IONIZATION LOSS OF HEAVY NONRELATIVISTIC PARTICLES IN MATTER

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Passage of a wide beam of fast heavy nonrelativistic particles through matter is discussed. Accurate distributions in energy and depth are found without assumption of small energy loss, in the case when the deflection due to multiple scattering can be neglected. The range distribution for heavy particles in matter is obtained. A simple approximate expression is obtained for the distribution function which is applicable for depths nearly equal to the total range. The limits of applicability of the results obtained are investigated.

1. The problem of passage of fast charged particles through matter has been discussed by a number of authors[1-7] for plane geometry of the scatterer and a wide beam. The distribution function f(x, T) as a function of the depth x and energy T has been calculated, at first for the case of thin absorbers and small energy loss[1,2], and later with refinements[3-5]. References to the unpublished work of Symon[6] for the case of thick absorbers with high energy loss can be found in Rossi's book[7] and in the article by Rosenzweig[8]. The results given there refer for the most part only to electrons. However, the explicit form of the distribution function itself has not been obtained up to the present time for heavy particles and thick absorbers. It is therefore of interest to find the distribution function in explicit form. We obtain below an analytic expression for the distribution function of heavy nonrelativistic particles without assuming that the energy losses are small, for the case of plane geometry, assuming that the deflection due to multiple scattering can be neglected. A simple approximate expression is obtained for the distribution function up to depths close to the total range. The distribution in particle range is found.

2. Let a wide monoenergetic flux of fast charged particles of mass $M \gg m$ with initial energy $T_0 \ll M/m$ be normally incident on the surface of a plane-parallel uniform plate (m is the electron mass, and T is the kinetic energy of the particles in units of mc^2). If the particle energy is

$$T \gg T_{\text{cr}} = 2\left(137\frac{m}{M}A^{-1/3}\right)^2, \tag{1}$$

where A is the atomic weight of the scatterer, then in a single collision event a particle cannot be scattered by an angle greater than $\vartheta_{\max}$:

$$\vartheta_{max} \approx 280A^{-1/3}\left(2\frac{M}{m}T\right)^{-1/2} \ll 1. \tag{2}$$

For this reason the mean square scattering angle $\langle\theta^2\rangle$ turns out to be small over the entire path of the particle up to its stopping, so that multiple scattering can be neglected. For protons, condition (1) is satisfied for $T > 10^{-3}$ MeV for aluminum, and for $T > 3 \times 10^{-4}$ MeV for lead.

We will designate by f(x, T)dT the number of particles per unit volume at depth x whose energy lies in the interval from T to T + dT. It is well known that f(x, T) satisfies the equation

$$\frac{\partial f}{\partial x} = -n_0\int_0^{\varepsilon_{max}} f(x,T)\frac{d\sigma(T,\varepsilon)}{d\varepsilon}d\varepsilon + n_0\int_0^{\varepsilon^*} f(x,T+\varepsilon)\frac{d\sigma(T+\varepsilon,\varepsilon)}{d\varepsilon}d\varepsilon + \frac{S_0}{v_0}\delta(x)\delta(T-T_0), \tag{3}$$

where n_0 is the atomic density of the medium, $d\sigma(T, \epsilon)$ is the scattering cross section corresponding to the transition from a state with energy T to a state with energy $T - \epsilon$, v_0 is the initial velocity of the particle, S_0 is the number of particles incident on a unit surface of the medium per unit time, ϵ^* is the lesser of the quantities $\Delta = T_0 - T$ and ϵ_{max} (Δ is the energy lost by the particle), and ϵ_{max} is the maximum energy which the particle can transfer to an atomic electron in the ionization process. In the nonrelativistic case[9]

$$\varepsilon_{max} = 4\frac{m}{M}T. \tag{4}$$

The function f(x, T) must satisfy the conditions: f(x, T) = 0 for x < 0 and f(x, T) = 0 for T < 0 and $T > T_0$.

The transport equation (3) has been solved in the small-energy-loss approximation[1,2], where $\Delta \ll T_0$. In this case in Eq. (3) we can approximately set $d\sigma(T, \epsilon) \approx d\sigma(T_0, \epsilon)$. If $\Delta \sim T_0$, replacement of T by T_0 in the transport equation is not permissible and the exact solution cannot be obtained in general form. However, if we take into account Eq. (4), we can expand the function $f(x, T + \epsilon)$ in a series in the small quantity ϵ. If we confine ourselves to the first three terms of the expansion in the region $\Delta > \epsilon_{max}$, we obtain instead of Eq. (3)

$$-\frac{1}{2}\overline{\varepsilon^2}(T)\frac{\partial^2 f}{\partial T^2} - \bar{\varepsilon}(T)\frac{\partial f}{\partial T} + \frac{\partial f}{\partial x} = \frac{S_0}{v_0}\delta(x)\delta(T-T_0), \tag{5}$$

where $\bar{\epsilon}(T)$ and $\overline{\epsilon^2}(T)$ are respectively the mean energy and mean square energy lost by the particle per unit path. In calculation of $\bar{\epsilon}(T)$ we can set $d\sigma(T + \epsilon, \epsilon) \approx d\sigma(T, \epsilon)$, since close collisions with large energy transfer occur relatively rarely and it is essential to include them in determination of the probable energy loss but not of the average energy loss.[10] Therefore $\bar{\epsilon}(T)$ is determined by the usual Bethe-Bloch formula[9]:

$$\bar{\varepsilon}(T) = n_0\int_0^{\varepsilon_{max}} \varepsilon\frac{d\sigma(T,\varepsilon)}{d\varepsilon}d\varepsilon = 2\pi n_0 Z r_e^2\frac{z^2 M}{Tm}\ln\frac{4mT/M}{I(Z)}, \tag{6}$$

where ze is the charge of the particle, Z is the atomic number of the medium, I(Z) is the total ionization potential of the atom, and r_e is the classical electron radius. Equation (6) was obtained on the assumption[11]

$$\frac{ze^2}{\hbar v} \ll 1, \quad T \gg \frac{M}{m} I(Z). \tag{7}$$

The first of the inequalities (7) is satisfied for protons if $T \gg 50 \times 10^{-3}$ MeV. From the second inequality (7) we obtain $T > 0.3$ MeV for aluminum and $T > 1.8$ MeV for lead. For μ mesons the second inequality (7) gives respectively $T > 0.03$ MeV and $T > 0.18$ MeV. Setting $T \approx T_0$ in the argument of the logarithm, in calculation of $\overline{\epsilon^2}(T)$ we take into account that the main contribution to the integral over ϵ is from the region of large energy transfers, so that the atomic electrons can be considered free[7,9]. Using the well known formula for the cross section for scattering of heavy nonrelativistic particles by free electrons[7]

$$\frac{d\sigma(T, \varepsilon)}{d\varepsilon} = \pi Z r_e^2 \frac{z^2 M}{Tm} \frac{1}{\varepsilon^2}, \tag{8}$$

we obtain

$$\overline{\varepsilon^2} = \int_0^{\varepsilon_{max}} \varepsilon^2 \frac{d\sigma(T+\varepsilon, \varepsilon)}{d\varepsilon} d\varepsilon = \pi n_0 Z r_e^2 \frac{z^2 M}{m} \ln\left(1 + 4\frac{m}{M}\right) \approx 4\pi n_0 Z r_e^2 z^2 \tag{9}$$

i.e., $\overline{\epsilon^2}$ does not depend on energy.

Equation (5) has been solved for small depths[7], when it is possible to set $T \approx T_0$ in Eq. (5). On this assumption we obtain for f(x, T) the usual Gaussian distribution ($\overline{\epsilon_0} = \overline{\epsilon}(T_0)$)

$$f(x, T) \approx \frac{S_0}{v_0} \frac{1}{\sqrt{2\pi x \overline{\varepsilon^2}}} \exp\left\{-\frac{(T_0 - T - \overline{\varepsilon}_0 x)^2}{2x\overline{\varepsilon^2}}\right\}. \tag{10}$$

3. In solution of Eq. (5) it is convenient to transform to the new variables

$$u = T/T_0, \quad \xi = x/R_0; \tag{11}$$

$$R_0 = \int_0^{T_0} \frac{dT}{\overline{\varepsilon}(T)} \approx \frac{m}{M} T_0^2 \left[4\pi n_0 Z r_e^2 z^2 \ln \frac{4mT_0/M}{I(Z)}\right]^{-1}. \tag{12}$$

In the new variables $\overline{\epsilon} = (2u)^{-1}$ and $\overline{\epsilon^2} = (2\nu)^{-1}$, where the parameter ν is defined by the expression

$$\nu = \frac{1}{2} \frac{M}{m} \ln \frac{4mT_0}{I(Z)M}. \tag{13}$$

Taking into account Eq. (13), we can transform the expression for the range (12) to the form

$$R_0 = 0{,}833 \frac{T_0^2}{\nu} \frac{A}{Zz^2} \frac{1}{\rho} \ [\mathrm{cm}], \tag{14}$$

where ρ is the density of the scatterer material in g/cm^3.

Defining a dimensionless distribution function $f(\xi, u)$ by the relation

$f(\xi, u) = 0$, if $\xi < 0$, or $f(\xi, u) = 0$, if $u < 0$ and $u > 1$.

we write the transport equation (5) in the form

$$-\frac{1}{4\nu} \frac{\partial^2 f(\xi, u)}{\partial u^2} - \frac{1}{2u} \frac{\partial f}{\partial u} + \frac{\partial f}{\partial \xi} = S_0 \frac{R_0}{v_0} \delta(\xi) \delta(1-u), \tag{15}$$

where $f(\xi, u) = 0$ if $\xi < 0$, and $f(\xi, u) = 0$ if $u < 0$ or $u > 1$.

As can be seen from Eq. (15), the dependence of the distribution function $f(\xi, u)$ on the properties of the medium and the properties of the particles being scattered is determined, in the approximation considered, by the single parameter ν. The value of ν in general does not depend on the charge of the scattered particles and depends very weakly on T_0 and Z. Since $\nu \sim M/m$, we have $\nu \gg 1$. For example, for a change of proton energy from 10 to 100 MeV the value of ν changes in the range 4600 to 6700 for aluminum and 2600 to 4800 for lead.

If we formally set $\nu = \infty$ in Eq. (15), we obtain the equation for the distribution function $f_0(\xi, u)$ in the continuous-loss approximation:

$$-\frac{1}{2u} \frac{\partial f_0}{\partial u} + \frac{\partial f_0}{\partial \xi} = S_0 \frac{R_0}{v_0} \delta(\xi) \delta(1-u). \tag{16}$$

The solution of this equation has the form ($u \geq 0$)

$$f_0(\xi, u) = 2S_0 v_0^{-1} R_0 \eta(1-u) \delta(1 - \xi - u^2), \tag{17}$$

where η is a unit function.

It can be seen from (17) that in this approximation the particle at depth ξ has with certainty an energy

$$u(\xi) = \sqrt{1-\xi}. \tag{18}$$

It will be shown below that the energy value (18) agrees rather well up to large depths $\xi \sim 1$ with the most probable energy u_{mp} at depth ξ. It is just this fact which is decisive for use of the continuous-loss model. However, the distribution function $f_0(\xi, u)$ does not describe the energy loss fluctuations and the spread in the particle range in the material. In this approximation all particles up to their stopping travel the same path length $\xi = 1$, i.e., $x = R_0$. Thus, R_0 has the meaning of range in the continuous-loss approximation.

The term with the second derivative in the transport equation takes into account the statistical spread in energy and, as a consequence, the spread in range. Since the coefficient of the second derivative with respect to energy in Eq. (15) is small, it is evident beforehand that the distribution function has a well defined peak, and only at great depth does the smearing of the distribution function become significant.

Let us look for a solution of Eq. (15) in the form

$$f(\xi, u) = S_0 \frac{R_0}{v_0} u^{-\nu + 1/2} \Phi(\xi_\nu; u), \quad \xi_\nu = \frac{\xi}{4\nu}. \tag{19}$$

For the function $\Phi(\xi, u)$ we obtain the equation

$$\frac{\partial^2 \Phi}{\partial u^2} + \frac{1}{u} \frac{\partial \Phi}{\partial u} - \frac{(\nu - 1/2)^2}{u^2} \Phi - \frac{\partial \Phi}{\partial \xi_\nu} = -\delta(\xi_\nu) \delta(1-u). \tag{20}$$

Using the Hankel transformation in the variable u, it is easy to show that in the region $\xi > 0$ for $0 \leq u \leq 1$

$$f(\xi, u) = S_0 \frac{R_0}{v_0} u^{-\nu + 1/2} \int_0^\infty J_{\nu - 1/2}(yu) J_{\nu - 1/2}(y) \exp\left\{-\frac{\xi}{4\nu} y^2\right\} y\, dy, \tag{21}$$

where J is the Bessel function. Carrying out the integration over y, we obtain the following expression for the distribution function:

$$f(\xi, u) = 2S_0 \frac{R_0}{v_0} \frac{\nu}{\xi} u^{-\nu + 1/2} I_{\nu - 1/2}\left(2\nu \frac{u}{\xi}\right) \exp\left\{-\nu \frac{1+u^2}{\xi}\right\}, \tag{22}$$

where $I_{\nu - 1/2}$ is the Bessel function of imaginary argument.

Let us calculate the distribution of particles in energy without regard to the depth:

$$N(u)\, du = du \int_0^\infty f(\xi, u)\, d\xi = 2S_0 \frac{R_0}{v_0} \frac{2\nu}{2\nu - 1} du. \tag{23}$$

We see that the distribution obtained does not depend on the particle energy. This result is the direct consequence of neglecting the deflection due to multiple scattering. The total number of particles in the entire half-space $\xi > 0$ is

$$N=\int_0^1 N(u)\,du = 2S_0\frac{R_0}{v_0}\frac{2\nu}{2\nu-1}. \tag{24}$$

Now we easily find the normalized probability distribution function:

$$F(\xi,u)=\frac{1}{N}f(\xi,u)=\frac{\nu-{}^1/_2}{\xi}u^{-(\nu-{}^1/_2)}\exp\left\{-\nu\frac{1+u^2}{\xi}\right\}I_{\nu-{}^1/_2}\left(2\nu\frac{u}{\xi}\right), \tag{25}$$

where $F(\xi, u)d\xi du$ is the probability of finding a particle in the depth interval from ξ to $\xi + d\xi$ and the energy interval from u to u + du. The function F satisfies the normalization equation

$$\int_0^\infty d\xi\int_0^1 du\,F(\xi,u)=1. \tag{26}$$

4. It is of interest to consider a number of limiting cases in which the expression for $F(\xi, u)$ is substantially simplified. In the region of small depths the energy loss is small, and therefore $u_{eff} \sim 1$. If the argument of the function $I_{\nu-1/2}$ is greater than the square of its index, i.e.,

$$0\leqslant\xi\ll 2/\nu, \tag{27}$$

then instead of $I_{\nu-1/2}$ we can use its asymptotic expression[12] and obtain

$$F(\xi,u)\approx\frac{1}{2}\left(\frac{\nu}{\pi\xi}\right)^{1/2}\exp\left\{-\nu\left[\ln u+\frac{(1-u)^2}{\xi}\right]\right\}. \tag{28}$$

In the region of effective energy values

$$1\geqslant u\geqslant 1-\nu^{-1/2} \tag{29}$$

Eq. (28) goes over to the Gaussian distribution (10), which in the variables ξ, u has the form

$$F(\xi,u)\approx\frac{1}{2}\left(\frac{\nu}{\pi\xi}\right)^{1/2}\exp\left\{-\nu\frac{[u-(1-\xi/2)]^2}{\xi}\right\}. \tag{30}$$

Thus, the inequalities (27) and (29) determine the region of applicability of the Gaussian distribution.

In the region of large depths, $u_{eff}\xi^{-1} \lesssim 1$. Therefore, using the well known formula of Meissel[13,14]

$$I_\mu(\mu x)\approx\frac{(\mu x)^\mu e^{-\mu}}{\Gamma(\mu+1)}\frac{\exp\{\mu(1+x^2)^{1/2}\}}{(1+x^2)^{1/4}[1+(1+x^2)^{1/2}]^\mu},\quad \mu\gg 1, \tag{31}$$

where $\Gamma(\mu + 1)$ is Euler's gamma function, after simple transformations we obtain

$$F(\xi,u)=\frac{1}{2}\left(\frac{\nu-{}^1/_2}{\pi\xi}\right)^{1/2}\frac{[1+(1+4u^2/\xi^2)^{1/2}]^{1/2}}{[1+4u^2/\xi^2]^{1/4}}\exp\{-\nu B(\xi,u)\}, \tag{32}$$

$$B(\xi,u)=\frac{1+u^2}{\xi}-\left(1+4\frac{u^2}{\xi^2}\right)^{1/2}+\ln\frac{\xi[1+(1+4u^2/\xi^2)^{1/2}]}{2}. \tag{33}$$

We see from (32) that since $\nu \gg 1$ the peak of the distribution function at depth ξ is determined mainly by the minimum of the expression (33). Solving the equation dB/du = 0, we find

$$u_{\text{mp}}(\xi)=(1-\xi)^{1/2},\quad \xi\leqslant 1, \tag{34}$$

which is identical with (18). Inclusion of the factor in front of the exponential in (32) gives a correction to (34) of the order ν^{-1}. Thus, to depths $\xi \leq 1$ the distribution function has a local maximum corresponding to an energy $u = (1-\xi)^{1/2}$. In the region of greater depths the distribution function generally does not have a local maximum, and the most probable energy is the energy u = 0. If in the region $0 \leq \xi < 1$ we expand $B(u, \xi)$ in a series in powers of the difference $[u-(1-\xi)^{1/2}]$ and retain the first three terms of this expansion, we obtain the following simple expression for the distribution function:

$$F(\xi,u)\approx\frac{1}{2}\left(\frac{2\nu}{\pi[1-(1-\xi)^2]}\right)^{1/2}\exp\left\{-2\nu\frac{1-\xi}{1-(1-\xi)^2}[u-(1-\xi)^{1/2}]^2\right\}. \tag{35}$$

The region of applicability of this function is determined by the condition

$$\xi\ll 1-1/\sqrt{2\nu}. \tag{36}$$

Thus, over a wide range of depths the distribution function is determined by Eq. (35) of the Gaussian type. We note that for small ξ Eq. (35) goes over to (30).

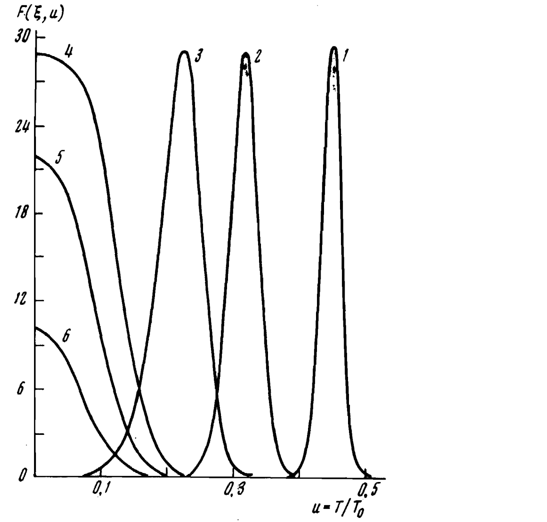

FIG. 1. Energy spectrum for protons with initial energy 20 MeV in aluminum at depths: 1–0.8 R_0, 2–0.9 R_0, 3–0.95 R_0, 4–R_0, 5–1.01 R_0, 6–1.02 R_0.

In Fig. 1 we have shown energy spectra for protons with initial energy T_0 = 20 MeV in aluminum (ν = 5240, $R_0 \approx 0.196$ cm). As can be seen, up to depths $x \sim 0.9\ R_0$ the curves are symmetric about the most probable energy. Therefore at such depths the particle distribution in energy can be quite accurately described by Eq. (35). With increasing scatterer thickness the asymmetry of the particle-energy-spectrum curves increases, so that Eq. (35) becomes inapplicable. In this case it is necessary to use Eq. (32) to find the energy spectrum.

Curves 5 and 6 of Fig. 1 show the energy spectrum of particles at depths greater than R_0, i.e., in the region which is inaccessible in the continuous-loss model. We see that the probability of observing a particle falls off rapidly at depths greater than R_0.

In Fig. 2 we have shown energy spectra for μ mesons with T_0 = 20 MeV for aluminum (ν = 820, $R_0 \approx 1.42$ cm). For mesons the asymmetry of the distribution function is appreciably more important and begins to appear at depths less than occurs for protons, so that the region of applicability of Eq. (35) is limited to depths $x \lesssim (0.7-0.8)R_0$. At large depths the distribution function has a characteristic tail in the region of energies less than the most probable energy. Since the energy spread of the particles is quite significant for mesons,

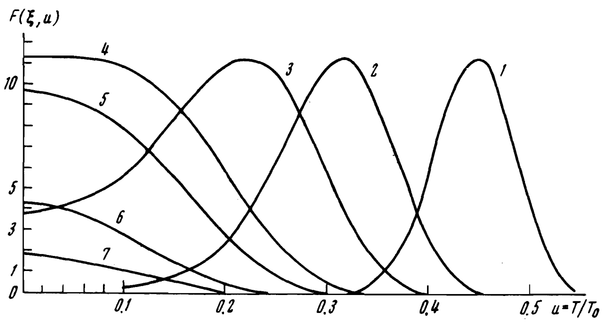

FIG. 2. Energy spectrum for μ mesons with initial energy 20 MeV in aluminum at depths: 1–0.8 R_0, 2–0.9 R_0, 3–0.95 R_0, 4–R_0, 5–1.02 R_0, 6–1.05 R_0, 7–1.07 R_0.

use of the continuous-loss model for these particles is limited to depths $x \lesssim 0.7-0.8$ of the range R_0.

5. It is well known that particles with the same energy, as the result of statistical fluctuations, traverse different paths in matter before coming to rest. Let W(r)dr be the probability that a particle completely loses its energy at a depth from r to r + dr; r is the range of the particle. As shown by Bohr[15], the range straggling of fast particles will be mainly determined by the initial part of the path, where the particle velocity is large and the principal energy-loss mechanism is the collision of particles with atomic electrons. Therefore the range distribution can be obtained if in Eq. (25) we formally go to the limit u → 0; taking into account that for small u

$$I_{\nu-1/2}\left(2\nu\frac{u}{\xi}\right)\approx\frac{1}{\Gamma(\nu+1/2)}\left(\frac{\nu u}{\xi}\right)^{\nu-1/2},$$

we find

$$W(r)=\frac{\nu^{\nu-1/2}}{\Gamma(\nu-1/2)}\frac{1}{\sqrt{r}}\exp\left\{-\nu\left[\frac{1}{r}+\ln r\right]\right\}. \qquad (37)$$

Making use of the fact that $\nu \gg 1$ and using the well known asymptotic expression for the gamma function

$$\Gamma(\nu-1/2)\approx(2\pi)^{1/2}\nu^{\nu-1}e^{-\nu},$$

we will write Eq. (37) in the form

$$W(r)=\left(\frac{\nu}{2\pi r}\right)^{1/2}\exp\left\{-\nu\left[\frac{1}{r}+\ln r-1\right]\right\}. \qquad (38)$$

Solving the equation dW/dr = 0, we find for the most probable range the value

$$r_{\text{mp}}=\frac{2\nu}{2\nu+1}\approx 1-\frac{1}{2\nu}<1. \qquad (39)$$

Using Eq. (37), let us calculate the moments of the distribution W(r). After simple calculations we obtain for the n-th moment

$$\overline{r^n}=\int\limits_0 r^n W(r)\,dr=\nu^n\frac{\Gamma(\nu-n-1/2)}{\Gamma(\nu-1/2)}. \qquad (40)$$

Setting n = 1 in Eq. (40), we obtain an expression for the mean range

$$\bar{r}=\nu\frac{\Gamma(\nu-3/2)}{\Gamma(\nu-1/2)}=\frac{\nu}{\nu-3/2}\approx 1+\frac{3}{2}\nu>1. \qquad (41)$$

Comparing (41) and (39), we see that $\bar{r} > r_{\text{mp}}$. This means that the function W(r) is more drawn out for values $r > r_{\text{mp}}$. For ranges less than the most probable range, W(r) drops more rapidly. Near the most probable value (39) the range distribution function W(r) can be written approximately in the form

$$W(r)\approx\left(\frac{\nu}{2\pi}\right)^{1/2}\exp\left\{-\nu\frac{(r-r_{\text{mp}})^2}{2}\right\}\left[1+\frac{2\nu}{3}(r-r_{\text{mp}})^3\right]. \qquad (42)$$

If we neglect the second term in the square brackets, we obtain the ordinary Gaussian distribution. The presence of the second term reflects the asymmetry in the distribution W(r). The difference of (38) from a Gaussian distribution is due to the fact that successive collisions of particles with the atoms of the medium are not statistically independent.

The mean square range (n = 2) is

$$\overline{r^2}=\frac{\nu^2}{(\nu-5/2)(\nu-3/2)}\approx 1+\frac{4}{\nu}. \qquad (43)$$

The relative range straggling (in %) is

$$\Omega=\frac{(\overline{r^2}-\bar{r}^2)^{1/2}}{\bar{r}}100=\frac{100}{\sqrt{\nu-5/2}}=\frac{100}{\sqrt{\nu}}. \qquad (44)$$

Thus, the relative straggling for heavy particles is small as the result of the large values of ν, while, since $\Omega \sim (m/M)^{1/2}$, the relative straggling increases with decreasing particle mass. It should be mentioned that Ω is a weak function of the initial particle energy T_0 and the ionization potential of the atoms of the medium I(Z). The quantity Ω was calculated by Symon[6] and his numerical results for iron are given by Rossi[7]. For protons with energy T_0 = 92 MeV, Ω = 1.26%; and for T_0 = 18 MeV, Ω = 1.48%. Calculating the value of Ω from Eq. (44), we obtain for the same energy values respectively Ω = 1.36% and Ω = 1.52%. The ionization potential of iron was taken as[10] I ≈ 330 eV.

For μ mesons (M/m = 210) with energy 30 MeV, Ω = 3.20% for beryllium (I = 60 eV) and Ω = 4.15% for lead (I ≈ 950 eV). The calculations of relative straggling for μ mesons made by Sternheimer[16] give Ω respectively as 3.03% and 3.84%.

6. In calculation of the distribution function (25), the following assumptions were made: scattering of particles was completely neglected; the integral transport equation (3) was replaced by the differential equation (5); in expansion of the function $f(x, T + \epsilon)$, terms $\partial^3 f/\partial T^3$ and so forth were discarded.

The deflection in scattering can be neglected if the mean square deflection angle is small in the region of depths ξ and energies u considered, i.e., $\langle\theta^2(\xi)\rangle \ll 1$. Since

$$\langle\theta^2(\xi)\rangle=R_0\int\limits_0^{\xi}\langle\theta_s^2(\xi')\rangle d\xi', \qquad (45)$$

where $\langle\theta_S^2\rangle$ is the mean square angle of deflection of a particle in a unit length at depth ξ, which according to Rossi[7] is equal to

$$\langle\theta_s^2(\xi)\rangle=\frac{1}{u^2(\xi)}\frac{4\pi n_0 Z(Z+1)z^2r_e^2}{T_0^2}\ln(183Z^{-1/3}), \qquad (46)$$

then, substituting (46) into (45), we obtain after integration

$$\langle\theta^2(\xi)\rangle=2\frac{R_0}{l_{tr0}}\ln\frac{1}{1-\xi}. \qquad (47)$$

Here $l_{\text{tr}_0} = 2\langle\theta_S^2\rangle_0^{-1}$ is the transport scattering length for particles with initial energy T_0. Taking into account that the maximum depth to which our discussion is applicable is limited by the condition $u_{eff} \gg MI(Z)/mT_0$, i.e.,

$$\xi \ll \xi_{max} = 1 - \left[\frac{M}{m}\frac{I(Z)}{T_0}\right]^2, \tag{48}$$

it is easy to see, substituting ξ_{max} into (47), that $\langle\theta^2(\xi_{max})\rangle \ll 1$. The integral equation (3) can be replaced by Eq. (5) if the "effective" width of the distribution function is much greater than ϵ_{max}. Using (4) and (35), we find

$$\xi \gg 16(m/M)^2\nu. \tag{49}$$

We note that inequality (49) is incompatible with (27), which determines the condition for which the distribution function (25) goes over to a Gaussian distribution in the form (30). Then, since in expansion of $f(x, T + \epsilon)$ in series in ϵ we retained only the first three terms of the expansion, it is necessary that

$$\left|\overline{\varepsilon^3}\frac{\partial^3 f}{\partial u^3}\right| \ll \left|\overline{\varepsilon^2}\frac{\partial^2 f}{\partial u^2}\right|. \tag{50}$$

By use of Eq. (8) it is easy to show that $\overline{\epsilon^3} = \overline{2\epsilon^2}mu/M$. By using the explicit expression for the function $f(\xi, u)$ we can show that when inequality (49) is satisfied, condition (50) is automatically satisfied.

7. Thus, the normalized distribution function in depth x and energy T for heavy nonrelativistic particles passing through a uniform medium has the form

$$F(x,T) = \frac{\nu - 1/2}{T_0 x}\left(\frac{T}{T_0}\right)^{-\nu+1/2} I_{\nu-1/2}\left(2\nu\frac{R_0 T}{T_0 x}\right)\exp\left\{-\nu\frac{R_0}{x}\left[1+\left(\frac{T}{T_0}\right)^2\right]\right\}, \tag{51}$$

where the quantities R_0 and ν are determined by Eqs. (13) and (14). At depths $x \ll R_0[1 - (2\nu)^{-1/2}]$ Eq. (51) is considerably simplified. At such depths the distribution function can be represented in the form

$$F(x,T) = \frac{1}{T_0}\left[\frac{\nu}{2\pi x(2R_0 - x)}\right]^{1/2}\exp\left\{-2\nu\frac{R_0(R_0-x)}{x(2R_0-x)}\left[\frac{T}{T_0} - \sqrt{1-\frac{x}{R_0}}\right]^2\right\}. \tag{52}$$

The expression (52) is symmetric about the most probable energy $T_{mp}(x)$ for depths $x < R_0$:

$$T_{mp}(x) = T_0(1 - x/R_0)^{1/2} \tag{53}$$

and the effective width δT of the distribution at depth x is

$$\delta T = |T - T_{mp}(x)|_{eff} \approx T_0\left[\frac{x}{\nu(R_0 - x)}\right]^{1/2}. \tag{54}$$

At greater depths $x \gtrsim R_0(1 - (2\nu)^{-1/2})$ the distribution function can be calculated from the formula

$$F(x,T) \approx \frac{1}{2T_0}\sqrt{\frac{\nu}{\pi R_0 x}}\frac{[1+\sqrt{1+4(R_0T/T_0x)^2}]^{1/2}}{[1+4(R_0T/T_0x)^2]^{1/4}}\exp\{-\nu B(x,T)\}, \tag{55}$$

where

$$B(x,T) = \frac{R_0}{x}\left[1+\left(\frac{T}{T_0}\right)^2\right] - \sqrt{1+4\left(\frac{R_0 T}{T_0 x}\right)^2} + \ln\frac{x}{2R_0}\left[1+\sqrt{1+4\left(\frac{R_0T}{T_0x}\right)^2}\right]. \tag{56}$$

It should be noted that Eqs. (51)—(56) describe the distribution of particles in depth and energy for the condition

$$x \gg 8R_0\frac{m}{M}\ln\frac{4mT_0/M}{I(Z)}. \tag{57}$$

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