Wilson's method in the static model of the nucleon

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An expansion in powers of $\epsilon = 2 - d$ (d is the number of space dimensions), to order ϵ^4, is obtained in the static model of the nucleon[4] for the exponent η of the nucleon Green's function. The expansion is in full agreement with the true value of η, which can be obtained in this model from the exact solution.

1. INTRODUCTION

Modern theory of phase transitions is based on the hypothesis of scale invariance (scaling) of the correlation functions over distances much larger than the interaction radius, but much smaller than the correlation radius. A direct consequence of this hypothesis is the power-law behavior of the two-point correlation functions

$$\langle \varphi(x)\varphi(0)\rangle \propto x^{-2\delta},$$

where δ is the so-called "anomalous dimensionality" of the field φ, and is not determined by scaling. Knowledge of the anomalous dimensionalities of various fields yields interesting information on the system near the critical point. In principle, anomalous dimensionalities can be determined from an infinite system of infinite equations of the unitarity condition[1]. This is impossible at present for systems that are more or less realistic.

Wilson[2] has proposed recently a practical method for calculating anomalous dimensionalities (critical exponents). He considered the $\lambda\varphi^4$ model, which describes the critical behavior of a Heisenberg ferromagnet and of the Baxter model[3]. Wilson's method makes use essentially of the fact that at a dimensionality of space d = 4 there is realized a zero-charge solution. The critical exponents are then determined in the form of expansions in powers of the deviation ϵ from the dimensions of 4-space.

In this paper we apply Wilson's procedure[2] to the static model of the nucleon[4]. In this model, at a certain ratio of the interaction constants (see the next section) there is scaling if the dimensionality of the space is d = 1, and a zero-charge solution is realized at d = 2. Calculation by Wilson's method leads to the following expression for the exponent η the nucleon Green's function $G(\omega) \propto \omega^{-1+\eta}$ (ω has the meaning of energy) and for the renormalized interaction constant λ_R:

$$\eta = {}^1/_{32}\varepsilon^2 + O(\varepsilon^3), \tag{1}$$

$$\lambda_R = -{}^1/_4\varepsilon + O(\varepsilon^4), \tag{2}$$

where $\epsilon = 2 - d$. The expansion (1) agrees fully with the exact value of the exponent η in the static model:

$$\eta_{\text{exact}} = \varepsilon^2/32. \tag{3}$$

2. STATIC MODEL OF THE NUCLEON IN A SPACE OF DIMENSIONALITY d = 2 – ϵ

The Hamiltonian of the static model of the nucleon in the case of interest to us is

$$H = \int \nabla\varphi^+(\mathbf{r})\nabla\varphi(\mathbf{r})\,d\mathbf{r} + \lambda\int\psi^+(\mathbf{r})\psi(\mathbf{r})\left\{\varphi^+(\mathbf{r})\varphi(\mathbf{r}) + \frac{\varphi^+(\mathbf{r})\varphi^+(\mathbf{r})+\varphi(\mathbf{r})\varphi(\mathbf{r})}{2}\right\}d\mathbf{r}, \tag{4}$$

where ψ is the field of the static nucleons and φ is the meson field.

As shown by Gribov et al.[4], the nucleon Green's function in the representation of the imaginary time ξ is expressed in terms of a certain function $\gamma_1(\omega, \xi)$[1)]:

$$G(\xi) = \theta(\xi)\exp\left\{\frac{1}{2}\int_0^\xi d\xi' \int\frac{d\omega}{2\pi i}[\gamma_1(\omega,\xi') - 1]\right\}. \tag{5}$$

The function $\gamma_1(\omega, \xi)$ satisfies the equation

$$\gamma_1(\omega,\xi) = 1 + \lambda^2\int\frac{d\omega_1'\,d\omega_2'}{(2\pi i)^2}\frac{D(\omega_1')}{1+\lambda D(\omega_1')}\frac{D(\omega_2')}{1+\lambda D(\omega_2')} \times\frac{\exp\{(\omega_1'+\omega_2')\xi\}\gamma_1(\omega_1',\xi)}{(\omega+\omega_2')(\omega_1'+\omega_2')}, \tag{6}$$

where $D(\omega)$ is a meson propagator with dimensionality d:

$$D(\omega) = \int\frac{1}{\omega+\mathbf{k}^2}\frac{d^d\mathbf{k}}{(2\pi)^d}, \tag{7}$$

$$d^d(\mathbf{k}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}k^{d-1}\,dk. \tag{8}$$

The dimensionality of space d is assumed to equal $2 - \epsilon$ throughout. This yields for the meson Green's function ($\epsilon \neq 0$)

$$D(\omega) = \omega^{-\varepsilon/2} \tag{9}$$

(we have omitted a coefficient, which we assume to be incorporated in the interaction constant λ). We note that the bare Green's function of the nucleon, as seen from (5), does not depend on the dimensionality of the space, and has the following form in the ω-representation

$$G_0(\omega) \equiv \int_0^\infty G_0(\xi)e^{-\omega\xi}\,d\xi = \frac{1}{\omega}. \tag{10}$$

The integral equation (6) with the propagator (9) was solved by the method used by Gribov et al.[4] to solve this equation in the case of one-dimensional space, i.e., at $\epsilon = 1$. The solution is

$$\gamma_1(\omega, \xi) = 1 - \varepsilon^2/16\omega\xi. \tag{11}$$

Using (5), we obtain an expression for the nucleon Green's function in the ω-representation:

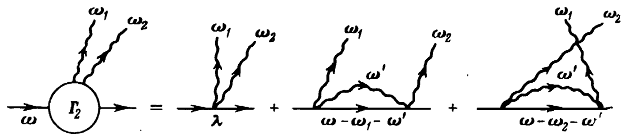
$$G(\omega) = \omega^{-1+\varepsilon^2/32}, \tag{12}$$

i.e., expression (3) for η.

3. CALCULATION OF THE EXPONENT η BY WILSON'S METHOD

In this section we obtain the exponent η to the first nonvanishing order in ϵ. The calculations in higher orders are given in the Appendix.

We consider the vertex $\Gamma_2(\omega_1, \omega_2, \omega)$ for the decay of a nucleon into two mesons and a nucleon in first-order perturbation theory:



or, in analytic form

$$\Gamma_2(\omega_1,\omega_2,\omega)=\lambda+\lambda^2\int\frac{d\omega'}{2\pi i}\frac{D(\omega')}{\omega-\omega_1-\omega'}+\lambda^2\int\frac{d\omega'}{2\pi i}\frac{D(\omega')}{\omega-\omega_2-\omega'},\qquad(13)$$

where $D(\omega)$ is the meson propagator at small deviations from two-dimensional space:

$$D(\omega)=\int\frac{1}{\omega+\mathbf{k}^2}\frac{d^{2-\varepsilon}\mathbf{k}}{(2\pi)^{2-\varepsilon}}$$

$$\approx-\ln\omega+\frac{\varepsilon}{4}\ln^2\omega-\frac{\varepsilon^2}{24}\ln^3\omega+\ldots\qquad(14)$$

(the cutoff parameter is set equal to unity; the general coefficient has been omitted since it can be incorporated in λ (see below)).

According to Wilson[2], the bare constant λ must be chosen equal to the renormalized value (physical charge), which, naturally, depends on the dimensionality of space. Sinze zero-charge obtains at d = 2, the expansion $\lambda(\epsilon)$ begins with the linear term:

$$\lambda(\varepsilon)=\lambda_1\varepsilon+O(\varepsilon^2).\qquad(15)$$

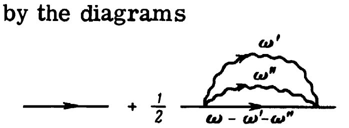
Retaining the zeroth-order terms in the meson propagators, we obtain the vertex to order ϵ^2:

$$\Gamma_2(\omega_1,\omega_2,\omega)=\lambda-\lambda^2\ln(\omega-\omega_1)-\lambda^2\ln(\omega-\omega_2).\qquad(16)$$

We shall henceforth be interested in the ratio $\Gamma_2/\tilde{\Gamma}_2$, where $\tilde{\Gamma}_2\equiv\Gamma_2(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{\omega})$, while $\tilde{\omega}_1/\tilde{\omega}=\omega_1/\omega$ and $\tilde{\omega}_2/\omega=\omega_2/\omega$. We have for this ratio

$$\frac{\Gamma_2}{\tilde{\Gamma}_2}=1-2\lambda\ln\frac{\omega}{\tilde{\omega}}.\qquad(17)$$

We now find Wilson's expansion for the nucleon Green's function. In first order of perturbation theory, $G(\omega)$ is given by the diagrams



(18)

and by the corresponding analytic expression

$$\frac{1}{\omega}+\frac{\lambda^2}{2\omega^2}\int\frac{d\omega'\,d\omega''}{(2\pi i)^2}\frac{\ln\omega'\ln\omega''}{\omega-\omega'-\omega''}+O(\varepsilon^3).\qquad(18')$$

Confining ourselves to the principal asymptotic term ($\omega\to 0$), we obtain for $G(\omega)$

$$G(\omega)=\omega^{-1}[1+{}^1/_2\lambda_1\varepsilon^2\ln\omega+O(\varepsilon^3)].\qquad(19)$$

On the other hand, we have from the scale invariance for the Green's function and the vertex

$$G(\omega)=\omega^{-1+\eta},$$

$$\Gamma_2(\omega_1,\omega_2,\omega)=\omega^{\varepsilon/2-\eta}f\left(\frac{\omega_1}{\omega},\frac{\omega_2}{\omega}\right)\qquad(20)$$

or

$$\Gamma_2/\tilde{\Gamma}_2=(\omega/\tilde{\omega})^{\varepsilon/2-\eta}.\qquad(21)$$

The exponent η in (20) and (31) is a function of the dimensionality of space:

$$\eta=\eta_0+\eta_1\varepsilon+\eta_2\varepsilon^2+O(\varepsilon^3).\qquad(22)$$

Taking this circumstance into account, we rewrite (20) in the form

$$G(\omega)=\frac{1}{\omega}[1+\eta_0\ln\omega+\eta_1\varepsilon\ln\omega+\eta_2\varepsilon^2\ln\omega$$

$$+{}^1/_2\eta_0^2\ln\omega+\eta_0\eta_1\varepsilon\ln\omega+{}^1/_2\eta_1\varepsilon^2\ln^2\omega+O(\varepsilon^3)].\qquad(23)$$

Comparing the expansions (19) and (23), we conclude that

$$\eta_0=\eta_1=0,\quad\eta_2={}^1/_2\lambda_1^2,\qquad(24)$$

i.e., the expansion for η begins with the ϵ^2 terms. Taking this into account, we obtain from (21) for the ratio $\Gamma_2/\tilde{\Gamma}_2$

$$\frac{\Gamma_2}{\tilde{\Gamma}_2}=1+\frac{\varepsilon}{2}\ln\frac{\omega}{\tilde{\omega}}+O(\varepsilon^2).\qquad(25)$$

Expanding the constant λ in (17) in powers of ϵ and comparing the series with the expansion (25), we get

$$\lambda=-\varepsilon/4+O(\varepsilon^2).$$

By virtue of (24) we have for η:

$$\eta=\varepsilon^2/32+O(\varepsilon^3).\qquad(26)$$

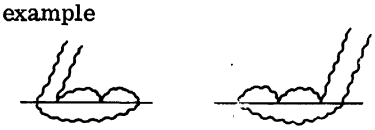
As shown in the Appendix, the new two terms in the expansions of λ and η are equal to zero. We thus obtain ultimately (1) and (2).

In conclusion, we are sincerely grateful to A. M. Polyakov for stimulating discussions and valuable remarks.

APPENDIX

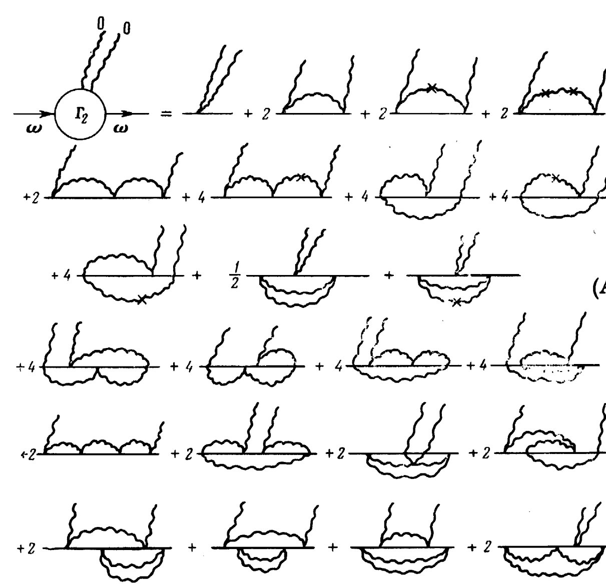
We present here Wilson's expansion for the constant λ and the exponent η to orders ϵ^3 and ϵ^4, respectively.

For the vertex $\Gamma_2(\omega_1,\omega_2,\omega)$, the perturbation-theory series, accurate to fourth-order terms, contains 44 diagrams. Without loss of rigor, however, we can consider the vertex Γ_2 at zero frequencies ω_1 and ω_2. Diagrams with interchanged external ends then make equal contributions, so that one doubled diagrams can be considered. Equal contributions are made also by symmetrical diagrams, for example



(A.1)

Taking these remarks into account, we obtain the following expansion:



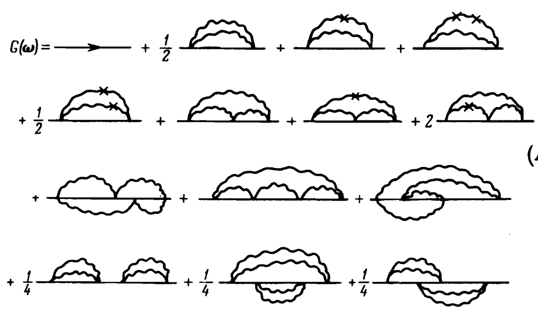
(A.2)

Simultaneously with the perturbation-theory expansion, we have expanded, with the required accuracy, the meson propagators in powers of ϵ: wavy lines without crosses correspond to $-\ln\omega$, those with one cross correspond to $(1/4)\epsilon\ln^2\omega$, and those with two crosses to $(-1/24)\epsilon^2\ln^3\omega$

(see (14)). After separating the principal asymptotic terms from the Feynman integrals, we obtain for the ratio $\Gamma_2/\tilde{\Gamma}_2$ ($\tilde{\omega} = 1$)

$$\frac{\Gamma_2}{\tilde{\Gamma}_2} = 1 - 2\lambda \ln \omega - \frac{1}{2}\lambda^2 \ln \omega + \left(\frac{\lambda\varepsilon}{4} + 3\lambda^2\right)\ln^2 \omega + \lambda^3 \ln^2 \omega - \left(\frac{\lambda\varepsilon^2}{12} + \frac{4}{3}\lambda^2\varepsilon + \frac{16}{3}\lambda^3\right)\ln^3 \omega - (\lambda^2\varepsilon + 4\lambda^3)\int \frac{d\omega'}{2\pi i}\frac{\ln(\omega - \omega')\ln^2 \omega'}{\omega - \omega'} \qquad \text{(A.3)}$$

For the Green's function we have the following expansion to order ϵ^4:



(A.4)

Again separating the entire logarithmic asymptotic expression, we obtain for the regularized Green's function

$$G(\omega) = \frac{1}{\omega}\left[1 + \frac{1}{2}\lambda^2 \ln \omega - \left(\frac{1}{4}\lambda^2\varepsilon + \lambda^3\right)(\ln^3 \omega - 2\ln \omega) + \left(\frac{\lambda^2\varepsilon^2}{12} + \frac{5}{6}\lambda^3\varepsilon + 2\lambda^4\right)(\ln^3 \omega - 3\ln^2 \omega + 6\ln \omega) + \frac{\lambda^4}{8}\ln^2 \omega\right]. \qquad \text{(A.5)}$$

Expanding the parameter λ in (A.3) and (A.5) in powers of ϵ and comparing these series with the expansions of (21) and (22), we arrive at (29) and (30).

[1)]Our expression for $G(\xi)$ contains the factor 1/2 in the argument of the exponential; omission of this factor led to an incorrect expression for η in [4].

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