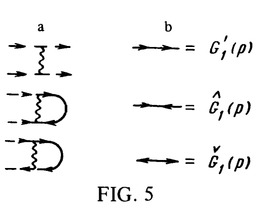

FIG. 5

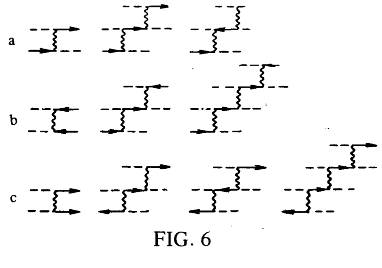

FIG. 6

$$\mu = n_0 U(0) + \int \frac{-\varepsilon_1(q) + q^2/2 + \Sigma_{11}^{(0)} - \mu - \Sigma_{02}^{(0)}}{2\varepsilon_1(q)} U(q)$$

$$= n_0 U(0) + \int_0^\infty \left[\left\{\left(\frac{q^2}{2} - \mu\right)^2 + 2n_0 U(q)\left(\frac{q^2}{2} - \mu\right)\right\}^{1/2}\right.$$

$$\left. + \frac{q^2}{2} - \mu\right] U(q) \left[\left(\frac{q^2}{2} - \mu\right)^2 + 2n_0 U(q)\left(\frac{q^2}{2} - \mu\right)\right]^{-1/2} \frac{d^2q}{(2\pi)^2}$$

For the momenta $q \lesssim 1/R$, the potential $U(q) \sim \beta\gamma \ll 1$. Consequently, the given region of integration makes a contribution that is much smaller than the fundamental contribution. Thus one can assume that for the expression under the integral

$$U(q) = \frac{\gamma}{1+q^2}.$$

Therefore the integral takes the form

$$J = \int \left[-\left\{\left(\frac{q^2}{2} - \mu\right)^2 + 2n_0\gamma \frac{q^2/2 - \mu}{1+q^2}\right\}^{1/2} + \frac{q^2}{2} - \mu\right] \frac{\gamma}{1+q^4}$$

$$\times \left[\left(\frac{q^2}{2} - \mu\right)^2 + 2n_0\gamma \frac{q^2/2 - \mu}{1+q^2}\right]^{-1/2} \frac{d^2q}{(2\pi)^2}.$$

We introduce the new variable of integration $\xi = q/(n_0\gamma)^{1/4}$. We then get

$$J = \frac{1}{(2\pi)^2}(n_0\gamma)^{1/4}\gamma \int_0^\infty \left[-\left\{\left(\frac{\xi^2}{2} - \frac{\mu}{(n_0\gamma)^{1/2}}\right)^2\right.\right.$$

$$\left.\left. + 2\frac{\xi^2/2 - \mu/(n_0\gamma)^{1/2}}{\xi^2 + 1/(n_0\gamma)^{1/2}}\right\}^{1/2} + \frac{\xi^2}{2} - \frac{\mu}{(n_0\gamma)^{1/2}}\right]$$

$$\times \left[\left(\frac{\xi^2}{2} - \frac{\mu}{(n_0\gamma)^{1/2}}\right)^2 + 2\frac{\xi^2/2 - \mu/(n_0\gamma)^{1/2}}{\xi^2 + 1/(n_0\gamma)^{1/2}}\right]^{-1/2} \frac{\xi^2\, d\xi}{\xi^2 + (n_0\gamma)^{-1/2}}$$

In what follows it will be shown that

$$\frac{1}{(n_0\gamma)^{1/2}} \ll 1, \quad \frac{\mu}{(n_0\gamma)^{1/2}} \ll 1.$$

In the expression for J we neglect these quantities. Then the expression for J can be written in the form

$$J = \frac{1}{(2\pi)^2}(n_0\gamma)^{1/4}\gamma \int_0^\infty \frac{-(\xi^4/4 + 1)^{1/2} + \xi^2/2}{(\xi^4/4 + 1)^{1/2}} d\xi.$$

It is evident that the basic contribution to the integral is made by $\xi \sim 1$. Here it turns out that $J \sim \gamma(n_0\gamma)^{1/4}$. And since $\xi \sim 1$ make the principal contribution to the integral, neglect of the terms $1/(n_0\gamma)^{1/2}$ and $\mu/(n_0\gamma)^{1/2}$ is obviously entirely legitimate.

Thus we have obtained the result that, although the principal contribution to individual diagrams of the sequence of diagrams is made by integration momenta $q \sim 1$, the principal contribution to the sum of these diagrams, i.e., to the expression for the chemical potential, is made by the momenta $q \sim (n_0\gamma)^{1/4}$. A similar situation arises in systems with Coulomb interaction between the particles, as is shown, for example, by Brueckner,[4] who studied a Bose gas with Coulomb interaction.

As a result, we obtain an expression for the chemical potential as a function of n_0:

$$\mu = n_0 U(0) - b\gamma(n_0\gamma)^{1/4},$$

$$b = \frac{1}{(2\pi)^2}\int_0^\infty \frac{(\xi^4/4 + 1)^{1/2} - \xi^2/2}{(\xi^4/4 + 1)^{1/2}} d\xi > 0.$$

It is evident here that b is a numerical coefficient larger than zero and of the order of unity.

We now calculate the thermodynamic functions of the system. For this purpose, we express the chemical potential μ as a function of the total particle number density n. Here we need to find the connection between the total particle number density n and the condensate particle number density n_0. It will be shown below that $n - n_0 \ll n_0$; therefore, we can assume that $n = n_0$ in first approximation in the expression for μ. Then the chemical potential as a function of the density takes the form

$$\mu = \beta\gamma n - \beta\gamma^{5/4} n^{1/4}.$$

This expression makes it possible to calculate all the thermodynamic functions of the system. The ground-state energy is

$$E_0 = \frac{1}{2}\beta\gamma\frac{N^2}{V} - \frac{4}{5}b\gamma^{5/4}\frac{N^{5/4}}{V^{1/4}},$$

where N is the total number of particles of the system, V the total volume of the system, and $n = N/V$. The pressure is

$$P = -\frac{\partial E_0}{\partial V} = \frac{1}{2}\beta\gamma n^2 - \frac{1}{5}b\gamma^{5/4}n^{5/4}.$$

The equilibrium density n_{eq} is determined as the value of the total particle number density for $P = 0$:

$$n_{eq} = \left(\frac{2}{5}\right)^{4/3} b^{4/3}\frac{\gamma^{1/3}}{\beta^{4/3}}.$$

The value of the chemical potential for $P = 0$ is determined from the expression

$$\mu(n_{eq}) = -\left[\left(\frac{2}{5}\right)^{1/3} - \left(\frac{2}{5}\right)^{4/3}\right]\frac{\gamma^{4/3}}{\beta^{1/3}} < 0.$$

The sound velocity c as a function of the density has the form

$$c^2 = n\frac{d\mu}{dn} = n\beta\gamma - {}^1/_4\, b\gamma^{5/4}n^{1/4}.$$

If the state of the system is stable, then, obviously, $c^2 > 0$. It is seen from the expression for c^2 as a function of the density that c^2 becomes less than zero for a sufficiently small density. This means that there exists a critical density n_{cr}, and that states with $n < n_{cr}$ are unstable. The value n_{cr} is determined by the expression $c^2(n_{cr}) = 0$. Then

$$n_{cr} = \left(\frac{1}{4}\right)^{4/3} b^{4/3}\frac{\gamma^{1/3}}{\beta^{4/3}}.$$

The states of the system with densities n such that

$$n_{eq} > n > n_{cr},$$

are metastable, since $P(n) < 0$ for them. The states with $n > n_{eq}$ are stable.

CALCULATION OF THE IRREDUCIBLE SELF-ENERGY PARTS $\Sigma_{11}(p)$, $\Sigma_{02}(p)$ AND THE GREEN'S FUNCTION

We make a selection of diagrams for the self-energy parts $\Sigma_{11}(p)$ and $\Sigma_{02}(p)$. Let the external momentum $p \gtrsim 1/R$, so that $U(p) \gg U(0)$. We choose diagrams for $\Sigma_{11}(p)$. We consider all possible diagrams for $\Sigma_{11}(p)$ and classify them by the number of lines of interaction along which integration is not carried out, i.e., the lines of interaction which do not enter into closed loops. They fall into three basic groups (Fig. 7). Generally, diagrams of another type also enter into $\Sigma_{11}(p)$. Examples of such diagrams are shown in Fig. 7e. But these diagrams contain an additional small factor $U(0) \ll 1$; therefore they make a much smaller contribution than the diagrams in Figs. 7a–c. As will be shown in the Appendix, the sum of the diagrams of Figs. 7b and c gives a much smaller contribution than the sum of the diagrams of Fig. 7a. Among the diagrams which do not contain lines of interaction along which integration is not carried out (except the two external lines) the principal sequence is separated in a manner similar to what was done in the selection of condensate diagrams for the calculation of μ. The single-loop diagrams will be the important ones.

Thus, $\Sigma_{11}(p) = n_0 U_{\text{eff}}(p)$ is the sum of the diagrams represented in Fig. 7a. Similarly, $\Sigma_{02}(p) = n_0 U_{\text{eff}}(p)$. In what follows, we shall denote $U_{\text{eff}}(p)$ graphically by a heavy wavy line. For the determination of $U_{\text{eff}}(p)$, we obtain a closed set of equations (Fig. 8). As is shown in [5]; $\hat{G}(p) = \check{G}(p)$, and from the equations of Figs. 8d and e we can obtain

$$G'(p) = \frac{\omega + p^2/2 + \Sigma_{11}(p) - \mu}{\omega^2 - \varepsilon^2(p) + i\delta}$$

$$\hat{G}(p) = -\frac{\Sigma_{02}(p)}{\omega^2 - \varepsilon^2(p) + i\delta}.$$

$$\varepsilon(p) = [(p^2/2 + \Sigma_{11}(p) - \mu)^2 - \{\Sigma_{02}(p)\}^2]^{1/2}.$$

The equation represented in Fig. 8a gives

$$U_{\text{eff}}(p) = \frac{U(p)}{1 - \Pi(p) U(p)},$$

$$\Pi(p) = i\int G'(p+q) \times G'(q) \frac{d^4q}{(2\pi)^4} + i\int \hat{G}(p+q) \times \hat{G}(q) \frac{d^4q}{(2\pi)^4}.$$

In the approximation in which $\Pi(p)$ does not depend on the momentum, we obtain

$$U_{\text{eff}}(p) = \frac{U(p)}{1 - \Pi(0) U(p)}$$

This approximation will be justified in what follows.

For $p \gg 1/R$ the potential $U(p) = \gamma/(p^2+1)$; furthermore, as will be shown, $\Pi(0) \gg 1$; therefore, for $p \gg 1/R$,

$$U_{\text{eff}}(p) = \frac{\gamma}{p^2 - \gamma\Pi(0)}.$$

We introduce the notation $\kappa^2 - \gamma\Pi(0)$. We now estimate $\Pi(0)$:

$$\Pi(0) = i\int \{[G'(q)]^2 + [\hat{G}(q)]^2\} \frac{d^4q}{(2\pi)^4},$$

$$i\int [G'(q)]^2 \frac{d^4q}{(2\pi)^4} = \frac{1}{8\pi}\int \frac{[-\varepsilon(q) + q^2/2 + \Sigma_{11}(q) - \mu]^2 \, d^3q}{\varepsilon^3(q)} \sim -\frac{(n_0\gamma)^{1/2}}{\varkappa};$$

$$\Sigma_{11} = \Sigma_{02} = \frac{n_0\gamma}{p^2 + \varkappa^2},$$

$$\varepsilon(q) = \left[\left(\frac{q^2}{2} - \mu\right)^2 + 2\frac{n_0\gamma}{q^2 + \varkappa^2}\left(\frac{q^2}{2} - \mu\right)\right]^{1/2}.$$

It is evident that the principal contribution to the integral is made by the momenta $q \sim \kappa$. Similarly, we obtain the result that

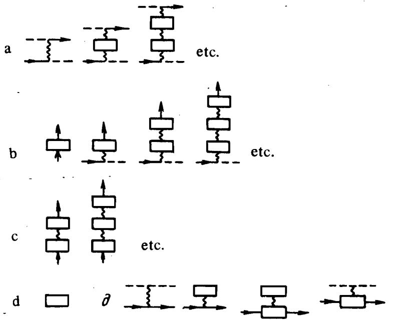

FIG. 7. Diagrams for $\Sigma_{11}(p)$, d–sum of all diagrams which do not contain an interaction line along which integration is not carried out.

$$i\int [\hat{G}(q)]^2 \frac{d^4q}{(2\pi)^4} \sim -\frac{(n_0\gamma)^{1/2}}{\varkappa}.$$

Therefore, $\Pi(0) \sim -(n_0\gamma)^{1/2}/\kappa$, which gives the final estimate for κ, namely $\kappa^2 \sim \gamma n_0^{1/2}$.

It is easy to see that $\Pi(0)$ depends weakly on the external momentum p when $p \ll \kappa$. This validates the approximation used. The solution for $U_{\text{eff}}(p)$ can be represented in the following form:

$$U_{\text{eff}}(p) = \frac{U(p)}{1 - \Pi(0) U(p)} \text{ for } p \sim \frac{1}{R},$$

$$U_{\text{eff}}(p) = \frac{\gamma}{p^2 + \varkappa^2} \text{ for } \frac{1}{R} \ll p \ll \varkappa,$$

$$U_{\text{eff}}(p) = \frac{\gamma}{p^2 - \gamma\Pi(p)} \text{ for } p \ll \frac{1}{R},$$

$$U_{\text{eff}}(p) = U(p) \text{ for } p \gg \varkappa.$$

We have obtained an expression for $\Sigma_{11}(p)$ and $\Sigma_{02}(p)$ for $p \gtrsim 1/R$. For $p \ll 1/R$, the quantity $\Sigma_{11}(p)$ is the sum of the diagrams of Fig. 9a and $\Sigma_{02}(p)$ is the sum of the diagrams of Fig. 9b. From the expressions for $\Sigma_{11}(p)$ and $\Sigma_{02}(p)$, at $p \ll 1/R$ it is seen that

$$\Sigma_{11}(0) - \Sigma_{02}(0) = n_0 U(0) + 2in_0 \int [G_1'(q) G_1'(-q) - \hat{G}_1(q) \hat{G}_1(-q)] \times U^2(q) \frac{d^4q}{(2\pi)^4} + i\int G_1'(q) U(q) \frac{d^4q}{(2\pi)^4} - i\int \hat{G}_1(q) U(q) \frac{d^4q}{(2\pi)^4}.$$

It is also easy to see that

$$G_1'(q) G_1'(-q) - \hat{G}_1(q) G_1(-q) = \frac{1}{\omega^2 - \varepsilon_1^2(q) + i\delta}$$

Therefore, we obtain

$$\Sigma_{11}(0) - \Sigma_{02}(0) = n_0 U(0) + i\int [G_1'(q) + \hat{G}_1(q)] U(q) \frac{d^4q}{(2\pi)^4}.$$

Thus, $\Sigma_{11}(0) - \Sigma_{02}(0) = \mu$, which is the well-known Hugenholtz Pines relationship.[5] It follows from this relation that the spectrum of elementary excitations is a phonon spectrum at small momenta (Fig. 10).

For different ranges of the momentum p, the spectrum of the elementary excitations has the form

$$\varepsilon(p) = p(\Sigma_{02}(0))^{1/2} = cp \quad \text{for } p \ll 1/R,$$
$$\varepsilon(p) = p(p^2/4 + n_0 U_{\text{eff}}(p))^{1/2} \quad \text{for } p \gtrsim 1/R,$$
$$\varepsilon(p) = p(n_0\gamma/\varkappa^2)^{1/2} \quad \text{for } 1/R \ll p \ll \varkappa.$$

It is not difficult to see that $n_{eq}\gamma/\kappa^2 \gg c^2$.

We calculate the density of supercondensate particles:

$$n = n_0 + i\int G'(q) \frac{d^4q}{(2\pi)^4}.$$

After integration over the frequency, we get

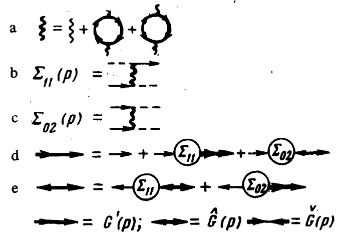

FIG. 8. Closed set of equations.

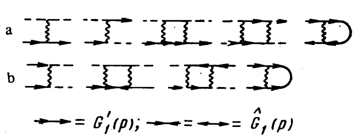

FIG. 9. Each solid line inside the loops of diagrams a and b should have two arrows. Where arrows are absent, they can be given either direction.

$$i\int G'(q)\frac{d^4q}{(2\pi)^4}=\int\frac{[-\varepsilon(q)+q^2/2+\Sigma_{11}(q)-\mu]}{2\varepsilon(q)}\frac{d^3q}{(2\pi)^3}.$$

The principal contribution to this integral is made by $q\sim(n_0\gamma)^{1/4}$, we therefore get

$$i\int G'(q)\frac{d^4q}{(2\pi)^4}=\frac{1}{(2\pi)^2}\int_0^\infty\frac{[-q^2(q^4/4+n_0\gamma)^{1/2}+q^4/2+n_0\gamma]dq}{(q^4/4+n_0\gamma)^{1/2}}$$

$$=(n_0\gamma)^{3/4}\frac{1}{4\pi^2}\int_0^\infty\frac{[\xi^2/2-(\xi^4/4+1)^{1/2}]^2d\xi}{(\xi^4/4+1)^{1/2}}$$

where $\xi=q/(n_0\gamma)^{1/4}$. Thus,

$$n-n_0=(n_0\gamma)^{3/4}b_1,$$

where $b_1\sim1$. It is seen from this expression that the number of supercondensate particles is small: $n-n_0\ll n_0$.

CONCLUSION

A general property of Bose systems with attraction is instability of the state of the system at densities $n<n_{cr}$. In a system with the pair-interaction potential model considered here,

$$n_{cr}\sim n_{eq}\sim\gamma^{1/3}/\beta^{4/3}\gg1.$$

This means that $n_{eq}\gg1$, i.e., there are many particles on the repulsive part of the pair interaction potential. It is understood that this is far from the real situation, being connected with the choice of the relation between the parameters of the attractive and repulsive parts of the interaction potential. Such a high density for a pressure equal to zero is obtained in connection with the fact that the repulsive part of the pair interaction potential is chosen to be soft.

It is seen from the expressions for n_{eq} that as $\beta\to0$ the density $n_{eq}\to\infty$, which means collapse of the system.

In the described solution, the Gaussian form of the attractive part of the interaction potential has not been used directly. Use is made only of the fact that the Fourier transform of the attractive part of the interaction potential is different from zero for sufficiently small momenta, which is valid for any sufficiently smoothly changing attraction with an effective radius greater than R_0.

The results obtained show that there is no manifestation of a bound state of a pair of particles. This is because a strong self-consistent field exists in the system; this appear in the large value of the chemical potential. The high density of the system leads to a renormalization of the potential, and the renormalized potential no longer leads to the appearance of a bound state (Fig. 2b).

In the coordinate representation, the attractive part of the pair interaction potential will have the form $U_0\exp[-(r/R)^2]$ and the attractive part of the renormalized potential will be

$$\frac{\gamma}{\varkappa^2R^3}\exp\left[-\left(\frac{r}{R}\right)^2\right]=U_{eff}^{(0)}\exp\left[-\left(\frac{r}{R}\right)^2\right],$$

where U_0 is the effective well depth in the pair interaction potential, and $U_{eff}^{(0)}$ is the effective well depth in the renormalized potential. In order to determine whether a bound state exists in the renormalized potential, we consider the quantity $U_{eff}^{(0)}R^2$:

$$U_{eff}^{(0)}R^2\sim\frac{\gamma R^2}{\varkappa^2R^3}\sim\frac{\gamma}{\varkappa^2R}\sim\frac{1}{(n_0R^3)^{1/3}}\ll1.$$

Thus it is seen that the Born criterion is satisfied for the well of the renormalized potential, i.e., the level certainly does not exist in them.

In conclusion, I consider it my duty to thank L. V. Keldysh for setting up the problem, discussion of the results and valuable advice.

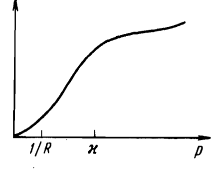

FIG. 10. Spectrum of elementary excitation.

APPENDIX

Let us show that the sum of diagrams of Figs. 7b, c is much smaller than the sum of the diagrams of Fig. 7a. For this purpose, we shall show that diagrams of the form of Fig. 11a give a contribution of order μ. With the exception of the first, diagrams of the form of Fig. 9b are principal diagrams of this type. Let us estimate their sum. It is not difficult to see that the sum is equal to

$$in_0\int\frac{d^4q}{(2\pi)^4}\{G_1'(q)+\hat{G}_1(q)+4G_1'(-q)\hat{G}_1(q)+3[\hat{G}_1(q)]^2$$
$$+[\hat{G}_1(-q)]^2\}U^2(q)+i\int\hat{G}_1U(q)\sim n_0\int q^2\,dq\,U^2(q)\frac{k(\omega,q)}{\varepsilon_1^3(q)},$$

where $\omega=-\epsilon_1(q)$. After several transformations, it is easy to show that the estimate of the integral gives a sum of diagrams of order

$$k(\omega,q)=2\left(\frac{q^2}{2}\right)^2+2\mu^2-\omega q^2+\omega\mu-2\mu q^2.$$

The fundamental contribution to the estimated integral is made by the momenta $q\sim(n_0\gamma)^{1/4}$. And, since for

$$n_0\gamma^2(\gamma n_0)^{1/2}(\gamma n_0)^{1/4}/(\gamma n_0)^{3/2}\sim\mu.$$

it is clear that these diagrams made a contribution that is much smaller than that of the diagram of the form of Fig. 11b.

Graphically, the sum of the principal diagrams of the form of Fig. 7b will have the form of the diagram of Fig. 11c, where the solid lines inside the loop correspond to the functions G_1', $\hat{G}_1$. We shall show that the sum of such diagrams makes a contribution much smaller than $n_0U_{eff}(p)$. The estimate of this sum for an external momentum p equal to zero is

$$1/R\leqslant p\leqslant\varkappa,\quad n_0U_{eff}\sim n_0\gamma/\varkappa^2\gg\mu,$$

The fundamental contribution to the estimated integral is made by the momenta $q\sim(n_0\gamma)^{1/4}$. It is understood that the integral will depend weakly on the external momentum p for $p\ll(n_0\gamma)^{1/4}$. Thus the diagrams considered will make a contribution of the order of

$$2in_0U_{eff}(0)\int\frac{d^4q}{(2\pi)^4}U(q)[G_1'(q)+\hat{G}_1(q)]^2\sim n_0U_{eff}(0)\frac{\gamma}{(\gamma n_0)^{1/4}}.$$

It is evident that the sum of diagrams of the form of Fig. 7c will also make a contribution that is much less than $n_0U_{eff}(p)$.

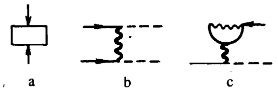

FIG. 11.

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Translated by R. T. Beyer
68