Contribution to the theory of local hydromagnetic stability of toroidal plasma configurations

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It is pointed out that despite the abundance of investigations of hydromagnetic stability of toroidal plasma configuration, the problem still remains unsolved. In particular, the meaning of the general geometric Mercier criterion, which plays an important role in modern calculations of stability of concrete systems, remains unclear. A general theory of local instability is developed in the present paper, from which it follows that the Mercier criterion is a necessary and sufficient condition for plasma stability with respect to a particular class of local perturbations, such as those for which the radial wavelength is small with respect to the wavelength along the small azimuth. Stability of a plasma confined by Shafranov-type configurations is also investigated with respect to local perturbations not described by the general geometric Mercier criterion. It is shown that the necessary and sufficient condition for local stability of such configurations is equivalent to the Mercier criterion simplified for the case of these configurations.

1. INTRODUCTION

In the theory of hydromagnetic instability of toroidal plasma configurations one investigates perturbations of two types, local perturbations concentrated near some magnetic surface, and nonlocal ones which affect the entire plasma as a whole. The reasons for the growth of the perturbations of both types are different. The cause of the growth of the local perturbations is the bending of the magnetic-field force lines and the associated magnetic drift of the particles. The instabilities that arise in this case are of the Rayleigh-Taylor type. The localizability of such perturbations is due to the shear of the magnetic field. The cause of the growth of the nonlocal perturbations is the current flowing along the plasma column. The growth of the nonlocal perturbations reflects the known tendency of a linear liquid current-carrying conductor to assume a helical shape.

Local perturbations are more sensitive to magnetic-field configuration details than nonlocal perturbations, so that the study of local perturbations occupies a principal place in the theory of stability of toroidal traps. The main problem in the theory of local hydromagnetic perturbations is to determine the conditions (criteria) for the stability of the plasma relative to such perturbations. In spite of the large number of papers devoted to the determination of such criteria, this process is still not quite clear. In particular, there are still different interpretations of the same stability criterion. An example is the interpretation of the general-geometric criterion of local stability, first derived by Mercier[1], and then obtained by different methods by Bineau[2], Greene and Johnson[3], and Solov'ev[4,5]. Mercier[1] and Bineau[2] called this criterion the necessary criterion of local stability, while Greene and Johnson[3] considered it to be necessary and sufficient. Solov'ev[5] calls it the necessary and sufficient criterion for local stability, while in[6] it is called the necessary criterion of local stability.

The lack of understanding of the meaning of the stability criteria can have far reaching consequences and in final analysis can lead to an incorrect estimate of the capabilities of various experimental devices. One of the purposes of the present paper is to introduce clarity in this process.

We shall show that Mercier's above-mentioned general-geometric criterion, which plays an important role in modern calculations of the stability of concrete systems, is the necessary and sufficient criterion for the stability of a plasma against a particular class of local perturbations, those whose radial wavelength is small in comparison with the wavelength along the minor azimuth. In the case of all other local perturbations, as will be shown below, Mercier's general-geometric criterion does not hold. For this reason, the Mercier criterion is a necessary criterion for local stability.

Is there a general-geometric necessary and sufficient local-stability criterion analogous to the Mercier criterion? The analysis that follows shows that such a criterion exists only in the trivial case of a plasma of very low pressure, when the perturbation of the magnetic field that localizes the magnetic surface is negligibly small. The plasma perturbations have in this case a flute-like character (the plasma displacement is constant along the force line), and the local stability, as was well known even before, is determined by the magnetic shear and by the average magnetic well. On the other hand, if the plasma pressure is not too low, when the magnetic-field perturbation plays an important role, the connection between the perturbation of the field and the displacement is nonlocal (in spite of the localized character of the perturbation near the corresponding magnetic surface). It is therefore impossible to obtain a necessary and sufficient criterion of local stability without additional assumptions concerning the properties of the magnetic configuration and the structure of the perturbation. The nonlocal character of the connection between the perturbed field and displacement is immaterial for the case of perturbations with radial wavelength smaller than the azimuthal length. This is precisely why Mercier was able to derive his general-geometric criterion[1]. As shown in the present paper, the necessary and sufficient criterion of local stability of the plasma can be obtained even in the case of nonlocal connection between the perturbation of the magnetic field and the cross section, if the equilibrium configuration satisfies certain additional conditions.

This problem can be solved for a rather extensive class of toroidal devices, including the principal modifications of Tokamaks and stellarators. We have in mind the configurations investigated in detail by Shafranov et al. in[7-11] and henceforth called configurations of the Shafranov type. Shafranov investigated also the local stability of such configurations on the basis of the Mercier criterion[7,9,10]. From the foregoing analysis it follows that the necessary and sufficient criterion obtained by us for local stability of Shafranov-type configurations is equivalent to a Mercier criterion simplified for the case of such configurations.

The very same configurations are sometimes investigated on the basis of sufficient stability criteria, and particularly on the basis of the Solov'ev sufficient criterion[5]. The result of such an investigation are shown in[6,12]. The conditions obtained in[6,12] for plasma stability in such configurations are always more stringent than is necessary and sufficient for local stability.

2. INITIAL EQUATIONS AND THEIR SIMPLIFICATION FOR THE CASE OF SMALL-CASE ALMOST-FLUTE-LIKE PERTURBATIONS

1. Equilibrium state. We assume that the force lines of the stationary magnetic field $\mathbf{B}_0$ form a system of imbedded magnetic surfaces. It is possible to introduce here, as was frequently done earlier (see, for example, [11]), a system of curvilinear coordinates $x^1 = a$, $x^2 = \theta$, and $x^3 = \zeta$, such that the coordinate a characterizes the distance from the magnetic axis to the corresponding magnetic surface, while θ and ζ characterize the angular coordinates on the surface a = const, the period of variation of which we shall assume equal to 2π. It is known[11] that the coordinates θ and ζ can be chosen such that the force lines of the field $\mathbf{B}_0$ are "straight." The contravariant components of $\mathbf{B}_0$ are then

$$B_0^1=0,\quad B_0^2=\chi'/2\pi\sqrt{g},\quad B_0^3=\Phi'/2\pi\sqrt{g},\tag{2.1}$$

where g is the determinant of the metric tensor g_{ik} of the coordinate system (a, θ, ζ), while $\chi'(a)$ and $\Phi'(a)$ are the derivatives with respect to a of the transverse (along θ) and longitudinal (along ζ) magnetic fluxes.

Assuming that the equilibrium state of the plasma is described by the equation

$$-\nabla p_0+[\mathbf{j}_0\mathbf{B}_0]=0,\tag{2.2)*$$

where $p_0 = p_0(a)$ and $\mathbf{j}_0$ are the equilibrium pressure and the equilibrium current in the plasma, and taking (2.1) into account, we conclude that $j_0^1 = 0$. The two other contravariant components of $\mathbf{j}_0$, which are connected by the condition div $\mathbf{j}_0 = 0$, can be represented in the form

$$j_0^2=\left(I'-\frac{\partial\nu}{\partial\zeta}\right)\Big/2\pi\sqrt{g},\qquad j_0^3=\left(J'+\frac{\partial\nu}{\partial\theta}\right)\Big/2\pi\sqrt{g}.\tag{2.3}$$

The functions I′(a) and J′(a) have the meaning of the derivatives, with respect to a, of the transverse and longitudinal currents flowing between the magnetic surfaces. As follows from (2.2) these quantities are connected with the pressure gradient by the relation

$$p_0'V'=I'\Phi'-J'\chi',\qquad V'=\int_0^{2\pi}\int_0^{2\pi}\sqrt{g}\,d\theta\,d\zeta,\tag{2.4}$$

where V′ is the derivative of the volume bounded by the corresponding magnetic surface with respect to a. Equation (2.2) leads also to an equation for the function ν:

$$\mathbf{B}_0\nabla\nu=2\pi p_0'(V'/4\pi^2\sqrt{g}-1).\tag{2.5}$$

Maxwell's equations $\mathbf{j}_0$ = curl $\mathbf{B}_0$ yield two additional relations between the currents I, J and the magnetic fluxes Φ, χ:

$$J=\left(\frac{g_{22}}{\sqrt{g}}\right)^{(0)}\chi'+\left(\frac{g_{23}}{\sqrt{g}}\right)^{(0)}\Phi',\qquad -I=\left(\frac{g_{23}}{\sqrt{g}}\right)^{(0)}\chi'+\left(\frac{g_{33}}{\sqrt{g}}\right)^{(0)}\Phi'.\tag{2.6}$$

The superscript 0 indicates here and below the part of the corresponding quantity averaged over θ and ζ:

$$(\cdots)^{(0)}=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(\cdots)\,d\theta\,d\zeta.\tag{2.7}$$

The equilibrium current $\mathbf{j}_0$ consists of two physically different parts, one along $\mathbf{B}_0$ and the other across $\mathbf{B}_0$,

$$\mathbf{j}_0=\alpha_0\mathbf{B}_0+\mathbf{j}_{0\perp},\tag{2.8}$$

where $\alpha_0 \equiv \mathbf{j}_0\cdot\mathbf{B}_0/B_0^2$, and according to (2.2) $\mathbf{j}_0$ is equal to

$$\mathbf{j}_{0\perp}=[\mathbf{B}_0/B_0^2,\nabla p_0].\tag{2.9}$$

It follows from the condition div $\mathbf{j}_0 = 0$ that the function α_0 satisfies a relation analogous to (2.5):

$$\mathbf{B}_0\nabla\alpha_0=-p_0'\,\mathrm{rot}^1(\mathbf{B}_0/B_0^2).\tag{2.10}$$

2. Initial equations for small-scale almost-flute-like perturbations. In the presence of perturbations, the magnetic field B, the current j, and the pressure p differ from their equilibrium values by small amounts which will henceforth be denoted $\tilde{\mathbf{B}}$, $\tilde{\mathbf{j}}$, and $\tilde{p}$, respectively. In other words, in the presence of perturbations we have

$$\mathbf{B}=\mathbf{B}_0+\tilde{\mathbf{B}},\quad \mathbf{j}=\mathbf{j}_0+\tilde{\mathbf{j}},\quad p=p_0+\tilde{p}.$$

Instead of $\tilde{p}$ we shall use the perturbed displacement ξ, which is defined by the relation $\tilde{p} = -p_0'\xi$. In the approximation of ideal magnetohydrodynamics with scalar pressure, and at zero perturbation frequency ($\partial/\partial t = 0$), the quantities B, j, and p satisfy the equations

$$\mathrm{rot}\,\mathbf{B}=\mathbf{j},\quad \mathrm{div}\,\mathbf{B}=0,\quad -\nabla p+[\mathbf{jB}]=0.\tag{2.11}$$

Putting $\mathbf{j} = \alpha\mathbf{B} + \mathbf{j}_\perp$, where $\alpha \equiv \mathbf{j}\cdot\mathbf{B}/B^2$ and $\mathbf{j}_\perp = [\mathbf{B}/B^2\times\nabla p]$, we obtain from (2.11) in the approximation linear in the perturbations

$$\mathbf{B}_0\nabla\tilde{\alpha}+\tilde{\mathbf{B}}\nabla\alpha_0+\mathrm{div}\,\tilde{\mathbf{j}}_\perp=0.\tag{2.12}$$

We simplify (2.12) by assuming the perturbations to be almost flute-like along $\mathbf{B}_0$:

$$\mathbf{B}_0\nabla\tilde{X}\ll|[\mathbf{B}_0,\nabla\tilde{X}]|,\tag{2.13}$$

and to be of small-scale across $\mathbf{B}_0$:

$$[\mathbf{B}_0,\nabla\ln\tilde{X}]\gg[\mathbf{B}_0,\nabla\ln X_0],\tag{2.14}$$

where $\tilde{X}$ and X_0 denote respectively any one of the perturbed and equilibrium values.

We start with the simplification of div $\mathbf{j}_\perp$. By definition,

$$\mathrm{div}\,\tilde{\mathbf{j}}_\perp=-p_0'\left\{\nabla\xi\,\mathrm{rot}\frac{\mathbf{B}_0}{B_0^2}-\mathrm{rot}^1\left(\frac{\tilde{\mathbf{B}}}{B_0^2}-\frac{2\mathbf{B}_0\tilde{\mathbf{B}}}{B_0^4}\mathbf{B}_0\right)\right\}.\tag{2.15}$$

We note that under the conditions (2.13) and (2.14)

$$\mathrm{rot}^1\tilde{\mathbf{X}}=\frac{2\pi}{\Phi'}\frac{\partial}{\partial\theta}(\mathbf{B}_0\tilde{\mathbf{X}}).\tag{2.16}$$

Therefore

$$\mathrm{rot}^1\left(\frac{\tilde{\mathbf{B}}}{B_0^2}-\frac{2\mathbf{B}_0\tilde{\mathbf{B}}}{B_0^4}\mathbf{B}_0\right)=-\frac{2\pi}{\Phi'B_0^2}\frac{\partial}{\partial\theta}(\mathbf{B}_0\tilde{\mathbf{B}}).\tag{2.17}$$

We now express the right-hand side of (2.17) in terms of ξ. To this end, we start with an equation for $\tilde{j}^1$ in the form

$$\tilde{j}^1 = -\frac{p_0'}{\sqrt{g}}\left(\frac{B_{02}}{\mathbf{B}_0^2}\frac{\partial \tilde{p}}{\partial \zeta} - \frac{B_{03}}{\mathbf{B}_0^2}\frac{\partial \tilde{p}}{\partial \theta}\right) - \alpha_0 \tilde{B}^1. \tag{2.18}$$

According to (2.11),

$$\tilde{B}^1 = \mathbf{B}_0 \nabla \xi. \tag{2.19}$$

On the basis of the assumption (2.13), we neglect in the right-hand side of (2.18) the derivatives along the line of force in comparison with the transverse derivatives. In this approximation we have

$$\tilde{j}^1 = \frac{2\pi}{\Phi'}\frac{p_0'}{\sqrt{g}}\frac{\partial \xi}{\partial \theta}.$$

Substituting this expression in the equation $\mathrm{curl}^1\, \tilde{\mathbf{B}} = \tilde{j}^1$, we obtain

$$\partial(\mathbf{B}_0\tilde{\mathbf{B}}) / \partial\theta = p_0' \partial\xi / \partial\theta. \tag{2.20}$$

Using (2.17) and (2.20), the entire right-hand side of (2.15) is expressed in terms of ξ.

We simplify next the first term in the curly brackets of (2.15). In the approximation (2.13), with allowance for the identity

$$B_0^2\frac{\partial}{\partial a}\frac{B_{02}}{\mathbf{B}_0^2} + B_0^3\frac{\partial}{\partial a}\frac{B_{03}}{\mathbf{B}_0^2} \equiv -\frac{1}{\mathbf{B}_0^2}\left(B_{02}\frac{\partial B_0^2}{\partial a} + B_{03}\frac{\partial B_0^3}{\partial a}\right)$$

and relation (2.10), we obtain

$$\nabla\xi\,\mathrm{rot}\frac{\mathbf{B}_0}{\mathbf{B}_0^2} = -\frac{\partial\xi}{\partial a}\frac{\mathbf{B}_0\nabla\alpha_0}{p_0'} + \frac{2\pi}{\Phi'}\frac{\partial\xi}{\partial\theta}\left\{\frac{1}{\mathbf{B}_0^2}\left(B_{02}\frac{\partial B_0^2}{\partial a} + B_{03}\frac{\partial B_0^3}{\partial a}\right) + \mathbf{B}_0\nabla\frac{B_{01}}{\mathbf{B}_0^2}\right\}. \tag{2.21}$$

Using (2.5), we get

$$\frac{1}{\mathbf{B}_0^2}\left(B_{02}\frac{\partial B_0^2}{\partial a} + B_{03}\frac{\partial B_0^3}{\partial a}\right) = -\frac{1}{(2\pi)^2\sqrt{g}\,p_0'}(\Omega - \mu'\Phi'^2\alpha_0) - \frac{1}{2\pi}\mathbf{B}_0\nabla\left(\frac{\nu}{p_0'}\right)', \tag{2.22}$$

where

$$\Omega = p_0'V'' + J'\chi'' - I'\Phi'', \quad \mu = \chi' / \Phi'. \tag{2.23}$$

Taking (2.17) and (2.20)–(2.22) into account, we express (2.15) in the form

$$\mathrm{div}\,\tilde{\mathbf{j}}_\perp = -\frac{2\pi}{\Phi'}\frac{\partial\xi}{\partial\theta}\left[\frac{p_0'^2}{\mathbf{B}_0^2} - \frac{1}{(2\pi)^2\sqrt{g}}(\Omega - \mu'\Phi'^2\alpha_0)\right] + \frac{\partial\xi}{\partial a}\mathbf{B}_0\nabla\alpha_0 - \frac{\partial\xi}{\partial\theta}\frac{p_0'}{\Phi'}\mathbf{B}_0\nabla\left[\left(\frac{\nu}{p_0'}\right)' + \frac{2\pi B_{01}}{\mathbf{B}_0^2}\right]. \tag{2.24}$$

We now simplify the expression for $\tilde{\alpha}$. By definition we have

$$\tilde{\alpha} = \frac{\mathbf{B}_0}{\mathbf{B}_0^2}\mathrm{rot}\,\tilde{\mathbf{B}} + \left(\frac{\tilde{\mathbf{B}}}{\mathbf{B}_0^2} - \frac{2\mathbf{B}_0(\mathbf{B}_0\tilde{\mathbf{B}})}{\mathbf{B}_0^4}\right)\mathrm{rot}\,\mathbf{B}_0.$$

The first term in the right-hand side contains derivatives of $\tilde{\mathbf{B}}$ in directions transverse to the lines of force of $\mathbf{B}_0$. There are no such derivatives in the second term. This term is small in comparison with the first by virtue of (2.14). We therefore have approximately

$$\tilde{\alpha} = (\mathbf{B}_0 / \mathbf{B}_0^2)\,\mathrm{rot}\,\tilde{\mathbf{B}}. \tag{2.25}$$

Expressing the right-hand side of (2.25) in terms of its components and using the conditions (2.13) and (2.14), we reduce (2.25) to the form

$$\tilde{\alpha} = \frac{\Phi'}{2\pi g\mathbf{B}_0^2}\left[(\mu g_{23} + g_{33})\frac{\partial\tilde{B}_2}{\partial a} - (\mu g_{22} + g_{32})\frac{\partial\tilde{B}_3}{\partial a} - \frac{4\pi^2 g\mathbf{B}_0^2}{\Phi'^2}\frac{\partial\tilde{B}_1}{\partial\theta} + (\mu g_{21} + g_{31})\frac{\partial}{\partial\theta}(\tilde{B}_3 + \mu\tilde{B}_2)\right]. \tag{2.26}$$

We express the quantities $\tilde{B}_\alpha$ in terms of ξ and $\tilde{B}_1$. To this end we express $\tilde{B}_\alpha$ first in terms of $\tilde{B}^\beta$, namely $\tilde{B}_\alpha = g_{\alpha\beta}\tilde{B}^\beta$ ($\alpha, \beta = 1, 2, 3$). In addition, we use the condition $\mathrm{div}\,\tilde{\mathbf{B}} = 0$, which means in the required approximation

$$\partial(\tilde{B}^2 - \mu\tilde{B}^3) / \partial\theta = -\partial\tilde{B}^1 / \partial a,$$

and Eq. (2.20), which denotes in explicit form

$$(\mu g_{22} + g_{32})\tilde{B}^2 + (\mu g_{23} + g_{33})\tilde{B}^3 = -(\mu g_{21} + g_{31})\tilde{B}^1 + 2\pi p_0'\xi / \Phi'\sqrt{g}.$$

As a result we get from (2.26)

$$\tilde{\alpha} = 2\pi\sqrt{g}\hat{L}_\perp\tilde{\psi} / \Phi', \tag{2.27}$$

where $\tilde{\psi} = \int \tilde{B}^1 d\theta$,

$$\hat{L}_\perp = -\left(\frac{\Phi'}{2\pi}\right)^2\frac{1}{\sqrt{g}B_0^2}\left(g^{11}\frac{\partial^2}{\partial a^2} - 2g^{1b}\frac{\partial^2}{\partial a\,\partial\theta} + g^{bb}\frac{\partial^2}{\partial\theta^2}\right) \tag{2.28}$$

$$g^{1b} \equiv g^{12} + \mu g^{13}, \quad g^{bb} \equiv g^{22} + 2\mu g^{23} + \mu^2 g^{33}, \tag{2.29}$$

$g^{ik} \equiv g^{-1}m_{ik}$ and M_{ik} is the minor of g_{ik}.

We see that $\tilde{\alpha}$ in (2.12) is proportional to the second derivatives of the perturbed quantities transverse to $\mathbf{B}_0$. There are no such large terms in the second term of the left-hand side of (2.12). The small-oscillation equation therefore reduces to

$$\mathbf{B}_0\nabla\tilde{\alpha} + \mathrm{div}\,\mathbf{j}_\perp = 0 \tag{2.30}$$

with $\mathrm{div}\,\tilde{\mathbf{j}}_\perp$ and $\tilde{\alpha}$ as defined by formulas (2.24) and (2.27).

3. EQUATION FOR THE FLUTE PART OF THE PERTURBED DISPLACEMENT

We represent the perturbed quantities in the form

$$\tilde{X} = e^{i(m\theta - n\zeta)}[\tilde{X}^{(0)}(a) + \tilde{X}^{(1)}(a, \theta, \zeta)],$$

where m and n are integers such that $n/m = \mu(a_0)$, and a_0 is the coordinate a of a certain rational magnetic surface. We put $|\tilde{X}^{(1)}| \ll |\tilde{X}^{(0)}|$, which corresponds to the assumption (2.13) that the perturbations are almost flute-like. Under this condition we have

$$\mathbf{B}_0\nabla\tilde{X} = \frac{\Phi'}{2\pi\sqrt{g}}e^{i(m\theta - n\zeta)}(im\mu' x\tilde{X}^{(0)} + \hat{L}\,\tilde{X}^{(1)})$$

where $\hat{L}_\parallel \equiv \mu\partial/\partial\theta + \partial/\partial\zeta$ and $x = a - a_0$.

In place of the functions $\tilde{X}^{(0)}(x)$ and $\tilde{X}^{(1)}(x)$ we shall deal with their Fourier components $\tilde{X}_k^{(0)}$ and $\tilde{X}_k^{(1)}$, defined by the relation

$$\tilde{X}(x) = \int e^{ik_x x}\tilde{X}_k\, dk_x.$$

In this case

$$(\mathbf{B}_0\nabla\tilde{X})_k = \frac{\Phi'}{2\pi\sqrt{g}}e^{i(m\theta - n\zeta)}\left(-m\mu'\frac{\partial\tilde{X}_k^{(0)}}{\partial k_x} + \hat{L}_\parallel\tilde{X}_k^{(1)}\right).$$

In the k_x representation, Eq. (2.30) takes the form

$$-m\mu'\partial\tilde{\alpha}_k^{(0)}/\partial k_x + \hat{L}_\parallel\tilde{\alpha}_k^{(1)} + im\xi_k W^{(0)} + \xi_k\hat{L}_\parallel A = 0, \tag{3.1}$$

where $\xi_k \equiv \xi_k^{(0)} + \xi_k^{(1)}$,

$$W^{(0)} = \frac{\Omega}{\Phi'^2} - \mu'\alpha_0^{(0)} - \left(\frac{2\pi p_0'}{\Phi'^2}\right)^2\left(\frac{\sqrt{g}}{\mathbf{B}_0^2}\right)^{(0)}; \quad A = i(k_x\alpha_0^{(1)} + m\beta_0^{(1)}), \tag{3.2}$$

$$\beta_0^{(1)} = \frac{p_0'}{\Phi'}\left[\left(\frac{\nu}{p_0'}\right)' + \frac{2\pi B_{01}}{\mathbf{B}_0^2}\right] + \frac{\mu'}{\Phi'^2}\hat{L}_\parallel^{-1}\alpha_0^{(1)} + (2\pi p_0')^2\hat{L}_\parallel^{-1}\left(\frac{\sqrt{g}}{\mathbf{B}_0^2}\right)^{(1)}. \tag{3.3}$$

It follows from (2.27) that

$$\tilde{\alpha}_k = 2\pi\sqrt{g}L_\perp\tilde{\psi}_k / \Phi', \tag{3.4}$$

where $L_\perp$ is the eigenvalue of the operator $\hat{L}_\perp$ and corresponds to the k_x-th Fourier harmonic:

$$L_\perp = \left(\frac{\Phi'}{2\pi}\right)^2 \frac{1}{\sqrt{g}\,B_0^2}(k_x^2 g^{11} - 2mk_x g^{1b} + m^2 g^{bb}), \qquad (3.5)$$

and the function $\tilde{\psi}_k$, as follows from the definition of $\tilde{\psi}$ and from (2.19), is connected with ξ_k by the relation

$$\tilde{\psi}_k = -\frac{i}{m}\frac{\Phi'}{2\pi\sqrt{g}}\left(-m\mu'\frac{\partial \xi_k^{(0)}}{\partial k_x} + \hat{L}_\parallel \xi_k^{(1)}\right). \qquad (3.6)$$

The part of (3.1) averaged over θ and ζ is

$$-m\mu'\frac{\partial \tilde{\alpha}_k^{(0)}}{\partial k_x} + im\xi_k^{(0)} W^{(0)} - (A\hat{L}_\parallel \xi_k^{(1)})^{(0)} = 0. \qquad (3.7)$$

The quantity $\tilde{\alpha}_k^{(0)}$ is determined from relations (3.4)–(3.6):

$$\tilde{\alpha}_k^{(0)} = -\frac{i}{m}\left[-m\mu'\frac{\partial \xi_k^{(0)}}{\partial k_x} L_\perp^{(0)} + (L_\perp^{(1)} \hat{L}_\parallel \xi_k^{(1)})^{(0)}\right], \qquad (3.8)$$

where $L_\perp^{(0)}$ is the part of $L_\perp$ averaged over θ and ζ, while $L_\perp^{(1)} = L_\perp - L_\perp^{(0)}$. The expression for $\hat{L}_\parallel \xi_k^{(1)}$, which enters in the right-hand side of (3.8) and also in the last term of the left-hand side of (3.7), is determined by the part of (3.1) which oscillates with respect to θ and ζ. This part of (3.1) reduces, under our assumption that $\xi^{(1)}/\xi^{(0)}$ is small, to the form

$$\tilde{\alpha}_k^{(1)} = -\xi_k^{(0)} A. \qquad (3.9)$$

We can substitute here the value of $\tilde{\alpha}_k^{(1)}$, which is determined by the part of (3.8) which oscillates with respect to θ and ζ:

$$\tilde{\alpha}_k^{(1)} = i\mu' L_\perp^{(1)}\frac{\partial \xi_k^{(0)}}{\partial k_x} - \frac{i}{m}[L_\perp \hat{L}_\parallel \xi_k^{(1)} - (L_\perp \hat{L}_\parallel \xi_k^{(1)})^{(0)}]. \qquad (3.10)$$

From (3.9) and (3.10) we get

$$\hat{L}_\parallel \xi_k^{(1)} = \frac{1}{L_\perp}\left[(L_\perp \hat{L}_\parallel \xi_k^{(1)})^{(0)} + m\mu'\frac{L_\perp^{(1)}}{L_\perp}\frac{\partial \xi_k^{(0)}}{\partial k_x} - im\xi_k^{(0)} A\right]. \qquad (3.11)$$

The left-hand side of this equation averaged over θ and ζ vanishes. A similar condition should be satisfied also by the right-hand side of (3.11). Therefore

$$(L_\perp \hat{L}_\parallel \xi_k^{(1)})^{(0)} = \frac{m}{Q^{(0)}}\left[-\mu'\frac{\partial \xi_k^{(0)}}{\partial k_x}(1 - L_\perp^{(0)} Q^{(0)}) + i\xi_k^{(0)}(QA)^{(0)}\right], \qquad (3.12)$$

where $Q \equiv L_\perp^{-1}$. Taking (3.12) into account, we obtain from (3.11) an expression for $\hat{L}_\parallel \xi_k^{(1)}$ as a function of $\xi_k^{(0)}$:

$$\hat{L}_\parallel \xi_k^{(1)} = m\left\{\mu'\frac{\partial \xi_k^{(0)}}{\partial k_x}\left(1 - \frac{Q}{Q^{(0)}}\right) - i\xi_k^{(0)}\left[QA - \frac{Q}{Q^{(0)}}(QA)^{(0)}\right]\right\}. \qquad (3.13)$$

Using (3.8), (3.12), and (3.13), we obtain from (3.7) a final expression for $\xi_k^{(0)}$:

$$\mu'^2\frac{\partial}{\partial k_x}\left(\frac{1}{Q^{(0)}}\frac{\partial \xi_k^{(0)}}{\partial k_x}\right) - \xi_k^{(0)}\left\{W^{(0)} + i\mu'\frac{\partial}{\partial k_x}\left[\frac{(QA)^{(0)}}{Q^{(0)}}\right] + (QA^2)^{(0)} - \frac{[(QA)^{(0)}]^2}{Q^{(0)}}\right\} = 0. \qquad (3.14)$$

From (3.13), we get the limits of applicability of our approximation $\xi^{(1)}/\xi^{(0)} \ll 1$:

$$\frac{mx}{a}\frac{a\mu'}{\mu} \ll 1, \quad \frac{mx}{a} ak \ll 1, \qquad (3.15)$$

where x is the characteristic dimension of the localization of the perturbation, a is the characteristic dimension of the plasma, and k is the curvature of the magnetic field.

We see that for (3.14) to be valid in the general case of perturbations with $mx \approx a$, we must assume $a\mu'/\mu \ll 1$ and $ak \ll 1$.

4. CRITERION FOR THE STABILITY OF A PLASMA WITH RESPECT TO PERTURBATIONS WITH $m/k_x \to 0$

We consider a perturbation with $m/k_x \to 0$, but continue nevertheless to assume that $m \gg 1$. It follows then from (3.3) and (3.5)

$$A = ik_x\alpha_0^{(1)}, \qquad (4.1)$$

$$Q = L_\perp^{-1} = \frac{1}{k_x^2}\left(\frac{2\pi V'}{\Phi'}\right)^2 \sqrt{g}\,\alpha_*, \qquad (4.2)$$

where

$$\alpha_* = B_0^2 / g^{11}V'^2 = B_0^2/|\nabla V|^2. \qquad (4.3)$$

In this case those quantities in (3.14) which are averaged over θ and ζ are

$$Q^{(0)} = \frac{V'^2}{k_x^2\Phi'^2}\langle\alpha_*\rangle, \quad \frac{(QA)^{(0)}}{Q^{(0)}} = ik_x\left(\frac{\langle\gamma_*\rangle}{\langle\alpha_*\rangle} - \alpha_0^{(0)}\right),$$
$$(QA^2)^{(0)} - \frac{[(QA)^{(0)}]^2}{Q^{(0)}} = \frac{V'^2}{\Phi'^2}\left(\frac{\langle\gamma_*\rangle^2}{\langle\alpha_*\rangle} - \langle\beta_*\rangle\right) + \frac{V'p_0'^2}{\Phi'^2}\left\langle\frac{1}{B_0^2}\right\rangle, \qquad (4.4)$$

where

$$\beta_* = j_0^2/|\nabla V|^2, \quad \gamma_* = j_0 B_0/|\nabla V|^2,$$
$$\langle(\ldots)\rangle = [\sqrt{g}(\ldots)]^{(0)}/(\sqrt{g})^{(0)}. \qquad (4.5)$$

The quantities α_*, β_*, and γ_* are the same as α, β, and γ in Shafranov's paper[9].

Substituting (4.4) in (3.14) and recalling the definition of $W^{(0)}$ (see (3.2)), we obtain

$$\mu'^2\frac{\partial}{\partial k_x}\left(k_x^2\frac{\partial \xi_k^{(0)}}{\partial k_x}\right) - \frac{V'^3}{\Phi'^2}\xi_k^{(0)}\left[\langle\alpha_*\rangle\frac{\Omega}{\Phi'^2} - \mu'\langle\gamma_*\rangle + \frac{V'^3}{\Phi'^2}(\langle\gamma_*\rangle^2 - \langle\alpha_*\rangle\langle\beta_*\rangle)\right] = 0. \qquad (4.6)$$

An equation of the type of (4.6) was investigated by Kadomtsev and Pogutse[13]. It has no solutions bounded in space if

$$\frac{\mu'^2}{4} + \frac{V'^3}{\Phi'^2}\left[\langle\alpha_*\rangle\frac{\Omega}{\Phi'^2} - \mu'\langle\gamma_*\rangle + \frac{V'^3}{\Phi'^2}(\langle\gamma_*\rangle^2 - \langle\alpha_*\rangle\langle\beta_*\rangle)\right] > 0, \qquad (4.7)$$

which, as is well known[13], is the necessary and sufficient condition for the stability of the plasma relative to the corresponding type of perturbations. The condition (4.7) is called the "Mercier stability criterion."

5. CRITERION FOR THE STABILITY OF SHAFRANOV-TYPE CONFIGURATIONS RELATIVE TO LOCAL PERTURBATIONS WITH ARBITRARY m/ak_x

Equation (3.14) enables us to investigate the stability of a plasma against local hydrodynamic perturbations with arbitrary m/k_x. Perturbations of this type were previously analyzed for the case of an axially-symmetrical Tokamak with round cross sections in the papers of Kadomtsev and Pogutse[13] and of Shafranov and Yurchenko[7]. We now consider toroidal configurations with elliptic cross sections of the magnetic surfaces, without assuming the system to have axial symmetry, but assume, just as in[9,10], that the curvature of the magnetic axis of the system is small and that the longitudinal magnetic field is large[1)].

We take the "radial" coordinate a to be the average radius of the cross section of the magnetic surface. In the case of an elliptic cross section of the magnetic surfaces and in the zeroth approximation in the curvature of the magnetic axis, this means that the coordinates a and θ are connected with the usual coordinates x and y by the relations $x = e^{-\eta/2} a \cos\theta$ and $y = e^{\eta/2} a \sin\theta$, where the parameter η characterizes the ellipticity and is defined by the relation $\epsilon = \tanh \epsilon$, where $\epsilon = (l_1^2 - l_2^2)/(l_1^2 + l_2^2)$ is the usual ellipticity parameter, and l_1 and l_2 are the major and minor semiaxes of the ellipse. From the presented relations between (x, y) and (a, θ) it follows that the components of the metric tensor g_{ik} with (i, k) = (1, 2) are given in the zeroth approximation in the curvature (with superscript 0) by

$$g_{11}{}^0 = \operatorname{ch}\eta - \operatorname{sh}\eta\cos 2\theta, \quad g_{12}{}^0 = g_{21}{}^0 = a\operatorname{sh}\eta\sin 2\theta,$$
$$g_{22}{}^0 = a^2(\operatorname{ch}\eta + \operatorname{sh}\eta\cos 2\theta). \tag{5.1}$$

In the case of axial symmetry (symmetry with respect to the coordinate ζ), the only nonzero element of g_{ik} besides those of (5.1) is g_{33}. The absence of axial symmetry entails the appearance of the metric cross coefficients $g_{\alpha 3}$ and $g_{3\alpha}$, $\alpha = 1, 2$. In accordance with[10] we have in the zeroth approximation in the curvature

$$g_{13}{}^0 = g_{31}{}^0 = 0, \quad g_{23}{}^0 = g_{32}{}^0 = -a^2\vartheta'(\zeta), \tag{5.2}$$

where $\vartheta'(\zeta) \equiv -\kappa(\zeta) + \delta'(\zeta)$, $\kappa(\zeta)$ is the torsion of the magnetic axis, and $\delta'(\zeta)$ is the angular velocity of the rotation of the coordinate system relative to the normal to the magnetic axis when moving along the magnetic axis. Just as in[10], we assume that $\vartheta'(\zeta)$ is constant.

To calculate the equilibrium and the stability it is necessary to take into account in the elements of g_{ik} the terms of first order of smallness in the curvature of the magnetic axis, and in the element g_{33} it is necessary to take into account also the second-order terms. It is not the purpose of this paper to calculate the equilibrium and the stability, but only to obtain a stability criterion without an exhaustive determination of all the elements g_{ik}. With respect to g_{33}, in particular, it suffices to know that the part of this element which does not depend on θ and ζ can be represented in the form

$$(g_{33})^{(0)} = R^2[1 + a^2 O(k^2)], \tag{5.3}$$

where $R = L/2\pi$, L is the length of the magnetic axis, and $O(k^2)$ are terms quadratic in the curvature of the magnetic axis $k(\zeta)$. In addition to the foregoing values of g_{ik}, we must also know $\sqrt{g}$ with allowance of terms of order $k(\zeta)$. An expression for $\sqrt{g}$, with accuracy sufficient for our purposes, has been derived in[10]:

$$\sqrt{g} = aR[1 - 2a(e^{-\eta/2}k\cos\delta\cos\theta + e^{\eta/2}k\sin\delta\sin\theta)]. \tag{5.4}$$

To calculate $Q \equiv L_\perp^{-1}$ (see (3.5)) it suffices to know g_{ik} in the zeroth approximation in ka. The square of the magnetic field V_0^2 can then be replaced by B_S^2, where $B_S = \Phi'/2\pi a$. Then

$$Q = aR / k_\perp^2 g_{22}{}^0(\theta - \lambda), \tag{5.5}$$

where $k_\perp^2 \equiv k_x^2 + (m/a)^2$, $\lambda = \tan^{-1}(m/ak_x)$, and $g_{22}^0(\theta - \lambda)$ denotes expression (5.1) for g_{22}^0, in which we put $\theta \to \theta - \lambda$.

The important terms in the calculation of A (see (3.3)) are those of order ka in the expression (5.4) for $\sqrt{g}$. The terms of higher order in ka can then be neglected. The approximate expression (3.3) for A then simplifies to

$$A = i\left[k_x\alpha_0{}^{(1)} - \frac{m}{\Phi'}p_0'\left(\frac{\nu}{p_0'}\right)'\right]. \tag{5.6}$$

From (2.3) we obtain

$$\alpha_0{}^{(1)} \equiv \left(\frac{\mathbf{j}_0\mathbf{B}_0}{\mathbf{B}_0{}^2}\right)^{(1)} = \frac{1}{\Phi'}\frac{\partial\nu}{\partial\theta}. \tag{5.7}$$

Recognizing that $p_0' \sim a$ and $\nu \sim a^2$, we find that $p_0'(\nu/p_0')' = \nu/a$. Therefore

$$A = \frac{ik_\perp}{\Phi'}\left(\frac{\partial\nu}{\partial\theta}\cos\lambda - \nu\sin\lambda\right). \tag{5.8}$$

Inasmuch as in accordance with (2.5) and (5.4) we have $\nu \sim \cos\theta, \sin\theta$, it follows from (5.8) that

$$A = \frac{ik_\perp}{\Phi'}\frac{\partial}{\partial\theta}\nu(\theta - \lambda). \tag{5.9}$$

With Q and A taking the forms (5.5) and (5.9) with g_{22}^0 as in (5.1), and with ν defined by (2.5) and (5.4), the mean values in (3.14) are

$$Q^{(0)} = R / k_\perp^2 a, \quad (QA)^{(0)} = 0, \quad (QA^2)^0 = -\lambda_{33}{}^{\theta\theta} / \Phi'^2, \tag{5.10}$$

where

$$\lambda_{33}{}^{\theta\theta} = aR\left[\frac{1}{g_{22}{}^0}\left(\frac{\partial\nu}{\partial\theta}\right)^2\right]^{(0)} \tag{5.11}$$

is a function that is used in[10]. Under the conditions (5.10), equation (3.14) signifies that

$$\mu'^2\frac{\partial}{\partial k_x}\left[\left(k_x^2 + \frac{m^2}{a^2}\right)\frac{\partial\xi_k^{(0)}}{\partial k_x}\right] - \xi_k^{(0)}\frac{R}{a}\left(W^{(0)} - \frac{\lambda_{33}{}^{\theta\theta}}{\Phi'^2}\right) = 0. \tag{5.12}$$

From this we get the stability criterion (cf.,[10]):

$$\frac{\mu'^2}{4} + \frac{R}{a}\left(W^{(0)} - \frac{\lambda_{33}{}^{\theta\theta}}{\Phi'^2}\right) > 0, \tag{5.13}$$

which reduces very small μ' to the condition

$$W^{(0)} - \lambda_{33}{}^{\theta\theta} / \Phi'^2 > 0. \tag{5.14}$$

We now obtain the explicit form of the function $W^{(0)}$. From (3.2) it follows that as $\mu' \to 0$

$$W^{(0)} = \frac{p_0'}{\Phi'^2}\left(V'' - \frac{\Phi''}{\Phi'}V' - p_0'V'\left\langle\frac{1}{\mathbf{B}_0{}^2}\right\rangle\right). \tag{5.15}$$

To find Φ''/Φ', we differentiate each of the equations in (2.6) with respect to a, after which the first result is multiplied by χ', the second by Φ', and the two are added. We then obtain, taking (2.4) into account,

$$\frac{\Phi''}{\Phi'} = -\frac{p_0'}{\langle\mathbf{B}_0{}^2\rangle} - \frac{\Phi'^2}{V'\langle\mathbf{B}_0{}^2\rangle}\left[\mu^2\left(\frac{g_{22}}{\sqrt{g}}\right)^{(0)} + 2\mu\left(\frac{g_{23}}{\sqrt{g}}\right)^{(0)} + \left(\frac{g_{33}}{\sqrt{g}}\right)^{(0)}\right]'.$$

Substituting this result in (5.15) and recognizing that $V'' = (2\pi)^2[(\sqrt{g})^{(0)}]'$, we get

$$W^{(0)} = \frac{p_0'}{(\sqrt{g})^{(0)}\langle\mathbf{B}_0{}^2\rangle}(\mu^2 g_{22}^{(0)} + 2\mu g_{23}^{(0)} + g_{33}^{(0)})' + \frac{p_0'^2V'}{\Phi'^2}\left(\frac{1}{\langle\mathbf{B}_0{}^2\rangle} - \left\langle\frac{1}{\mathbf{B}_0{}^2}\right\rangle\right) \tag{5.16}$$

The difference $1/\langle\mathbf{B}_0^2\rangle - \langle 1/\mathbf{B}_0^2\rangle$ is of the order of k^2a^2/B_S^2, so that the contribution of the term with $p_0'^2$ to the right-hand side of (5.16) is of order of smallness $\beta \equiv 2p_0/B_S^2$, so that this term should be left out. Replacing $g_{22}^{(0)}$ and $g_{23}^{(0)}$ by their approximate values obtained from (5.1) and (5.2), and putting $(\sqrt{g})^{(0)} \approx aR$ and $\langle\mathbf{B}_0^2\rangle \sim B_S^2$, we arrive at a final result for $W^{(0)}$:

$$W^{(0)} = \frac{4\pi^2 ap_0'}{R\Phi'^2}(\mu^2a^2\operatorname{ch}\eta - 2\mu a^2\vartheta'(\zeta) + g_{33}^{(0)})'. \tag{5.17}$$

The inequality (5.14) with $\lambda_{23}^{\theta\theta}$ and $W^{(0)}$, defined by formulas (5.11) and (5.17), is none other than the stability criterion obtained by Shafranov and Yurchenko[10] on the basis of the general-geometric criterion (4.7). We emphasize that the general-geometric criterion has been derived under the assumption $m/ak_x \ll 1$. The present analysis shows that in the particular case of the concrete geometry considered here, the criterion for the stability of a plasma with respect to perturbations with $m/ak_x \ll 1$ coincides with the criterion for the stability with respect to perturbations with arbitrary m/ak_x.

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*$[j_0 B_0] \equiv j_0 \times B_0$.

[1)]For the case of a stellarator with slightly elliptic cross section, the results presented below were obtained by another method jointly with N. D. Borisov.

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