Nonlinear theory of the parametric instability of waves in a plasma

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The nonlinear stage of the instability of an external high-frequency electric field, $\mathbf{E}(t) = \mathbf{E}_0 \cos \omega_0 t$, with respect to decay into Langmuir (LW) and ion-sound (ISW) waves ($\omega_0 = \omega_{\mathbf{k}} + \Omega_{-\mathbf{k}}$) is studied. It is shown that the correlation of the LW and ISW phases that appears in the pair $\omega_{\mathbf{k}}$ and $\Omega_{-\mathbf{k}}$ at the linear stage of the instability also determines the nonlinear behavior of the wave system. Therefore, the kinetic equation for waves is in principle not applicable in this problem. Equations are formulated in which the statistical problem of the interaction between parametrically excited waves is reduced to the dynamical problem of pair interactions. They differ from the linear dynamical equations for decay instability only by a self-consistent renormalization of the wave spectrum and by the nature of their interaction with the external field. The qualitative picture of the nonlinear behavior of the waves essentially depends on the frequency and amplitude of the external field. The amplitudes of the parametrically excited waves, their distribution in k-space, and the energy flux into the plasma are estimated.

The amplitudes of ion-sound and Langmuir oscillations parametrically excited in a nonisothermal plasma by a high-frequency electric field[1] often become such that the behavior of the system is largely determined by the interaction of the waves among themselves. Interest in this problem[2] is excited not only by the importance of the practical applications, but also by the diversity of the physical phenomena that arise, which explains the difficulty encountered in the study of the problem.

Even the linear theory of parametric instabilities in uniform high-frequency fields, $\mathbf{E} = \mathbf{E}_0 \cos \omega_0 t$, which has been thoroughly developed in the paper[3] by Andreev et al., is quite nontrivial. However, for not too large amplitudes E_0 when the characteristic increment is smaller than the ion-sound frequency and the dispersion of the Langmuir waves, the parametric instabilities in a nonisothermal plasma have the simple meaning of decay instabilities of the field E(t) of the first[4]

$$\omega_0 = \omega_{\mathbf{k}} + \Omega_{-\mathbf{k}} \tag{1}$$

and second orders[5]

$$2\omega_0 = \omega_{\mathbf{k}} + \omega_{-\mathbf{k}}, \tag{2}$$

where $\Omega_{\mathbf{k}}$ and $\omega_{\mathbf{k}}$ are the dispersion laws for the ion-sound (ISW) and Langmuir (LW) waves respectively.

In the present paper we study diverse nonlinear phenomena arising in the development of the instability (1), which has the minimum threshold. The problem is considerably simplified by the fact that the excited waves can be described by the hydrodynamic equations (4) and (5). In (6) in Sec. 1 these equations are written in terms of the canonical variables $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$, which are the complex amplitudes of the running Langmuir and ion-sound waves[6]. The structure of the canonical equations (10) does not depend on the nature of the interacting waves: all the concrete information about the system (plasma) is contained in the wave-dispersion laws in the form of matrix elements. In the canonical formulation, our problem gets close to the thoroughly studied problem of the behavior of spin waves parametrically excited by a uniform high-frequency magnetic field in ferromagnets[7].

In Sec. 2 we briefly state the results of the investigation of the linear stage of the parametric instabilities (1) and (2) in the framework of the canonical equations (10). Special attention is paid to the phase relations between the excited waves which play the important role at the nonlinear stage of the process.

For not too large excesses over the threshold, the primary nonlinear mechanism determining the behavior of the parametrically excited waves is their dynamical four-wave interaction that proceeds via induced intermediate beats. The equations for the beats are easily integrated, and, eliminating them, we arrive in Sec. 3 at the canonical equations (7) with the four-wave interaction Hamiltonian (14) which includes only the parametrically excited waves $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$. In Sec. 3 we make the primary approximation of the problem: assuming random phases for the individual waves, we go over to the statistical description in the language of pair correlation functions

$$n_s(\mathbf{k}) = \langle b_{\mathbf{k}} b_{\mathbf{k}}^* \rangle, \quad n_L(\mathbf{k}) = \langle a_{\mathbf{k}} a_{\mathbf{k}}^* \rangle, \quad \sigma(\mathbf{k}) = \langle a_{\mathbf{k}} b_{-\mathbf{k}} e^{i\omega_0 t} \rangle. \tag{3}$$

Such an approximation corresponds to a simplification of the interaction Hamiltonian (14) to a form that is diagonal with respect to wave pairs, and is essentially a self-consistent field approximation; the statistical problem of the interaction of the parametrically excited waves then reduces to the self-consistent dynamical problem of the pair interaction. It is shown at the end of Sec. 3 that the correlation of the phases of the $a_{\mathbf{k}}$- and $b_{-\mathbf{k}}$-waves that results from the process (1) is, in this approximation, fully preserved at the nonlinear stage of the process, i.e., $|\sigma(\mathbf{k})|^2 = n_L n_S$. In a number of papers[8-10] the wave-particle interaction leading to increased dissipation in the plasma was assumed to be the primary cause of the limitation of the growth of the noise amplitude during a parametric instability. We show in the present paper that phase correlation leads to the growth of the characteristic increments of the four-wave processes up to the value $\sim \omega_p W/nT$. As a result, the energy flux from the pumping into the system of parametrically excited waves decreases considerably, and this is often the primary cause of the amplitude limitation.

In Sec. 4 we investigate the steady-state solutions of Eqs. (17) for n_L, n_S, and σ, and study their stability. It is shown that, as in[11], the stable distributions lie on a surface in k-space. This surface is described by formula (1), where the pertinent nonlinear corrections are taken into account in the wave-dispersion laws. The characteristics of the system—in particular, the distribution of the waves on this surface—depend essentially on the wave vectors of the excited waves. We shall consider only the case when $kr_d < \frac{2}{9}(m/M)^{1/2}$, i.e., when the parametrically excited waves cannot merge with a natural LW. In this case the wave distribution is singular right up to sufficiently large excesses (27): $n(\mathbf{k}) \neq 0$ only at the "pole" $\mathbf{k} \parallel \mathbf{E}$ where the coupling between the waves and the external field is strongest. The steady-state amplitudes and phases of the pairs—formulas (24)—are determined in Sec. 4.

In Sec. 5 it is shown that in the framework of the unaveraged equations this monochromatic distribution turns out to be modulated largely in the transverse direction with the characteristic dimension $\kappa_\perp \ll k_0$ (see (29)). The modulation pattern is a dynamical one and changes completely within a time interval of the order of the reciprocal of the increment of the decay instability (1). In the course of the turbulent motion the modulation amplitude may turn out to be large enough for the self-action of the waves to be stronger than the divergence due to diffraction, and these regions collapse just like high-intensity light beams in a dielectric[12].

In conclusion, we consider the stability of the physical phenomena described in the paper with respect to induced scattering by particles—the primary nonlinear mechanism that arises outside the framework of the hydrodynamic description.

1. BASIC EQUATIONS

Let us consider a homogeneous nonisothermal plasma ($T_e \gg T_i$) located in a uniform alternating electric field, $\mathbf{E} = \mathbf{E}_0 \cos \omega_0 t$, of frequency ω_0 close to the plasma frequency ω_p. The resulting homogeneous periodic motion of the electrons and ions in the field E(t) turns out to be unstable with respect to the excitation of long-wave LW and ISW ($kr_d \ll 1$)[3]. Assuming that the plasma in the ISW is quasi-neutral, we have for the ion density:

$$n_i = n + \delta n, \quad \delta n \ll n$$

and the electron density:

$$n_e = n + \delta n + \delta n_e, \quad \delta n_e \ll n,$$

where δn_e is the change in the electron density due to the LW. We apply to them the linearized system of hydrodynamic equations

$$\begin{gathered} \partial \delta n_e / \partial t + n \operatorname{div} \mathbf{V}_e + \operatorname{div} \delta n\, \mathbf{V}_e = -\operatorname{div} \delta n\, \mathbf{V}_{0e}, \\ \frac{\partial \mathbf{V}_e}{\partial t} + \frac{e}{m} \nabla \varphi_e + 3 V_{Te}{}^2 \frac{\nabla \delta n_e}{n} = 0, \quad \Delta \varphi_e = 4\pi e \delta n_e. \end{gathered} \tag{4}$$

Here the electron velocity $\mathbf{V} = \mathbf{V}_e + \tilde{\mathbf{V}}_{0e}$, where $\mathbf{V}_e$ is their velocity in the LW, $\tilde{\mathbf{V}}$ corresponds to the slow motion of the electrons, and $\mathbf{V}_{0e}$ is the electron-oscillation velocity in the uniform electric field, $V_{0e} = eE_0/m\omega_0$. The motion of the ions under the action of a high-frequency pressure is described by the equations

$$\begin{gathered} \partial \delta n / \partial t + n_0 \operatorname{div} \mathbf{v} = 0, \\ \frac{\partial \mathbf{v}}{\partial t} + c_s{}^2 \frac{\nabla \delta n}{n} = \frac{1}{8\pi\rho_0} \nabla [\overline{(\nabla \varphi_e)^2} + 2 \overline{(\mathbf{E} \nabla \varphi_e)}], \end{gathered} \tag{5}$$

where $\mathbf{v}$ is the velocity of the ions and c_s is the speed of the ISW.

Following Zakharov[6], we go over in Eqs. (4) and (5) to the canonical variables:

$$\begin{gathered} \delta n_e = \frac{1}{(2\pi)^{3/2}} \int k \left(\frac{n}{2m\omega_p} \right)^{1/2} (a_\mathbf{k} + a_{-\mathbf{k}}{}^*) e^{i\mathbf{kr}}\, d\mathbf{k}, \\ \mathbf{V}_e = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathbf{k}}{k} \left(\frac{\omega_p}{2mn} \right)^{1/2} (a_\mathbf{k} - a_{-\mathbf{k}}{}^*) e^{i\mathbf{kr}}\, d\mathbf{k}, \\ \delta n = \frac{1}{(2\pi)^{3/2}} \int \left(\frac{nk}{2Mc_s} \right)^{1/2} (b_\mathbf{k} + b_{-\mathbf{k}}{}^*) e^{i\mathbf{kr}} d\mathbf{k}, \\ \mathbf{v} = \frac{1}{(2\pi)^{3/2}} \int \mathbf{k} \left(\frac{2Mnk}{c_s} \right)^{1/2} (b_\mathbf{k} - b_{-\mathbf{k}}{}^*) e^{i\mathbf{kr}} d\mathbf{k}, \end{gathered} \tag{6}$$

in terms of which the equations assume the form

$$\partial a_\mathbf{k} / \partial t = -i \delta H / \delta a_\mathbf{k}{}^*, \quad \partial b_\mathbf{k} / \partial t = -i \delta H / \delta b_\mathbf{k}{}^*, \tag{7}$$

where the Hamiltonian

$$\begin{gathered} H = \int (\omega_\mathbf{k} a_\mathbf{k} a_\mathbf{k}{}^* + \Omega_\mathbf{k} b_\mathbf{k} b_\mathbf{k}{}^*)\, d\mathbf{k} + H_p + H_{int}^{(3)}, \\ \omega_\mathbf{k} = \omega_p (1 + {}^3/_2 (kr_d)^2), \quad \Omega_\mathbf{k} = c_s k \end{gathered}$$

are the LW- and ISW-dispersion laws, and $a_\mathbf{k}$ and $b_\mathbf{k}$ are the normal amplitudes of these waves.

The Hamiltonian $H_{int}^{(3)}$ of the interaction between the $a_\mathbf{k}$- and $b_\mathbf{k}$-waves is given by

$$H_{int}^{(3)} = \int [V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} b_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}{}^* + \text{c. c.}]\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)\, d\mathbf{k}_1\, d\mathbf{k}_2\, d\mathbf{k}_3, \tag{8}$$

where

$$V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = \frac{1}{(2\pi)^{3/2}} \frac{\omega_p}{2^{3/2} (\rho_0 c_s)^{1/2}} k^{1/2} \frac{\mathbf{k}_2 \mathbf{k}_3}{k_2 k_3},$$

and the Hamiltonian for the interaction between the wave and the external ("pumping") field is

$$H_p = \int [e^{-i\omega_0 t} V_\mathbf{k} a_\mathbf{k} (b_\mathbf{k} + b_{-\mathbf{k}}{}^*) + \text{c.c.}]\, d\mathbf{k}, \tag{9}$$

where

$$\begin{gathered} V_\mathbf{k} = \frac{\mathbf{k}\mathbf{V}_0}{4} \left(\frac{m}{M} \right)^{1/4} (kr_d)^{-1/2} = \frac{\omega_0}{2^{3/2}} \left(\frac{m}{M} \right)^{1/4} (kr_d)^{1/2} \left(\frac{W_{ext}}{nT} \right)^{1/2} \frac{\mathbf{k}\mathbf{V}_0}{kV_0}, \\ \mathbf{V}_0 = e\mathbf{E}_0 / m\omega_0, \quad V_{-\mathbf{k}} = -V_\mathbf{k}. \end{gathered}$$

We have discarded in Eqs. (4) and (5) the terms that strictly describe the electronic and ionic nonlinearities. These terms lead in the interaction Hamiltonian to corrections which are insignificant in one of the parameters

$$m/M \ll 1, \quad kr_d (m/M)^{1/2} \ll 1, \quad (kr_d)^2 \ll 1.$$

With allowance for damping, Eqs. (7) assume the form

$$\begin{gathered} \dot{a}_\mathbf{k} + (i\omega_\mathbf{k} + \gamma_\mathbf{k}) a_\mathbf{k} + i V_\mathbf{k} (b_\mathbf{k} + b_{-\mathbf{k}}{}^*) e^{-i\omega_0 t} \\ = -i \int [V_{\mathbf{k}'\mathbf{k}''\mathbf{k}} b_{\mathbf{k}'} + V_{-\mathbf{k}'\mathbf{k}''\mathbf{k}} b_{-\mathbf{k}'}^*] \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'')\, d\mathbf{k}'\, d\mathbf{k}'', \\ \dot{b}_\mathbf{k} + (i\Omega_\mathbf{k} + \Gamma_\mathbf{k}) b_\mathbf{k} + i V_\mathbf{k} (e^{i\omega_0 t} a_\mathbf{k} - e^{-i\omega_0 t} a_{-\mathbf{k}}{}^*) \\ = -i \int V_{\mathbf{k}\mathbf{k}'\mathbf{k}''} a_{\mathbf{k}'} a_{\mathbf{k}''}{}^* \delta(\mathbf{k}' - \mathbf{k} - \mathbf{k}'')\, d\mathbf{k}'\, d\mathbf{k}'', \end{gathered} \tag{10}$$

where $\gamma_\mathbf{k}$ and $\Gamma_\mathbf{k} = (\pi m/8M)^{1/2} \Omega_\mathbf{k}$ are the damping constants for the LW and ISW respectively.

2. PARAMETRIC INSTABILITY

The linear phase of the parametric instability of a plasma located in a uniform field has been thoroughly investigated in a number of papers[1-3]. Here we state briefly the pertinent results in the hydrodynamic approximation, i.e., in the framework of the Eqs. (10). The zero-order solution of Eq. (10) may turn out to be unstable with respect to the growth of the waves:

$$\begin{gathered} a_\mathbf{k} \sim \exp[-i(\omega_0 + \omega)t], \quad a_{-\mathbf{k}}{}^* \sim \exp[-i(\omega - \omega_0)t], \\ b_\mathbf{k},\ b_{-\mathbf{k}}{}^* \sim \exp(-i\omega t). \end{gathered}$$

We obtain for the complex frequency ω the dispersion equation

$$[(\omega_\mathbf{k} - \omega_0)^2 - (\omega + i\gamma_\mathbf{k})^2][\Omega_\mathbf{k}{}^2 - (\omega + i\Gamma_\mathbf{k})^2] = 4\Omega_\mathbf{k} V_\mathbf{k}{}^2 (\omega_\mathbf{k} - \omega_0). \tag{11}$$

For not too long waves the first-order decay instability has the lowest threshold $V_c = (\gamma\Gamma)^{1/2}$[4]. Its increment is a maximum on the sphere (1)

$$\nu_{max} = {}^1/_2[-\gamma - \Gamma + ((\gamma - \Gamma)^2 - 4V^2)^{1/2}].$$

Two pairs of waves $a_{\mathbf{k}}b_{-\mathbf{k}}$ and $a_{-\mathbf{k}}b_{\mathbf{k}}$ grow with this increment, the amplitudes and phases of the waves in each pair being connected by the relations

$$i(\nu_{max} + \gamma_{\mathbf{k}})a_{\mathbf{k}} = V_{\mathbf{k}}b_{-\mathbf{k}}^*, \quad i(\nu_{max} + \gamma_{\mathbf{k}})a_{-\mathbf{k}} = -V_{\mathbf{k}}b_{\mathbf{k}}^*.$$

In the other region of **k**-space a second-order decay instability exists near the sphere (2). In the vicinity of (2), Eq. (11) is transformed into

$$(\omega + i\gamma_{\mathbf{k}})^2 = (\omega_{\mathbf{k}} - \omega_0)[\omega_{\mathbf{k}} - \omega_0 + 4V_{\mathbf{k}}^2/\Omega_{\mathbf{k}}],$$

with the maximum increment attained at

$$\omega_{\mathbf{k}} = \omega_0 + 2V_{\mathbf{k}}^2/\Omega_{\mathbf{k}} \tag{12}$$

and equal to

$$\nu_{max} = 2V^2/\Omega - \gamma.$$

The LW pair $a_{\pm\mathbf{k}}$ for which $a_{-\mathbf{k}}^* = ia_{\mathbf{k}}$ grows with this increment.

For sufficiently large wave amplitudes, when $V_{\mathbf{k}} \geq \Omega_{\mathbf{k}}$, the frequency separation of the surfaces (2) and (1) becomes smaller than the characteristic nonlinearity of the problem, and there arise combined instabilities with a nontrivial dependence of the increment on the amplitude of the external field. We shall henceforth assume that $V_{\mathbf{k}} < \Omega_{\mathbf{k}}$, i.e., that

$$W_{ext}/nT < 8(kr_d)(m/M)^{1/2}. \tag{13}$$

This implies that the instabilities (2) and (1) develop in different regions of **k**-space, and that their nonlinear phase can be considered independently.

3. THE NONLINEAR EQUATIONS FOR THE DECAY INSTABILITY

The instability of an external field with respect to decay into LW and ISW is characterized by the largest increment, and it is natural to consider in the first instance the nonlinear phase of its development. We emphasize that there arises at the linear stage a correlation between the sum of the phases of the waves in a pair and the phase of the pumping field. At sufficiently large wave amplitudes, this phase correlation will remain during the nonlinear stage[11]. Therefore, the four-wave interaction in the equations of motion no longer vanishes in first order when averaged over the phases and will play an important role, which is decisive in a number of cases. Four-wave processes are not present in the interaction Hamiltonian (8), but they will appear in the second-order perturbation theory in $H_{int}^{(3)}$. Indeed, as a result of the development of the instability, there occurs a buildup of LW and ISW pairs with wave vectors **k** close to (1). In conformity with $H_{int}^{(3)}$, they serve as the inducing force for the ion-sound and Langmuir beats with $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$ and frequencies $\omega(\mathbf{k}_1) - \omega(\mathbf{k}_2)$ and $\omega(\mathbf{k}_1) \pm \Omega(\mathbf{k}_2)$ respectively. These beats may in turn generate new pairs of waves with other $\mathbf{k}_1'$ and $\mathbf{k}_2'$ ($\mathbf{k}_1' + \mathbf{k}_2' = \mathbf{k}$).

The equations of motion for the beats can be assumed to be linear in the beat amplitude (see[13]); they are easily integrated, and, after eliminating the beats, we again arrive at the "canonical" equations (7) with the new interaction Hamiltonian

$$H_{int}^{(4)} = \frac{1}{2}\int \tilde{T}_{12,34}a_1^*a_2^*a_3a_4\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\mathbf{k}_4 \tag{14}$$
$$+ \int T_{12,34}a_1^*b_2^*a_3b_4\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\mathbf{k}_4,$$

where

$$\tilde{T}_{12,34} = \tilde{T}_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} = -\frac{4\Gamma^2}{c_s}\frac{(\mathbf{k}_1\mathbf{k}_2)(\mathbf{k}_3\mathbf{k}_4)}{k_1k_2k_3k_4}, \quad \Gamma^2 = \frac{\omega_p^2}{8(2\pi)^3\rho_0c_s},$$
$$T_{12,34} \equiv T_{\mathbf{k}_1\mathbf{k}_2,\mathbf{k}_3\mathbf{k}_4} = \frac{V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1+\mathbf{k}_2}V_{\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_3+\mathbf{k}_4}}{\Omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) - \omega(\mathbf{k}_1 + \mathbf{k}_2) + i\gamma(\mathbf{k} + \mathbf{k}_2)} \tag{15}$$
$$+ \frac{V_{\mathbf{k}_3,\mathbf{k}_2-\mathbf{k}_3,\mathbf{k}_2}V_{\mathbf{k}_1,\mathbf{k}_2-\mathbf{k}_3,\mathbf{k}_4}}{\omega(\mathbf{k}_2) - \Omega(\mathbf{k}_3) - \omega(\mathbf{k}_2 - \mathbf{k}_3) + i\gamma(\mathbf{k}_2 - \mathbf{k}_3)}.$$

The damping term in the denominator of $T_{12,34}$ should be taken into account only when $kr_d > {}^2/_9(m/M)^{1/2}$, when the decay conditions

$$\omega(\mathbf{k}_1) \pm \Omega(\mathbf{k}_2) = \omega(\mathbf{k}). \tag{16}$$

can be fulfilled. In the Hamiltonian $H_{int}^{(4)}$ then arises an anti-Hermitian part that describes the nonlinear damping of the waves. The physical cause of this is the linear damping of the LW which are not connected with the pumping waves, but which are generated as a result of the merging of the parametrically excited LW and ISW (16).

In the case when a sufficiently wide wave packet is excited in the development of the parametric instability, with the result that the individual phases of the waves are random (when the correlation of the sum of the phases in a pair is conserved!), we can go over from the dynamical equations for the waves to a statistical description in terms of the pair correlation functions:

$$n_L(\mathbf{k}) = \langle a_{\mathbf{k}}a_{\mathbf{k}}^*\rangle, \quad n_s(\mathbf{k}) = \langle b_{\mathbf{k}}b_{\mathbf{k}}^*\rangle,$$
$$\sigma(\mathbf{k}) = \langle a_{\mathbf{k}}b_{-\mathbf{k}}e^{-i\omega_0t}\rangle.$$

The conditions of applicability of such a description have been discussed in a paper by Zakharov and one of the present authors[14]. We shall consider in Sec. 5 in what sense the results of the statistical description can be kept for narrow packets.

Splitting up the quaternary correlations in terms of the pair correlations according to the rule:

$$\langle a_1^*a_2^*a_3a_4\rangle = n_L(\mathbf{k}_1)n_L(\mathbf{k}_2)[\delta(\mathbf{k}_1 - \mathbf{k}_3)\delta(\mathbf{k}_2 - \mathbf{k}_4) + \delta(\mathbf{k}_1 - \mathbf{k}_4)\delta(\mathbf{k}_2 - \mathbf{k}_3)],$$
$$\langle a_1^*b_2^*a_3b_4\rangle = \sigma^*(\mathbf{k}_1)\sigma(\mathbf{k}_3)\delta(\mathbf{k}_1 + \mathbf{k}_2)\delta(\mathbf{k}_3 + \mathbf{k}_4)$$
$$+ n_L(\mathbf{k}_1)n_s(\mathbf{k}_2)\delta(\mathbf{k}_1 - \mathbf{k}_3)\delta(\mathbf{k}_2 - \mathbf{k}_4),$$
$$\langle a_1^*b_2a_3a_4\rangle = n_L(\mathbf{k}_1)\sigma(\mathbf{k}_2)[\delta(\mathbf{k}_1 - \mathbf{k}_3)\delta(\mathbf{k}_2 + \mathbf{k}_4) + \delta(\mathbf{k}_1 - \mathbf{k}_4)\delta(\mathbf{k}_2 + \mathbf{k}_3)],$$

we arrive at the system of equations:

$$\begin{aligned} {}^1/_2\dot{n}_L(\mathbf{k}) + (\gamma_{\mathbf{k}} + \bar{\gamma}_{\mathbf{k}})n_L(\mathbf{k}) + \operatorname{Im} P_{\mathbf{k}}^*\sigma_{\mathbf{k}} = 0,\\ {}^1/_2\dot{n}_s(\mathbf{k}) + (\Gamma_{\mathbf{k}} + \bar{\Gamma}_{\mathbf{k}})n_s(\mathbf{k}) + \operatorname{Im} P_{\mathbf{k}}^*\sigma_{\mathbf{k}} = 0,\end{aligned} \tag{17}$$

$$[d/dt + (\gamma_{\mathbf{k}} + \Gamma_{\mathbf{k}} + \bar{\gamma}_{\mathbf{k}} + \bar{\Gamma}_{\mathbf{k}}) + i(\tilde{\omega}_{\mathbf{k}} + \tilde{\Omega}_{\mathbf{k}} - \omega_0)]\sigma(\mathbf{k})$$
$$= -iP_{\mathbf{k}}(n_L(\mathbf{k}) + n_s(\mathbf{k})).$$

Here $P_{\mathbf{k}}$ is the self-consistent pumping:

$$P_{\mathbf{k}} = V_{\mathbf{k}} + \int S_{\mathbf{k}\mathbf{k}'}\sigma(\mathbf{k}')\,d\mathbf{k}',$$

$\tilde{\omega}_{\mathbf{k}}$ and $\tilde{\Omega}_{\mathbf{k}}$ are the renormalized (owing to the interaction) LW and ISW frequencies,

$$\tilde{\omega}_{\mathbf{k}} = \omega_{\mathbf{k}} + 2\int \tilde{T}_{\mathbf{k}\mathbf{k}'}n_L(\mathbf{k}')\,d\mathbf{k}' + \operatorname{Re}\int T_{\mathbf{k}\mathbf{k}'}n_s(\mathbf{k}')\,d\mathbf{k}',$$

$$\tilde{\Omega}_{\mathbf{k}} = \Omega_{\mathbf{k}} + \operatorname{Re}\int T_{\mathbf{k}\mathbf{k}'}n_L(\mathbf{k}')\,d\mathbf{k}',$$

and $\bar{\gamma}_{\mathbf{k}}$ and $\bar{\Gamma}_{\mathbf{k}}$ describe the nonlinear damping:

$$\bar{\gamma}_{\mathbf{k}} = -\operatorname{Im}\int T_{\mathbf{k}\mathbf{k}'}n_s(\mathbf{k}')\,d\mathbf{k}', \quad \bar{\Gamma}_{\mathbf{k}} = -\operatorname{Im}\int T_{\mathbf{k}\mathbf{k}'}n_L(\mathbf{k}')\,d\mathbf{k}'$$

The functions $S_{\mathbf{k}\mathbf{k}'}$, $T_{\mathbf{k}\mathbf{k}'}$, and $\tilde{T}_{\mathbf{k}\mathbf{k}'}$ are defined by the formulas (19).

Notice that such a procedure for splitting up the correlations is equivalent to the reduction of the exact Hamiltonian $H_{int}^{(4)}$ to a form which is diagonal with respect to wave pairs

$$H_{int}^{(4)} = \int \{[T_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}a_{\mathbf{k}}^*b_{\mathbf{k}'}^*b_{\mathbf{k}'} + S_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}^*b_{-\mathbf{k}}^*a_{\mathbf{k}'}b_{-\mathbf{k}'}]$$

$$+ \tilde{T}_{\mathbf{kk}'} a_{\mathbf{k}}^{*} a_{\mathbf{k}} a_{\mathbf{k}'}^{*} a_{\mathbf{k}'} \} d\mathbf{k}\, d\mathbf{k}'. \tag{18}$$

Here

$$T_{\mathbf{kk}'} \equiv T_{\mathbf{kk}',\mathbf{kk}'}$$
$$= -\left\{ \frac{1+\cos\alpha}{1-\beta(1-2\cos\alpha)+i\gamma_+/\Omega} + \frac{1-\cos\alpha}{1+\beta(1-2\cos\alpha)-i\gamma_-/\Omega} \right\} \frac{\Gamma^2}{2c_s},$$
$$S_{\mathbf{kk}'} \equiv T_{\mathbf{k},-\mathbf{k},\mathbf{k}',-\mathbf{k}'} = -\frac{\Gamma^2}{2c_s} \frac{(1+\cos\alpha)}{1+\beta(1+2\cos\alpha)+i\gamma_+/\Omega}$$
$$\tilde{T}_{\mathbf{kk}'} \equiv \tilde{T}_{\mathbf{kk}';\mathbf{kk}'} = -\frac{2\Gamma^2}{c_s}, \qquad \beta = \frac{3kr_d}{2}\left(\frac{M}{m}\right)^{1/2},$$
$$\gamma_\pm = \gamma(\mathbf{k} \pm \mathbf{k}'), \qquad \cos\alpha = \mathbf{kk}'/kk'. \tag{19}$$

In the computation of $S_{\mathbf{kk}'}$ we dropped the $\mathbf{k} = 0$ and $\omega = \omega_0$ terms connected with the Langmuir beat. They describe the inverse effect of the excited waves on the pumping field. To take it into account, we must write out an additional equation connecting the external field inside the plasma. We shall not do this, but assume that the E(t) determining $V_{\mathbf{k}}$ in accordance with (9) is the amplitude of the internal field in the plasma.

Notice that from Eqs. (17) follows

$$[{}^1\!/_2 d/dt + (\gamma_{\mathbf{k}} + \bar{\gamma}_{\mathbf{k}} + \Gamma_{\mathbf{k}} + \bar{\Gamma}_{\mathbf{k}})](|\sigma(\mathbf{k})|^2 - n_L(\mathbf{k}) n_s(\mathbf{k})) = 0. \tag{20}$$

This implies that $|\sigma(\mathbf{k})|$ relaxes to $(n_L n_s)^{1/2}$ within a time interval of the order of $(\gamma_{\mathbf{k}} + \Gamma_{\mathbf{k}})^{-1}$. This permits us to henceforth consider only the class of solutions $|\sigma(\mathbf{k})|^2 = n_L n_s$, i.e., those solutions in which a strict phase correlation has been established between the waves in a pair.

4. THE STEADY STATES AND THEIR STABILITY IN THE FRAMEWORK OF THE DIAGONAL HAMILTONIAN

The steady states of the wave system described by Eqs. (17) are concentrated in **k**-space in a narrow layer near the surface (1). This permits us to replace all the coefficients in (17), except $\tilde{\omega}_{\mathbf{k}}$ and $\Omega_{\mathbf{k}}$, by their values on this surface. The system (17) consists, in the steady-state case, of linear equations with the coefficients $(\gamma_{\mathbf{k}} + \bar{\gamma}_{\mathbf{k}})$, $\Gamma_{\mathbf{k}} + \bar{\Gamma}_{\mathbf{k}}$, $P_{\mathbf{k}}$, and $\tilde{\omega}_{\mathbf{k}} + \tilde{\Omega}_{\mathbf{k}}$. For each direction of **k** the equations have a nontrivial solution for only one or two values of $(\tilde{\omega}_{\mathbf{k}} + \tilde{\Omega}_{\mathbf{k}})$:

$$(\tilde{\omega}_{\mathbf{k}} + \tilde{\Omega}_{\mathbf{k}} - \omega_0)^2 = (\gamma_{\mathbf{k}} + \bar{\gamma}_{\mathbf{k}} + \Gamma_{\mathbf{k}} + \bar{\Gamma}_{\mathbf{k}})^2 \left[\frac{|P_{\mathbf{k}}|^2}{(\gamma_{\mathbf{k}} + \bar{\gamma}_{\mathbf{k}})(\Gamma_{\mathbf{k}} + \bar{\Gamma}_{\mathbf{k}})} - 1 \right]. \tag{21}$$

It can be shown[15] that a steady state in the form of two surfaces is unstable with respect to wave excitation in the space between them. Therefore, the state can be stable only when the two surfaces merge into one:

$$\tilde{\omega}_{\mathbf{k}} = \tilde{\Omega}_{\mathbf{k}} = \omega_0. \tag{22}$$

In this case

$$|P_{\mathbf{k}}|^2 \leqslant (\gamma_{\mathbf{k}} + \bar{\gamma}_{\mathbf{k}})(\Gamma_{\mathbf{k}} + \bar{\Gamma}_{\mathbf{k}}). \tag{23}$$

This condition determines partially the distribution of the waves on the surface (22), namely, $n(\mathbf{k}) \neq 0$ for those **k** to which the sign of equality in (23) applies.

Let us show that for not too large excesses over the threshold only two pairs of waves are excited. Assuming this, we obtain for them from (17), (20), and (23) the expressions:

$$\begin{gathered} n_L(\mathbf{k}_0) = n_L(-\mathbf{k}_0) = (\Gamma/\gamma)^{1/2}(V^2 - \gamma\Gamma)^{1/2}/|S|, \\ n_s(\mathbf{k}_0) = n_s(-\mathbf{k}_0) = (\gamma/\Gamma)^{1/2}(V_{\mathbf{k}_0}{}^2 - \gamma\Gamma)^{1/2}/|S|, \\ \varphi_{\mathbf{k}_0} = \varphi_{-\mathbf{k}_0} + \pi, \quad V_{\mathbf{k}_0} \sin\Phi_{-\mathbf{k}_0} = (\gamma\Gamma)^{1/2}, \quad S_{\mathbf{k}_0\mathbf{k}_0} = S. \end{gathered} \tag{24}$$

Using (24), we compute $|P_{\mathbf{k}}|^2$ in conformity with (17) and, equating it to $\gamma_{\mathbf{k}}\Gamma_{\mathbf{k}}$, we find the amplitudes n_L and n_S (or the excess $V_{\mathbf{k}}$) which correspond, according to (23), to the generation threshold for the second group of pairs with a certain angle between **k** and $\mathbf{k}_0$:

$$\frac{V_{\mathbf{k}_0}{}^2}{\gamma\Gamma} = 1 + S^2 \frac{V_{\mathbf{k}_0}{}^2 - V_{\mathbf{k}}{}^2}{[SV_{\mathbf{k}} - S'_{\mathbf{k}_0\mathbf{k}} V_{\mathbf{k}_0}]^2}, \qquad S'_{\mathbf{kk}'} = \frac{S_{\mathbf{kk}'} - S_{\mathbf{k},-\mathbf{k}'}}{2}. \tag{25}$$

We assume here that

$$\beta \equiv \frac{3}{2} kr_d \left(\frac{M}{m}\right)^{1/2} < \frac{1}{3},$$

so that the decay processes (16) are forbidden and there is no nonlinear damping, i.e., the constants $\bar{\gamma}$ and $\bar{\Gamma}$ vanish. Below we shall consider just this case; the behavior of the parametrically excited waves will be essentially different for $\beta > 1/3$.

Substituting $S_{\mathbf{kk}'}$ from (19) into (25), we obtain

$$\frac{V_{\mathbf{k}_0}{}^2}{\gamma\Gamma} = 1 + \frac{1}{4\beta^4} \frac{[(1+\beta)^2 - 4\beta^2 x]^2}{4x(1-x)}, \qquad x = \cos^2\alpha.$$

For $\beta < 1/3$ the minimum of this expression is located near $x = 1/2$, which corresponds to the generation of pairs with $\alpha = 45°$, the generation threshold being given by

$$V_{\mathbf{k}_0}/(\gamma\Gamma)^{1/2} \approx (1 + 2\beta - \beta^2)/2\beta^2 \geqslant 7.$$

At such excesses, however, the weak-nonlinearity condition is already violated

$$Sn_L(\mathbf{k}_0) < {}^3\!/_2\, \omega_p (kr_d)^2 \quad \text{or} \quad W_L/nT < (kr_d)^2, \tag{26}$$

In fact, writing

$$\frac{V_{\mathbf{k}_0}}{(\gamma\Gamma)^{1/2}} \approx \frac{S(n_L n_s)^{1/2}}{(\Gamma\gamma)^{1/2}} = \frac{Sn_L}{\Gamma} = \frac{Sn_L}{{}^3\!/_2 \omega_p (kr_d)^2} \frac{\Omega\beta}{\Gamma}$$

we obtain from (26)

$$V/(\gamma\Gamma)^{1/2} < \Omega\beta/\Gamma < 10. \tag{27}$$

5. STRONG TURBULENCE OF PAIRS OF LANGMUIR AND ION-SOUND WAVES

We have shown above that in the framework of the diagonal Hamiltonian (18) the only stable steady state of the system is the state in which the two pairs of waves (24) are excited. For the diagonal Hamiltonian approximation to be applicable, however, it is necessary for the phases to be random, for which it is sufficient to have wave packets that are broad in **k**-space. But the solution (24) is a set of four monochromatic waves. Therefore, it is necessary to ascertain what meaning the results of the preceding section can have.

Notice that the state (24) is also a solution to Eqs. (7) with the exact Hamiltonian (14), but that in the framework of the exact Hamiltonian, the stability condition (23) obtained above is only a necessary, and not a sufficient condition.

We have previously[15] shown that the state (24) is unstable with respect to scattering of the parametrically excited waves by each other with the conservation laws

$$\begin{gathered} \omega_{\mathbf{k}_0} + \omega_{-\mathbf{k}_0} = \omega_{\mathbf{k}_0+\varkappa} + \omega_{-\mathbf{k}_0-\varkappa}, \qquad 2\omega_{\mathbf{k}_0} = \omega_{\mathbf{k}_0+\varkappa} + \omega_{\mathbf{k}_0-\varkappa}, \\ \Omega_{\mathbf{k}_0} + \omega_{\mathbf{k}_0} = \omega_{\mathbf{k}_0+\varkappa} + \Omega_{\mathbf{k}_0-\varkappa}, \qquad \Omega_{\mathbf{k}_0} + \omega_{-\mathbf{k}_0} = \omega_{-\mathbf{k}_0+\varkappa} + \Omega_{\mathbf{k}_0-\varkappa}. \end{gathered}$$

It can be seen that this instability develops in a narrow layer near (1). The region along the surface where the increment is positive is, because of the decrease of the coupling V_K with the pumping field, restricted by some $\kappa_c \ll k_0$, in the same way as in the symmetric situation[16].

At the linear stage of the development of the instability there is a buildup of noise in a broad region of **k**-space where $\nu > 0$. However, the noise amplitude eventually becomes large in a narrow region of **k**-space:

$$(\varkappa_\perp / k_0)^2 \leqslant (\tilde{\varkappa}_\perp / k_0)^2 \approx Sn_L / c_s k_0, \tag{28}$$

$$(\varkappa_\parallel / k_0) \leqslant (\tilde{\varkappa}_\parallel / k_0) \approx Sn_L / c_s k_0. \tag{29}$$

Indeed, for wave packets which are sufficiently broad compared to (28), the characteristic frequency difference $\omega''\tilde{\kappa}_\perp$ in the packet is large compared to the reciprocal of the nonlinear-interaction time Sn_L and, consequently, the individual phases of the Langmuir waves become randomized. Therefore, the time evolution of such packets can be studied in the framework of the "diagonal" equations (17). In the preceding section we showed that these equations have a unique, stable, steady state (24) in the form of two pairs of monochromatic waves. Therefore, an arbitrary packet will grow narrow in the same way as happens in the symmetric situation[16] right up to the limit of applicability of the "diagonal" equations (17). On the other hand, the packet cannot be narrower than (28) and (29) because such a packet will be unstable with respect to perturbations with $\kappa \sim \tilde{\kappa}$, for which the packet will play the role of a monochromatic wave. The resulting turbulence is characterized by a strong correlation of the phases of the waves, with the result that the turbulence is strong. Because of the fact that $\tilde{\kappa} \ll k_0$, this turbulent state represents two pairs of monochromatic waves whose amplitudes and phases are slowly modulated in the longitudinal and transverse directions with the characteristic dimensions $1/\tilde{\kappa}_\parallel$ and $1/\tilde{\kappa}_\perp$ respectively.

The modulation pattern is a dynamical one and substantially changes within a time interval of the order of $1/V_{\mathbf{k}_0}$. In order of magnitude the average level of the amplitude of the waves coincides with the quantity (24) which follows from the diagonal equations (17). Against the background of this turbulence there develops an interesting phenomenon: the collapse of the quasi-monochromatic waves. In fact, there evolve in the course of the turbulent motion regions with a longitudinal dimension $1/\tilde{\kappa}_\parallel$ that exceeds the transverse dimension $1/\tilde{\kappa}_\perp$ by a factor of $(Sn_L/\Omega)^{1/2}$, and with an amplitude which is several times larger than the average level. In these regions we can neglect the effect of damping and pumping and verify that the nonlinear interaction leading to self-focusing of the waves exceeds the divergence due to diffraction[12]. As a result of the self-focusing, the amplitude at the center of this filament increases many times over during the finite time $\sim 1/Sn_L$, while the energy of the waves in this region of space is rapidly dissipated. The "dumping" dissipation mechanism described here is another reason why the average turbulence level cannot considerably exceed (24). Such strong turbulence of the envelope waves has previously been considered for the symmetric situation[16].

As the amplitude of the pumping increases, the modulation dimensions $1/\tilde{\kappa}_\parallel$ and $1/\tilde{\kappa}_\perp$ decrease and tend to the wavelength $1/k_0$ when the conditions (26) and (27) are violated. The strong turbulence of the envelope waves is then converted into the strong turbulence of the LW considered by Zakharov[6].

In conclusion, let us estimate the energy absorbed by the plasma during the parametric excitation of long ($\beta < 1/3$) (see (27)) waves. Differentiating with respect to time the wave-pumping field interaction Hamiltonian H_p (see (9)), we obtain for the energy absorbed per unit time,

$$Q = 2\omega_0 \operatorname{Im} \int V_{\mathbf{k}} a_{\mathbf{k}} b_{-\mathbf{k}} e^{i\omega_0 t}\, d\mathbf{k} = 2\omega_0 V \overline{|\sigma| \sin \Phi}. \tag{30}$$

Substituting the values of $|\sigma|$ and $\sin\Phi$ (see (24)) obtained from the diagonal equations, we have for the estimate of Q the expression

$$Q \approx \omega_0 (\gamma\Gamma)^{1/2} \frac{(V^2 - \gamma\Gamma)^{1/2}}{S} \approx (\gamma\Gamma)^{1/2} n_0 T \left(\frac{W_{ext} - W_c}{W_c} \right)^{1/2} \left(\frac{m}{M} \right)^{1/4} (kr_d)^{1/2}. \tag{31}$$

For larger excesses over the threshold the decisive role is played by absorption, which is neglected above, of energy in the regions of collapse. As shown in[12], a collapse occurs in a region of dimension $1/\tilde{\kappa}_\perp$ within a time interval $\sim 1/V$, and an energy $\omega_0 V / S\kappa_\perp^2$ is dissipated at the same time in a unit volume. This dissipation should be compensated by external field's energy flux into the system. Taking this into account, we obtain

$$Q \approx \omega_0 V^2 / S \approx \omega_0 k r_d (m/M)^{1/2} W_{ext}. \tag{32}$$

Let us estimate the role of the kinetic effects, the principal one of which is induced scattering of the LW by ions. Notice first of all that the scattering of the parametrically excited waves by particles does not contribute to the nonlinear interaction, since all of them have the same frequency. Therefore, induced scattering can only lead to the loss of stability of the single-frequency turbulence and to the excitation of a weak multifrequency turbulence. When the increment γ^{nL} of the induced LW scattering by ions exceeds γ, a dynamical turbulence is excited, and the results of Sec. 5 are generally speaking inapplicable. Using the expression for γ^{nL} given in[17], we find that this occurs at excesses

$$(W_{ext} - W_c) / W_c > (T_e\gamma / T_i\Gamma)^2.$$

Thus, the nonlinear behavior of a nonisothermal, parametrically excited plasma can basically be described hydrodynamically. In an isothermal plasma, the kinetic effects apparently play a major role. In particular, they lead to the transfer of energy across the weak-turbulence spectrum to the region of small k, just as happens in the relaxation of a beam in a plasma[18]. This circumstance has been ignored in a number of papers[9,10], and this had led to an incorrect estimate for the energy flux into the plasma.

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