EFFECT OF DISPERSION ON THE NONLINEAR EVOLUTION OF QUASI-MONOCHROMATIC HELICAL WAVES IN A MAGNETOACTIVE PLASMA

Ya. N. ISTOMIN and V. I. KARPMAN

Institute of Terrestrial Magnetism, the Ionosphere, and Propagation of Radio Waves, USSR Academy of Sciences

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The concurrent manifestation of dispersion and nonlinear effects in the evolution of quasi-monochromatic helical waves in a magnetoactive plasma is investigated. (Implied here are the nonlinear effects due to cyclotron interaction between the wave and resonance particles.) It is shown that the role of dispersion becomes important at sufficiently large wave amplitudes. The stationary waves of the envelopes are investigated; in the case under consideration these waves have the form of shock waves with an oscillatory structure behind the wave front.

1. THE BASIC EQUATIONS

DURING the propagation of circularly polarized waves along a magnetic field, the dominant nonlinear effect (for not too large amplitudes) is the resonance cyclotron interaction of the wave with a group of particles whose velocity is close to the "resonance" velocity $v_R = (\omega - \omega_c)/k$, where ω_c is the cyclotron frequency (we consider, for definiteness, electron-cyclotron waves ("whistlers"); the extension of the corresponding results to the case of ion-cyclotron waves is apparent). If a packet of such waves has a sufficiently small spectral width, such that we can neglect dispersion, then its evolution is described by the theory developed in[1].

If, however, the wave front is sufficiently steep (such a situation may, as has been shown in[1], arise owing to the nonlinear twisting of the front due to the cyclotron interaction between the wave and the resonance particles), then the dispersive spreading of the front becomes important and should be taken into account together with the nonlinear effects. The present paper is devoted to the investigation of this problem.

For waves propagating along a magnetic field (the Z axis), Maxwell's equations are easily reduced to a single equation which is conveniently written in the form

$$\frac{\partial^2 \mathscr{E}_\alpha}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 \mathscr{D}_\alpha}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial (j_\alpha - j_{L\alpha})}{\partial t}, \tag{1}$$

where on the right-hand side stands the difference between the total current vector

$$\mathbf{j} = -en\int \mathbf{v} f(t, Z, \mathbf{v})\,d\mathbf{v} \tag{2}$$

and the same vector $\mathbf{j}_L$ in the linear approximation, obtained by replacing the total distribution function f in the expression (2) by the solution f_L, of the linearized equation[1)]. We require then, in addition, that the poles in f_L be circled in the principal-value sense. Thus, f_L determines the Hermitian part of the permittivity tensor, and, consequently, in the matter equation $\mathscr{D}_\alpha = \epsilon_{\alpha\beta}\mathscr{E}_\beta$ we should take $\epsilon_{\alpha\beta}$ to mean a Hermitian tensor, as is henceforth assumed.

Taking into account the fact that the components of the field in a circularly polarized wave have the form

$$\mathscr{E}_x = A\cos(kZ - \omega t + \varphi); \quad \mathscr{E}_y = -A\sin(kZ - \omega t + \varphi),$$

where $A = A(Z, t)$ is a real amplitude and $\varphi = \varphi(Z, t)$ is the slowly varying part of the phase which determines the correction to the frequency and the wave number, $\delta\omega = -\partial\varphi/\partial t$, $\delta k = \partial\varphi/\partial Z$, it is convenient to introduce the following complex quantities:

$$E = Ae^{i\varphi}, \quad \tilde{\mathscr{E}} = \mathscr{E}_x - i\mathscr{E}_y = E\exp(ikZ - i\omega t), \quad \tilde{\mathscr{D}} = \mathscr{D}_x - i\mathscr{D}_y,$$
$$\delta j = -en\int (v_x - iv_y)(f - f_L)\,d\mathbf{v}. \tag{3}$$

Then Eq. (1) can be easily reduced to the form

$$\frac{\partial^2 \tilde{\mathscr{E}}}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 \tilde{\mathscr{D}}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial(\delta j)}{\partial t}, \tag{4}$$
$$\tilde{\mathscr{D}} = N^2(\hat{\omega})\tilde{\mathscr{E}}, \quad \hat{\omega} = i\frac{\partial}{\partial t},$$

where $N(\omega)$ is the refractive index for the whistlers

$$N^2(\omega) = \omega_p^2/\omega(\omega_c - \omega). \tag{5}$$

The expansion

$$\frac{\partial^2 \tilde{\mathscr{D}}}{\partial t^2} = -\left\{\omega^2 N^2(\omega)E + i\frac{\partial(\omega^2 N^2)}{\partial\omega}\frac{\partial E}{\partial t} - \frac{1}{2}\frac{\partial^2(\omega^2 N^2)}{\partial\omega^2}\frac{\partial^2 E}{\partial t^2} + \ldots\right\}\exp(ikZ - i\omega t), \tag{6}$$

where the dotted line stands for terms with third and higher-order derivatives with respect to the complex amplitude $E(Z, t)$, may be used in the transformation of Eq. (4). We shall henceforth consider $\partial E/\partial t$ and $\partial E/\partial Z$ as terms of first order in smallness, and the second derivatives on the right-hand side of Eq. (4), as second-order terms. Then, the usual relation $(\partial/\partial t + v_g\partial/\partial Z)E = 0$, where $v_g = d\omega/dk$ is the group velocity, follows, in the first approximation, from (4) and (6). Retaining terms of second order in smallness, we obtain from (4) and (6) the following equation:

$$i\left(\frac{\partial E}{\partial t} + v_g\frac{\partial E}{\partial Z}\right) + \frac{1}{2}v_g'\frac{\partial^2 E}{\partial Z^2} = -2\pi i\frac{v_g\omega}{kc^2}\delta j\exp(-ikZ + i\omega t), \tag{7}$$

where $v_g' = dv_g/dk = d^2\omega/dk^2$ (here, $v_g^2\partial^2 E/\partial Z^2$ has been substituted in place of $\partial^2 E/\partial t^2$).

[1)]Such an approach was also used in a recently published paper [2] in which the nonlinear frequency shift for Langmuir waves of constant amplitude was investigated.

It follows from Eq. (7) that $\varphi(Z, t)$ and $A(Z, t)$ satisfy the following system of equations:

$$\left(\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial Z}\right)\varphi+\frac{v_g'}{2}\left(\frac{\partial\varphi}{\partial Z}\right)^2-\frac{v_g'}{2A}\frac{\partial^2 A}{\partial Z^2} = -2\pi\frac{\omega v_g}{kc^2}\frac{\mathrm{Im}(\tilde{\mathscr{E}}^*\delta j)}{|\tilde{\mathscr{E}}|^2}, \quad (8)$$

$$\left(\frac{\partial}{\partial t}+v_g\frac{\partial}{\partial Z}\right)U+v_g'\frac{\partial}{\partial Z}\left(U\frac{\partial\varphi}{\partial Z}\right)=-\mathrm{Re}(\tilde{\mathscr{E}}^*\delta j), \quad (9)$$

where U is the energy density of the wave, which can, after simple computations, be represented in the form

$$U=\frac{c^2(\omega_c-\omega)A^2}{2\pi v_g^2\omega_c}=\frac{B_0^2\omega_c}{8\pi(\omega_c-\omega)}h^2. \quad (10)$$

Here, $h = B/B_0$ (B is the amplitude of the field of the wave and B_0 is the amplitude of the constant magnetic field). We can replace the expression on the right-hand side of Eq. (9) by

$$\mathrm{Re}(\tilde{\mathscr{E}}^*\delta j)=j_x\mathscr{E}_x+j_y\mathscr{E}_y=\frac{d\delta T}{dt}=\left(\frac{\partial}{\partial t}+v_R\frac{\partial}{\partial Z}\right)\delta T, \quad (11)$$

where δT is the mean kinetic energy of the resonance particles of the plasma, for which the following expressions were obtained in[1]:

$$\delta T=\frac{B_0^2\omega_c}{8\pi(\omega_c-\omega)}r(h), \quad (12a)$$

$$r(h)\equiv\begin{cases}-{}^4/_3 a h^{3/2}h_m^{1/2}\\ -{}^4/_3 a h^{3/2}h_m^{1/2}\left[1+{}^9/_2\int\limits_1^{(h_m/h)^{1/2}}dy\,yR(y)\right].\end{cases} \quad (12b)$$

The first of the expressions (12b) is valid for those points of the front of the packet which are situated before the maximum of the amplitude $h = h_m$, the second, for those points behind the amplitude maximum,

$$a=\frac{64}{3\pi^2}\gamma_L\tau_m\frac{W_T^{1/2}\int\limits_0^\infty f_0'W^{5/2}dW}{\int\limits_0^\infty f_0'W^3dW}\sim\gamma_L\tau_m. \quad (13)$$

Here, γ_L is the linear increment,

$$\tau_m=(h_m k\omega_c W_T)^{-1/2}, \quad (14)$$

τ_m is the characteristic nonlinear time, W_T is the transverse thermal velocity,

$$f_0'=\left(\frac{\partial f}{\partial v_z}+\frac{\omega_c}{kW}\frac{\partial f}{\partial W}\right)_{v_z=v_R},$$

and the function $R(y)$ is given parametrically: $R = 1/\kappa K(\kappa)$, $y = E(\kappa)/\kappa$ ($K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kinds).

Equation (11) is a consequence of the energy-conservation law and the fact that on the average the wave does work on the resonance particles, which propagate with the average resonance velocity $v_R = (\omega - \omega_c)/k$ (for the whistlers this velocity is always directed in the direction opposite to that of the motion of the packet, since $\omega < \omega_c$). The relation (11) can, of course, be obtained directly from the expression (2) for δj and the kinetic equation for the resonance particles in the packet. Using (10) and (12a) and going over to a coordinate system moving with the group velocity, $z = -(Z - v_g t)$ (the z axis in this system is directed into the packet, i.e., opposite the direction of its motion), we finally write Eq. (9) in the following form:

$$\frac{\partial[h^2+r(h)]}{\partial t}+v_g'\frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}h^2\right)+v_0\frac{\partial r(h)}{\partial z}=0, \quad (15)$$

where $r(h)$ is defined in (12) and v_0 is the velocity of the resonance electrons in the reference frame under consideration:

$$v_0=v_g-v_R=v_g(1+\omega_c/2\omega). \quad (16)$$

Equation (15), without the second term on the left-hand side, which describes the dispersion spreading, was the basic equation for the investigation of the evolution of the packet in[1]. Let us now ascertain the cases in which dispersion becomes important. Dispersion clearly becomes important when the wave front is sufficiently steep, so that the second term in (15) is comparable with the third. Denoting the characteristic width of the front by l and setting $\partial\varphi/\partial Z \sim 1/l$, we obtain by equating the orders of magnitude of the second and third terms in (15) and taking (12b) into account,

$$l=v_g'/v_0\gamma_L\tau_m. \quad (17)$$

The length l over which the dispersion effects are comparable with the nonlinear effects will henceforth play an important role. It can be seen from (17) and (14) that l is proportional to the square root of the wave amplitude.

On the other hand, since we use the expression (12b), which was obtained in[1] in the adiabatic approximation, it is necessary that the length (17) be considerably greater than the nonlinear length $l_N = v_0\tau_m$. This imposes the following limitation on the wave amplitude:

$$(\gamma_L\tau_m)^2\ll\left|\frac{\gamma_L\omega}{\omega_c^2}\left(1-\frac{4\omega}{\omega_c}\right)\right|. \quad (18)$$

It can be seen from (14) and (18) that in order for the dispersion to manifest itself under the conditions of the adiabatic approximation, the wave should be sufficiently intense[2)]. In the opposite limiting case when $l \ll v_0\tau_m$, the dispersion effects are of little importance compared to the nonlinear modulation effects considered in[1] for a packet with a steep leading edge (see Sec. 6 of the paper cited).

Let us turn now to the analysis of Eq. (8). Let us set

$$\Delta\omega=\frac{2\pi\omega v_g}{kc^2}\frac{\mathrm{Im}(\tilde{\mathscr{E}}^*\delta j)}{|\tilde{\mathscr{E}}|^2} \quad (19)$$

(this designation is connected with the fact that for a wave with a constant—in space—amplitde, the quantity (19) is, as can be seen from (8), equal to $-\partial\varphi/\varphi t$, i.e., coincides with the frequency shift). We can, on the basis of the kinetic equation, show after simple but tedious calculations that $\Delta\omega$ depends only on the amplitude h of the field (and not on its derivatives, as obtains in (11) and that in order of magnitude

$$\Delta\omega\sim\beta\gamma_L\sim\beta a/\tau_m, \quad (20)$$

where β is the small parameter given in the footnote[2)].

[2)]On the other hand, it should be borne in mind that the whole theory (both in [1] and in the present paper) is valid for not very intense waves, namely, when

$$\beta\equiv(h\omega_c^3/k^3W_T^3)^{1/2}=\omega_c/k^2W_T^2\tau_m\ll1.$$

It can be seen from (20) that $\Delta\omega$ is, generally speaking, an extremely small quantity. More precisely, it follows from (8) and (20) that under the condition that

$$v_0\tau_m \gg \beta l, \tag{21}$$

which we shall henceforth assume to be fulfilled[3)], the right-hand side of Eq. (8) can be neglected.

2. HELICAL SHOCK WAVES

As the simplest solution to Eqs. (15) and (8) which gives some idea about the nature of the effects due to the concurrent manifestation of dispersion and nonlinear resonance interaction between the waves and the particles, let us consider a steady-state solution of the form

$$h = h(z - vt), \quad \varphi = \varphi(z - vt). \tag{22}$$

Substituting (22) into (8) and integrating, we obtain

$$\varphi' = (vh^2 - v_0 r(h) + C) / v_g' h^2, \tag{23}$$

where C is the constant of integration and a prime denotes a derivative. Substituting (23) in (15), we shall have

$$h'' + \{v^2h^4 - [v_0 r(h) - C]^2\} / h^3 v_g'^2 = 0. \tag{24}$$

Let us choose the constant C such that the resulting wave propagates into the unperturbed region, i.e., $h \to 0$, $\varphi' \to 0$ as $z \to -\infty$ (we recall that our z axis is directed against the motion of the packet). It should, however, be borne in mind here that the expression for r(h) is, strictly speaking, not valid for very small h, since it was obtained in[1] under the condition that $\gamma_L\tau_m \ll 1$ (which is, in a sense, contrary to the linear-approximation condition). To overcome this difficulty, we replace the above-indicated boundary conditions by the following conditions:

$$h \to h_0, \quad \varphi' \to 0 \ \text{ for } \ z \to -\infty, \tag{25}$$

where h_0 is neglibly small compared to the maximum amplitude h_m, but, on the other hand, $\gamma_L\tau_0 \ll 1$ (τ_0 is the nonlinear time (14) in which h_0 has been substituted for h_m).

It follows from (23)—(25) that

$$C = -vh_0^2 + v_0 r(h_0). \tag{26}$$

Substituting this in (24) and considering the region of the wave in front of the maximum, we must use the first of the expressions of (12b). Assuming that $h \gg h_0$, we obtain the following equation for the amplitude ($h_m = h_{max}$):

$$h'' + \frac{v^2}{v_g'^2}h - \frac{16v_0^2a^2h_m}{9v_g'^2} = 0. \tag{27}$$

The solution of Eq. (27) has the form

$$h = \frac{16v_0^2a^2h_m}{9v^2} + H\cos\left(\frac{v}{v_g'}\zeta\right), \tag{28}$$

where $\zeta = z - vt + \text{const}$, and H is the constant of integration. Let us assume that the maximum value of the amplitude is attained at $\zeta = 0$. Then, it follows from (28) that

$$H = h_m\left(1 - \frac{16v_0^2a^2}{9v^2}\right) + O(h_0). \tag{29}$$

In order to satisfy the boundary condition (25), we should consider the solution to the more exact equation (24), where the constant is defined in (26). We can, however, proceed more simply by matching the approximate solution (28) with zero in such a way that the continuity of the first derivative is preserved. For this purpose we should require that $h(\zeta) = h'(\zeta) = 0$ when $\zeta \le \zeta_0$. Hence, we obtain $\zeta_0 = -\pi|/v_g'|v|$,

$$h = \frac{h_m}{2}\left(1 + \cos\frac{v}{v_g'}\zeta\right), \quad \zeta_0 < \zeta < 0, \tag{30}$$

$$h = 0, \quad \zeta < \zeta_0,$$

$$v = \pm(\sqrt{32}/3)v_0 a \sim \pm v_0\gamma_L\tau_m. \tag{31}$$

The approximate solution (30) is quite a good substitute for the exact solution, since it differs from the latter in a region of little importance where $\gamma_L\tau \gtrsim 1$.[4)]

The two different signs in expression (31) for v, indicate that two types of waves—fast (whose velocity is higher than the group velocity by the amount $|v|$) and slow (whose velocity is smaller by the same amount)—can exist independently.

The length of the front part of the wave where h changes from 0 to h_m is, as was to be expected, equal to

$$\pi\frac{v_g'}{v} = \frac{3\pi}{\sqrt{32}}\frac{v_g'}{v_0 a} \sim l.$$

Let us now investigate the profile of the wave behind the point where the amplitude is a maximum. In this region the second of the expressions in (12b) should be chosen for the function r(h). Let us write Eq. (24) (where the constant C can, as before, be neglected) in terms of the dimensionless variables $\eta = h/h_m$, $\lambda = v\zeta/v_g'$, $\rho(\eta) = 9r^2(h)/16a^2h^3h_m$:

$$\partial^2\eta / \partial\lambda^2 - {}^1/_2\rho(\eta) + \eta = 0. \tag{32}$$

Equation (32) coincides in form with the equation of motion of a particle in a force field with the potential energy:

$$P(\eta) = \frac{1}{2}\left\{\eta^2 - 1 - \int_1^\eta \rho(\eta')\,d\eta'\right\}. \tag{33}$$

The arbitrary constant in $P(\eta)$ is chosen such that the potential is continuous at the point $\eta = 1$. The shape of $P(\eta)$ is shown in Fig. 1 (the integral in (33) was computed numerically). The amplitude of the envelope of the wave will behave in the same way as the coordinate of a "particle" oscillating in the potential shown; therefore, the profile of the wave is periodic, undamped oscillations with a spatial period $\sim l$. The resulting profile of the amplitude is shown in Fig. 2. It has the form of a shock wave with undamped oscillations behind the front, the characteristic period of these oscillations being the same as the width of the transition layer, i.e., of the order of the length (17) over which the nonlinear effects are comparable with the dispersion effects. The modulation index behind the wave front is quite high: it is equal to the distance ab in Fig. 1, i.e., of the order of half the maximum amplitude.

[3)]It follows from (21) and (18) that the amplitude of the wave should, in our case, satisfy the condition $l \gg v_0\tau_m \gg \beta l$.

[4)]The auxilliary constant h_0 then drops out from the solution.

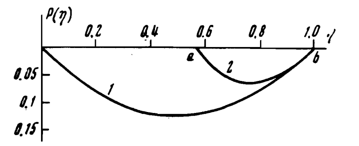

FIG. 1. "Potential energy" $P(\eta)$: 1—before the first maximum of the envelope of the shock-wave amplitude; 2—behind the wave front after the maximum.

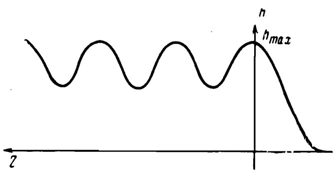

FIG. 2. Profile of the shock wave.

The periodic modulation behind the wave front leads to the appearance of an effective mean shift of the wave number by the amount $\delta k = -\overline{\varphi}'$. It follows from formula (23) (where we can set $h \sim h_m$ and $C = 0$) that $\delta k \sim v/v_g' \sim \pm 1/l$. Correspondingly, the average frequency shift $\delta\omega = v_g \delta k \sim \omega_c \gamma_L \tau_m$. This quantity is considerably higher than the nonlinear shift (20): $\delta\omega/\Delta\omega \sim (\tau_m \omega_c)/\beta \gg 1$. A frequency broadening of the spectrum $\Delta f \sim 1/\tau_m$[3] should be simultaneously observed.

Let us finally consider the results obtained from the point of view of the experiments on the observation of monochromatic whistlers emitted by ground-level transmitters with frequency $f = 15$ kHz and propagating along the lines of force of the geomagnetic field[4]. Reports of the experimental observation of this kind of amplitude modulations, which arise in the process of propagation and are apparently nonlinear in nature, have recently been published[5,6]. The frequency broadening of the spectrum in such experiments is $\Delta f = 10^2$ Hz. We must consequently assume that $\tau \sim 10^{-2}$ sec. The generation time for the satellites (which, according to[3], is of the order of γ_L^{-1}) is approximately 0.1 sec.[5)]

Thus, it is reasonable to assume $\gamma_L \tau_m \sim 0.1$. The cyclotron frequency in the equatorial region for the line of force along which the signal propagates (this region is roughly three times the earth's radius away from the earth's center), $f_c \sim 30$ kHz. The mean group velocity computed from the time lag of the signal, $v_g \sim 4 \times 10^9$ cm/sec. The length (17) computed from these data is $l \sim 2 \times 10^5$ cm. On the other hand, the nonlinear length $l_N = v_0 \tau \sim 8 \times 10^7$ cm. Thus, $l \ll l_N$ in the experiments discussed and, consequently, the observed amplitude modulation apparently corresponds to the mechanism, considered in[1], in which dispersion is unimportant. The dispersion-nonlinear mechanism considered in the present paper appears in the case of waves of much higher intensity.

In conclusion the authors take the opportunity to express their gratitude to V. L. Ginzburg for stimulating comments.

5)This time can be estimated in the following way. Ground-level transmitters emitted dot-dash Morse-code type of signals. The trigger radiations were observed almost exclusively in the dash-type pulses (duration 150 μsec); they were very rarely observed when the dots (duration ~ 50 μsec) were transmitted. Hence it follows that the generation time for the trigger radiation is ~ μsec.

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