THE "IMAGINARY-TIME" METHOD IN PROBLEMS CONCERNING THE IONIZATION OF ATOMS AND PAIR PRODUCTION

V. S. POPOV

Institute of Theoretical and Experimental Physics

Submitted May 31, 1972

Zh. Eksp. Teor. Fiz. **63**, 1586–1598 (November, 1972)

The "imaginary-time" method is formulated in a form which is most general and convenient for applications. The relation between atomic ionization processes induced by an intense light wave and pair production from vacuum due to the influence of an electric field F(t) is clarified. A closed expression for the ionization probability w, valid for arbitrary dependence of F(t) on the time, is found for the case of linear polarization. The problem of calculating the Coulomb corrections to the probability w is discussed.

1. INTRODUCTION

THE "imaginary-time" method was originally proposed[1,2] in order to solve the problem of the ionization of nonrelativistic bound systems (atoms, ions) in the field of an intense light wave. Being a generalization of the quasiclassical WKB approximation to the case of time-dependent fields, this method describes the tunneling transition of an electron from a bound state to the continuum by using the classical equations of motion, but with an imaginary time. Such an approach has also turned out to be useful in connection with investigations of ionization in Coulomb collisions.[3] Furthermore, it has been established that the "imaginary-time" method works successfully in calculating the probability for pair production from vacuum due to an external field.[4,5] However, the relation between these two problems has remained unclear.

It is shown in the present article that the ionization of atoms and the production of e^+e^- pairs from vacuum correspond to two limiting cases of the general relativistic problem of the ionization of a level with arbitrary energy E ($mc^2 > E \geq -mc^2$) due to the influence of an external electric field. Such a generalization, which is of interest by itself, possesses a number of advantages. In particular, by going to the nonrelativistic limit one can obtain a closed expression for the atomic ionization probability w in an arbitrary electric field (see formulas (15) and (21) given below), a result which had not previously been obtained. These formulas greatly simplify the calculation of the probability w, and allow us to investigate its dependence on the shape of the pulse, which is of interest in connection with the development of laser technology and the production of ultrashort pulses of light (see the review article by B. Zel'dovich and Kuznetsova[6]).

2. THE "IMAGINARY-TIME" METHOD

Let F denote the amplitude of the electric field and let ω be its characteristic frequency; let us set s = ωt,

$$F(t) = F\varphi(s), \qquad (1)$$

where $|\varphi(s)| \leq \varphi(0) = 1$ (it is convenient to assume that t = 0 is the instant when the field is maximum; at this instant the particle emerges from the barrier[1,2]). For simplicity we confine our attention to the case when the field F(t) is uniform and does not change its direction in space (the case of linear polarization). The probability for the ionization of a level with energy E is determined by the barrier penetration and is given by (in the case of an oscillating barrier)

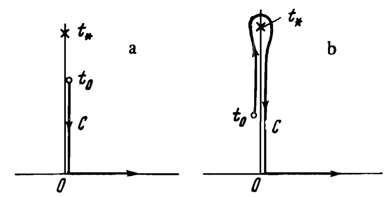

FIG. 1. The variation of the "time" t during the course of the particle's subbarrier motion: a—for $\epsilon_0 > 0$, b—for $\epsilon_0 < 0$. The point t_0 is the initial instant of time, and t_* denotes the branch point of the action W.

$$w_p = D \exp\{-2\,\mathrm{Im}\,W(\mathbf{p})\}, \qquad (2)$$

$$W(\mathbf{p}) = W(0, t_0) = \int_{t_0}^{0} (L+E)\,dt - \mathbf{pr}\,|_{t_0}^{0}, \qquad (3)$$

where L is the Lagrangian. The integration here is taken along the subbarrier trajectory connecting the initial state (an electron localized near an atom) with the final state (an electron emitted with momentum **p**).

Taking into consideration that $\dot{\mathbf{p}} = e\mathbf{F}(t)$ in a uniform field, we obtain[1)]

$$W(0, t_0) = m \int_{t_0}^{0} dt[\varepsilon_0 - \varepsilon(t)], \quad \varepsilon = (1 - \mathbf{v}^2)^{-1/2}. \qquad (4)$$

In the subbarrier motion the "time" t flows along the contour C (see Fig. 1). The initial moment of time, t_0, is determined from the conditions

$$\varepsilon(t_0) = \varepsilon_0 = E/m, \qquad (5)$$

where E denotes the energy of the level ($-1 < \epsilon_0 < 1$).

As long as $\epsilon_0 > 0$ the picture is generally similar to the nonrelativistic situation. For $\epsilon_0 < 0$ the contour C

[1)]Here and also in what follows ħ = c = 1, and it is assumed that $F \ll F_0$, $\omega \ll m$. The Coulomb interaction of the electron with the atomic core is neglected (until sec. 6), i.e., the derived formulas pertain to the ionization of negative ions of the type He^-, etc.

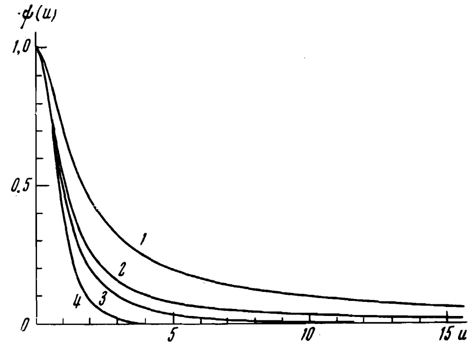

FIG. 2. Shape of the function $\psi(u)$. The numericals 1–4 used to label the curves correspond to examples 1–4 from Table I.

passes around the branch point $t = t_*$ of the function W, this being the point at which

$$\varepsilon(t_*) = 0, \text{ or } p(t_*) = im. \tag{6}$$

As $\epsilon_0 \to -1$ the contour C takes the form of a loop encompassing the point t_*, which corresponds to pair production from the vacuum.[4] Introducing the parameter γ characterizing the adiabatic nature of the subbarrier motion (ω_t denotes the tunneling frequency),

$$\gamma = \omega/\omega_t = m\omega/eF, \tag{7}$$

and assuming $p(t) = imu$, we transform expression (2) to the form

$$w_p = D\exp\left\{-\frac{F_0}{F}\left[g(\gamma) + c_1(\gamma)\frac{p_\perp^2}{m^2} + c_2(\gamma)\frac{p_\parallel^2}{m^2}\right]\right\}, \tag{8}$$

where $F_0 = (m^2/e)K(\epsilon_0)$ is the field strength which is characteristic for a level whose binding energy is $m(1-\epsilon_0)$. The frequency dependence of the ionization probability is determined by the function $g(\gamma)$:

$$g(\gamma) = \frac{2}{K(\varepsilon_0)}\int_0^{u_0} du\,\psi(\gamma u)\left[(1-u^2)^{1/2} - \varepsilon_0\right],$$
$$K(\varepsilon) = \arccos\varepsilon - \varepsilon(1-\varepsilon^2)^{1/2}, \tag{9}$$

where $\epsilon_0 = (1-u_0^2)^{1/2}$. Here the dependence of the probability w on the shape of the external field only enters through the function $\psi(u)$, whose definition and fundamental properties we investigate in the Appendix. For a constant field $\gamma = 0$, $g(0) = 1$, and as a rule the function $\psi(u)$ monotonically decreases with increasing values of u (see Fig. 2).

Sometimes it is possible to evaluate the integral (9) in closed form. We shall confine our attention to two examples.

1) Let $\varphi = \cos\omega t$. In the nonrelativistic case this corresponds to monochromatic laser light, and in the relativistic case—it corresponds to the field in a standing light wave.[2)] Here

$$\psi(u) = (1+u^2)^{-1/2},$$
$$g(\gamma) = \frac{2\sqrt{1+\gamma^2}}{\gamma^2 K(\varepsilon_0)}\{\mathrm{F}(\alpha,k) - \mathrm{E}(\alpha,k) - k\varepsilon_0\chi(\gamma\sqrt{1-\varepsilon_0^2})\}, \tag{10}$$

where F and E are the elliptic integrals of the first and second kind,

$$\alpha = \arccos\frac{\varepsilon_0}{(1+\gamma^2-\varepsilon_0^2\gamma^2)^{1/2}},\quad k = \frac{\gamma}{(1+\gamma^2)^{1/2}},\quad \chi(x) = \operatorname{Arsh}x - \frac{x}{(1+x^2)^{1/2}}.$$

When $\epsilon_0 = -1$ we have: $\alpha = \pi$, and expression (10) agrees with formula (25) from[4]. In the nonrelativistic limit expression (10) goes over into the well-known Keldysh formula.[8]

2) $\varphi = (\cosh s)^{-2}$ is an example of a pulsed field. This case is unusually simple from the analytical point of view:

$$g(\gamma) = \frac{2}{\gamma^2 K(\varepsilon_0)}\left[\alpha(1+\gamma^2)^{1/2} - \alpha_0 - \varepsilon_0\gamma\alpha_1\right] \tag{11}$$

(α is defined above, $\alpha_0 = \cos^{-1}\epsilon_0$, and $\alpha_1 = \tan^{-1}(\gamma\sqrt{1-\epsilon_0^2})$).

Expressions (9)–(11) determine the dominant (exponential) factor in the probability w. In order to determine this factor it was sufficient to investigate the extremal subbarrier trajectory on which Im W reaches its minimum value (and the probability of tunneling is maximum). In order to determine the factor D appearing in front of the exponential and the coefficients c_1 and c_2 in formula (8), in addition to the extremal trajectory one must also consider a beam of classical trajectories lying close to it. In this connection the initial point t_0 is displaced from the imaginary axis; however, in the quasiclassical case this displacement is small (see Eq. (A.10)), and as a consequence of this we can apply the "imaginary-time" method as usual. In view of the somewhat cumbersome nature of the expressions which appear, we shall not give them in explicit form, but we shall cite the results for the two limiting cases ($\epsilon_0 \to \pm 1$) which are of the greatest physical interest.

3. IONIZATION OF ATOMS BY AN INTENSE LIGHT WAVE[3)]

The theory of this process has been investigated by many authors, starting with Keldysh[8] (see[9-12] and also[1,2]). The results of the previous section enable us to obtain a general formula for the ionization probability w, which is applicable for an arbitrary field. Since the electron motion is nonrelativistic, and the wavelength of the light is much larger than the atom's radius, then the field of the wave (of arbitrary polarization and spectral composition) effectively amounts to an oscillating electric field F(t). Since the electron velocity in the subbarrier motion is much smaller than c, then one should redefine the quantities ω and γ according to Keldysh:[8]

$$\omega_t = \frac{eF}{\varkappa},\quad \gamma = \frac{\omega\varkappa}{eF} = 0{,}75\frac{\omega}{I}\frac{F_0}{F} \tag{7'}$$

($I = \kappa^2/2$ is the ionization potential and $F_0 = (2/3)\kappa^3$).

We obtain the following result from Eq. (9) in the nonrelativistic limit $\epsilon_0 \to 1$:

$$g(\gamma) = \frac{3}{2}\int_0^1 \psi(\gamma u)(1-u^2)\,du, \tag{12}$$

where $\psi(u)$ is the same function as above (see the Appendix). A simplification arises due to the fact that in

[2)]In the relativistic problem it is generally impossible to neglect the influence of the magnetic field on the subbarrier trajectory. For example, in the limiting case $\epsilon_0 = -1$ (that is, for the case of pair production), the influence of the magnetic field leads to the result that w = 0 in the case of a plane wave. [7] However, a field of the type we are considering is formed at the antinodes of the high-intenisty standing wave which can be obtained by using two laser beams.

[3)]Atomic units are used in this section: $e = \hbar = m = 1$ (m denotes the electron mass). Here the field strength is measured in units of $F_a = m^2e^5/\hbar^4 = 5.1 \times 10^9$ V/cm.

the nonrelativistic regime the point t_0 is located much closer to the real axis than t_* is. We determine the momentum spectrum of the emitted electrons from the expression

$$W(0,t_0) = -\frac{1}{2}\int_{t_0}^{0}(\mathbf{p}^2+\varkappa^2)\,dt. \tag{13}$$

Assuming that $\mathbf{r}(t) = \mathbf{r}_0 + \delta\mathbf{r}$, where $\mathbf{r}_0(t)$ represents the extremal trajectory, we find: $\delta\dot{\mathbf{r}}(t) = \delta\mathbf{p}(t) = \text{const}$; therefore the variation of the action W is given by

$$\delta W = W - W_0 = [\mathbf{r}_0(t_0) - \mathbf{r}_0(0)]\,\delta\mathbf{p} + {}^1/_2 c_{ij}\delta p_i \delta p_j + \ldots$$

The linear term in $\delta\mathbf{p}$ must vanish at the point corresponding to the minimum of Im W. Hence follows the condition determining the extremal trajectory:

$$\operatorname{Im}[\mathbf{r}(0) - \mathbf{r}(t_0)] = 0. \tag{14}$$

In connection with the evaluation of the c_{ij}, it is not only necessary to take the variation of the integrand in Eq. (13) into consideration, but the displacement of the initial instant t_0 (see formula (A.10)) should also be taken into account. Finally we obtain

$$\delta W = (\mathbf{p}_0\delta\mathbf{p})\,\delta t_0 + {}^1/_2[t_0(\delta\mathbf{p})^2 + \mathbf{p}_0\dot{\mathbf{p}}_0(\delta t_0)^2]. \tag{15}$$

For the case of linear polarization it follows from Eq. (14) that the extremal trajectory is one-dimensional and is directed along the field **F**. In this case w_p is determined by Eq. (2), where

$$2\operatorname{Im} W(\mathbf{p}) = \frac{2I}{\omega}\left\{f(\gamma) + c_1(\gamma)\frac{p_\perp^2}{\varkappa^2} + c_2(\gamma)\frac{p_\parallel^2}{\varkappa^2}\right\}, \tag{16}$$

$$f(\gamma) = {}^2/_3\gamma g(\gamma) = \tau(\gamma) - \frac{1}{\gamma^2}\int_0^\gamma \psi(u)\,u^2\,du, \tag{16$'$}$$

$$c_1 = \tau(\gamma) = \int_0^\gamma \psi(u)\,du, \quad c_2 = \left(1 - \gamma\frac{d}{d\gamma}\right)c_1. \tag{16$''$}$$

Here $\mathbf{p} = (p_\parallel, p_\perp)$ is the electron's momentum upon its escape from the barrier (at t = 0); the extremal trajectory corresponds to $\mathbf{p} = 0$. Upon satisfying the conditions $\omega \ll I$ and $F \ll F_0$, the effective values of p are very much smaller than κ, and therefore the next terms of the expansion in (16) are unimportant.

Let us investigate the behavior of the possibility w as the frequency ω of the light increases. In the adiabatic limit $\gamma \ll 1$, the ionization primarily takes place at the instant when the electric field is maximum, and the ionization current is characterized by sharp peaks. Assuming for $s \ll 1$ ($s = \omega t$) that

$$\varphi(s) = F(t)/F(0) = 1 - a_1 s^2 + a_2 s^4 + \ldots, \quad a_1 \geqslant 0, \tag{17}$$

with the aid of Eqs. (A.7) and (16) we find

$$g(\gamma) = 1 - {}^1/_5 a_1\gamma^2 + {}^1/_7(a_1^2 - {}^3/_5 a_2)\gamma^4 + \ldots,$$
$$c_1 = \gamma - {}^1/_3 a_1\gamma^3 + \ldots, \quad c_2 = {}^2/_3 a_1\gamma^3 + \ldots\ . \tag{18}$$

Here $c_1 \gg c_2$, that is, the distribution in $p_\parallel$ is much broader than the distribution in $p_\perp$: $p_\perp/\kappa \sim \gamma p_\parallel/\kappa \sim (F/F_0)^{1/2}$.

As a consequence of the fact that $a_1 > 0$, the function $g(\gamma)$ decreases with increasing values of γ, and the probability w begins to increase. This increase becomes especially noticeable for $\gamma \gg 1$. In this case, as is clear from Table I, the function $f(\gamma)$ either approaches a constant limit (examples 3–6) or else increases logarithmically (examples 1 and 2). Since $\psi(u)$ decreases rapidly for $u \gg 1$ (see Fig. 2a; also see formula (A.8)), it follows from Eq. (16′) that

$$f(\gamma) = \tau(\gamma) - {}^1/_2 a, \quad a = \lim_{x\to\infty}[x\psi(x)]. \tag{19}$$

for $\gamma \gg 1$. Hence, within logarithmic accuracy $c_1 \approx c_2 \approx f(\gamma)$ and

$$w(\mathbf{p}) = D\exp\left\{-\frac{2I}{\omega}f(\gamma)\left(1 + \frac{\mathbf{p}^2}{\varkappa^2}\right)\right\} = D\exp\left\{-\frac{2\varepsilon}{\omega}f(\gamma)\right\}, \tag{20}$$

where $\epsilon = (p^2 + \kappa^2)/2$. Thus, the distribution of the emitted electrons with respect to the momentum **p** becomes isotropic for $\gamma \gg 1$ and has a width $\Delta p \sim \kappa(\omega/I)^{1/2} \ll \kappa$.

In conclusion let us cite the complete formula for the ionization probability w, with the pre-exponential factor D included:

$$w = \int \frac{d^3p}{(2\pi)^3}w_{\mathbf{p}} = \frac{D}{8c_1c_2^{1/2}}\left(\frac{\omega}{\pi}\right)^{3/2}\exp\left\{-\frac{2I}{\omega}f(\gamma)\right\}, \tag{21}$$

where $D = 4\pi^2|A|^2/F|\varphi(i\tau_0)|$, $I = \kappa^2/2$, and A is the numerical coefficient in the asymptotic expression for the bound state wave function:

$$\psi_i(r) = A(\varkappa/2\pi)^{1/2}e^{-\varkappa r}/r \quad \text{for} \quad \varkappa r \gg 1$$

(for simplicity the case $l = 0$ is taken here).

Formula (21) determines the total probability of ionizing the atom during the entire time interval in which the field acts, and is applicable to that case when F(t) has the shape of a single pulse. However, if the field is periodic, then the electron's energy takes a discrete set of values, and it is necessary to introduce a δ-function under the integral sign in Eq. (21), in order to take the law of energy conservation into account. For example, in the case $\varphi(s) = \cos s$ this factor has the form

$$\frac{\omega^2}{\pi}\sum_n \delta\left(\frac{p^2 - p_n^2}{2}\right),$$
$$p_n = [2\omega(n - \nu)]^{1/2}, \quad \nu = \frac{I}{\omega}\left(1 + \frac{1}{2\gamma^2}\right),$$

where n is the number of quanta absorbed. In this con-

Example No.	$\varphi(s)$, $s = \omega t$	$\psi(u)$	$\tau(\gamma)$	$f(\gamma)$	$l(\gamma)$
1	$\cos s$	$(1+u^2)^{-1/2}$	$\operatorname{Arsh}\gamma$	$\ln 2\gamma - 1/2$	$2/\gamma$
2	$\exp(-s^2)$	см. (П. 12)	—	$(\ln\gamma)^{1/2}$	$[\gamma\ln^{1/2}\gamma]^{-1}$
3	$\operatorname{ch}^{-2}s$	$(1+u^2)^{-1}$	$\operatorname{arctg}\gamma$	$\pi/2$	$2\gamma^{-2}\ln\gamma$
4	$(1+s^2)^{-1}$	$\operatorname{ch}^{-2}u$	$\operatorname{th}\gamma$	1	$2\gamma^{-2}\ln 2$
5	$\operatorname{ch}^{-1}s$	$\operatorname{ch}^{-1}u$	$\operatorname{arctg}(\operatorname{sh}\gamma)$	$\pi/2$	$A\gamma^{-2}$
6	$(1+s^2)^{-3/2}$	$(1+u^2)^{-3/2}$	$\gamma(1+\gamma^2)^{-1/2}$	1	$2\gamma^{-2}$

Note. The quantities f and l refer to the nonrelativistic case $\gamma \gg 1$: f is the argument of the exponential appearing in Eq. (21), and l is the barrier width defined by Eq. (24). $A = 2\int_0^\infty (x\,dx/\sinh x) = 3.66\ldots$

nection w retains the meaning of the ionization probability per unit time, averaged over the period of the field F(t).

We note that formulas (12) and (16)—(21) in this section are new in comparison with previous work;[1-3] these formulas are applicable for an arbitrary electric field F(t). Their use does not require a preliminary determination of the Green's function for an electron in the external field and appreciably simplifies the calculation of the probability w. The results of calculations for several characteristic fields are collected together in Table 1.

4. PAIR PRODUCTION IN AN ELECTRIC FIELD ($\epsilon_0 = -1$)

This process has been investigated by Schwinger[7] for the case of a constant field, by Brezin and Itzykson[13] for a field of the form F cos ωt, and by Narozhnyĭ and Nikishov[14] and also by the author[15] for a field of the form $F(t) = F/(\cosh^2 \omega t)$. The quasiclassical approach, based on the method of "imaginary time" and applicable to a wide class of fields, was developed in articles[4,5]. For the case of linear polarization, the quantities appearing in Eq. (8) have the following form:

$$g(\gamma)=\frac{2}{\pi}\int_{-1}^{1}\psi(\gamma u)(1-u^2)^{1/2}du,$$

$$c_1(\gamma)=\left(1+\frac{\gamma}{2}\frac{d}{d\gamma}\right)g(\gamma),\quad c_2(\gamma)=-\gamma c_1'(\gamma),\quad D=1. \tag{22}$$

Let us emphasize that the function $\psi(u)$ is determined solely by the external field $\varphi(s)$, that is, it coincides with the function appearing in Eq. (12) (the parameter γ is now given by expression (7)).

The general behavior of the curves $g(\gamma)$ and $c(\gamma)$ is quite similar to the nonrelativistic case. In the region $\gamma \gg 1$ we find

$$f(\gamma)=\frac{\pi\gamma}{4}g(\gamma)=\int_0^{\gamma}\psi(x)\left(1-\frac{x^2}{\gamma^2}\right)^{1/2}dx\approx\tau(\gamma)-(1-\ln 2)a \tag{19'}$$

and $c_1 \approx c_2 \approx (1/2)g(\gamma)$. Therefore, the distribution of the components of the pair with respect to the momentum p becomes almost isotropic for $\gamma \gg 1$:

$$w_p=\exp\left\{-\frac{\pi m^2}{eF}g(\gamma)\left(1+\frac{p^2}{2m}\right)\right\}=\exp\left\{-\frac{2\varepsilon}{\omega}f(\gamma)\right\}, \tag{20'}$$

where $\epsilon = 2(m^2+p^2)^{1/2}$ is the total energy of the created pair (the effective values of p are $\sim(m\omega)^{1/2} \ll m$). The similarity between these formulas and Eqs. (19) and (20) is obvious.

Phenomena analogous to the production of e^+e^- pairs from vacuum develop in the Coulomb field of the nucleus containing Z protons near the critical charge $Z_c = 170$. If the K-shell is not filled, then the Coulomb field spontaneously creates positrons if $Z > Z_c$ (see[16] for further details). The spontaneous creation of positrons is impossible as long as $Z < Z_c$; however, for $|1+\epsilon_0| \ll 1$ such a system is extremely close to the "critical" state. This reveals itself in the fact that the threshold frequency for pair production, $\omega_{th} = 1+\epsilon_0$, tends to zero. If such an atom (having an unfilled K-shell) is placed in an external electric field F(t), then the process of positron creation can occur in a subbarrier fashion, where the motion of the hole in the lower continuum is nonrelativistic. The formulas given in Section 3 are valid for the probability w of positron production in the model we are considering (a level bound by short-range forces), provided κ is understood as $(1-\epsilon_0^2)^{1/2}$ (that is, $\epsilon_0 = -1+\alpha^2\kappa^2/2$, where $\alpha = 1/137$). If $\kappa \sim 1$, then the production of positrons begins when $F \sim F_a \sim 10^{10}$ V/cm, which is six orders of magnitude smaller than the "electrodynamic" field $F_0 = m^2/e$. Unfortunately, in the actual case as $Z \to Z_c$ the Coulomb barrier becomes very substantial, sharply reducing the probability w.

5. SUBBARRIER TRAJECTORY OF THE ELECTRON

By integrating the equation of motion with the initial conditions $x(t_0) = 0$ and $\dot{x}(t_0) = i\kappa$, we obtain the following result for the extremal trajectory:

$$x(t)=eF\omega^{-2}[X(\tau_0)-X(\tau)];$$
$$X(\tau)=\int_0^{\tau}(\tau-\tau')\varphi(i\tau')d\tau',\quad \tau=-is=-i\omega t. \tag{23}$$

The particle stops at t = 0, changing the imaginary values of the momentum p into real values. The turning point x(0) = b defines the width of the barrier:

$$b=\frac{\kappa^2}{2F}l(\gamma), \tag{24}$$

$$l(\gamma)=\frac{2}{\gamma^2}X(\tau_0)=2\int_0^1\psi(\gamma u)u\,du. \tag{25}$$

Using the expansion (A.7), we find the following result for $\gamma \ll 1$:

$$l(\gamma)=1-{}^1\!/_2a_1\gamma^2+{}^1\!/_9(5a_1^2-3a_2)\gamma^4+\ldots. \tag{26}$$

Since $u_1 > 0$, the barrier starts to contract as the frequency ω increases. The behavior of $l(\gamma)$ in the region $\gamma \gg 1$ is of particular interest. In this case $c_1\gamma^{-1} > l(\gamma) > c_2\gamma^{-2}$ for all of the fields we considered, where c_1 and c_2 are certain constants (see Table 1). If $l(\gamma) \approx c/\gamma$, then κb is reduced from the static limit $F_0/2F$ to a value $\sim I/\omega$ for $\gamma \gg 1$, that is, b still appreciably exceeds the atomic radius; however, if $l(\gamma) \approx c/\gamma^2$, then for $\gamma \sim (F_0/F)^{1/2}$ the barrier contracts to atomic dimensions.

Formulas (23)—(26) refer to the nonrelativistic case (the ionization of atoms); however, the picture is not qualitatively changed in going over to the relativistic regime. Thus, in the problem of pair production the effective width of the barrier between the lower and upper continua is given by

$$b=\frac{2m}{eF}l_1(\gamma),$$
$$\gamma=\frac{m\omega}{eF},\quad l_1(\gamma)=\int_0^1\psi(\gamma u)\frac{u}{(1-u^2)^{1/2}}du. \tag{27}$$

In the adiabatic region, instead of Eq. (26) we obtain

$$l_1(\gamma)=1-\frac{2}{3}a_1\gamma^2+\frac{8}{3}\left(\frac{a_1^2}{3}-\frac{a_2}{5}\right)\gamma^4+\ldots, \tag{28}$$

and for $\gamma \gg 1$ we have $l_1(\gamma) \approx (1/2)l(\gamma)$ provided that $u\psi(u) \to 0$ as $u \to \infty$.

Thus, shrinkage of the barrier for $\omega \gg \omega_t$ is a general effect. This qualitatively explains the sharp increase of the tunneling probability w in the region $\gamma \gg 1$

Table II

Atom	I, eV	η	Atom	I, eV	η
H	13.6	1.00	Ar	15.8	0.93
He	24.6	0.74	Kr	14.0	0.99
Ne	21.6	0.79	Xe	12.1	1.06

(associated with one and the same value of the field amplitude F).

6. ALLOWANCE FOR THE COULOMB INTERACTION

Let us make a few remarks about taking the Coulomb interaction between the emitted electron and the atomic core into account. The importance of introducing this correction is already clear from the formula for a constant field F:

$$w(F) = (3F_0/F)^{2\eta} w_{\text{s.r.}}(F), \quad F_0 = {}^2/_3\varkappa^3. \tag{29}$$

Here η is the Coulomb parameter:

$$\eta = Z/\varkappa = Z(I/I_0)^{-1/2} \tag{30}$$

(Z is the charge of the atomic core, and $I_0 = 13.6$ eV), and the probability of ionizing a level having the same energy $E = -I = -\kappa^2/2$, but bound by a short-range potential, is denoted by $w_{s.r.}(F)$. The parameter η is usually close to unity (see Table II), and for $F \ll F_0$ the Coulomb correction is extremely important.[4)] Therefore, the question arises concerning the generalization of Eq. (29) to the case of a variable field.

Let us consider the ratio $F_{Coul}/F(t) = \sigma$ at the end of the subbarrier trajectory ($t = 0$):

$$\sigma = \frac{Z}{b^2 F} = \frac{8\eta}{3l^2(\gamma)} \frac{F}{F_0}.$$

As long as $\sigma \ll 1$, one can neglect the influence of the Coulomb interaction on the trajectory, and take its contribution into account by using perturbation theory. By virtue of the fact that $F \ll F_0$ this is, in any case, true for $\gamma \lesssim 1$; if $l(\gamma) \sim \gamma^{-\alpha}$ as $\gamma \to \infty$, then the approximation under consideration is valid right up to $\gamma \sim \gamma_c$, where

$$\gamma_c = (\eta F/F_0)^{-1/2\alpha} \gg 1 \tag{31}$$

(actually $1 \le \alpha \le 2$, see Table I). In first order perturbation theory the contribution of the Coulomb potential $V(r) = -Ze^2/r$ to the action W is given by

$$\delta W = -i\eta \ln \varkappa r_1 - \int_{t_1}^{0} V(r(t))\, dt,$$

where r(t) is the extremal trajectory, and r_1 is the point where the quasiclassical solution is matched with the atomic wave function ($\kappa^{-1} \ll r_1 \ll b$). Hence for $\gamma \ll \gamma_c$ we find (r_1 drops out of the answer):

$$w(F, \omega) = \left[\frac{3F_0}{F} C(\gamma)\right]^{2\eta} w_{\text{s.r.}}(F, \omega). \tag{32}$$

Here $w_{s.r.}(F, \omega)$ denotes the probability for ionizing the level in a short-range potential, which was calculated in Sec. 3. The frequency dependence of the Coulomb correction is determined by the function $C(\gamma)$:

$$C(\gamma) = l(\gamma) \exp\left\{\int_0^{\tau_0} d\tau \frac{\dot{X}(\tau_0) - \dot{X}(\tau)}{X(\tau_0) - X(\tau)} - 2\ln 2\right\}; \tag{33}$$

$X(\tau)$ is defined in Eq. (23) and $l(\gamma)$ is defined in Eq. (25).

These formulas are valid for an arbitrary electric field F(t). In the case $\gamma \ll 1$, by using expansions (17) and (26), and after rather cumbersome calculations we find

$$C(\gamma) = 1 + {}^1/_9(a_2 - {}^1/_6 a_1^2)\gamma^4 + O(\gamma^6). \tag{34}$$

Thus, the corrections $\sim\gamma^2$ completely cancel each other, and the coefficient associated with γ^4 is numerically small. This leads to the result that, in the adiabatic region one can take the Coulomb correction in the form (29).

Let us mention two simple cases when the correction $C(\gamma)$ can be calculated in closed form:

1) For the field $\varphi = \cos \omega t$ (laser light), by taking the value of the integral

$$\int d\tau \frac{\operatorname{sh}\tau_0 - \operatorname{sh}\tau}{\operatorname{ch}\tau_0 - \operatorname{ch}\tau} = \ln[\operatorname{ch}(\tau_0 + \tau) - 1],$$

into consideration, we have $C(\gamma) \equiv 1$. As a result of the accidental cancellation, the Coulomb correction for $\gamma > 0$ has the same form as for the case of a constant field.[17]

2) In the pulsed field $\varphi(s) = (1 + s^2)^{-3/2}$ the integral (23) is evaluated by making the substitution $\tau = \sin t$, which gives

$$C(\gamma) = \frac{1}{1+\gamma^2} \exp(\gamma \operatorname{arc\,tg} \gamma).$$

The values of the function $C(\gamma)$ are shown in Fig. 3. It is clear from this figure that a rapid increase of $C(\gamma)$ is possible in the region $1 < \gamma < \gamma_c$, that is, even within the region where perturbation theory is valid the Coulomb correction to the probability w may differ substantially from the static limit (29).

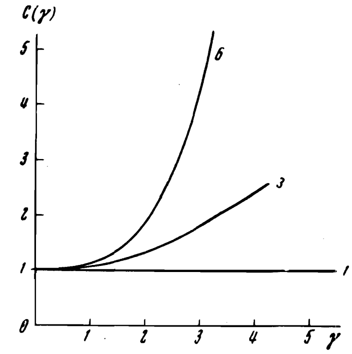

FIG. 3. The Coulomb correction C(γ) from Eq. (32). The numerals on the curves correspond to the example numbers in Table I.

To compare the theory with experiment it is necessary to determine the behavior of $C(\gamma)$ not only for $\gamma \ll \gamma_c$ but also for $\gamma \sim \gamma_c \gg 1$. This calculation runs into substantial difficulties since both the field F(t) of the wave and the Coulomb interaction with the atomic core influence to an equal extent the subbarrier trajectory of the electron when $\gamma \sim \gamma_c$ (in other words, they affect the wave function of the bound state). An attempt was made by Perelomov et al.[18] to calculate $C(\gamma)$ outside the framework of perturbation theory; however, it was not possible to determine the form of $C(\gamma)$ over the entire range of optical frequencies. At the present time

4)The reason for the appearance of the large factor $(3F_0/F)^{2\eta}$ in (29) is as follows: a pre-exponential factor $(\kappa r)^\eta$ appears in the wave function of the bound electron as a consequence of the Coulomb interaction, and this factor increases the density of the electron cloud for $\kappa r \gg 1$. Assuming $r \sim b = \kappa^2/2F$, we obtain expression (29) to within order of magnitude.

the problem of taking the Coulomb interaction into account for $\gamma \gtrsim \gamma_c$ has not been solved, and this problem is of great interest from the experimental point of view.

7. CONCLUSION

It is convenient to treat the ionization of a bound-state level in a variable electric field by the "imaginary-time" method. Such an approach includes, as special cases, the theory of the ionization of atoms and ions in the field of an intense light wave and also the theory of the production of e^+e^- pairs from the vacuum. The relation between the two limiting cases is intuitively clear from Fig. 1, which establishes the shape of the "imaginary-time" contour. The specific characteristic of the problem under consideration (a uniform field F(t)) is the fact that the equations of motion for the classical particle can be integrated in analytic form. Thanks to this, it is possible to obtain the closed expressions (15) and (22) for the ionization probability w; the external field enters into these formulas only through the function $\psi(u)$, which greatly simplifies the calculation of w. It is easy to determine the explicit form of this function if the time dependence of F(t) is known. These formulas give the complete solution of the problem and allow us to investigate the dependence of the probability w on F, on ω, and on the shape of the external field. In our opinion these results effectively demonstrate the potentialities of the "imaginary-time" method.

The requirement that the external field be homogeneous and the restriction to the case of linear polarization are not necessary in principle in order to apply the "imaginary-time" method (although, of course, they appreciably simplify the formulas). Thus, an elliptically polarized wave was treated in[1,2], and certain types of inhomogeneous fields were treated in[3]. Apparently other problems exist for which this method might be useful (for example, in the theory of atomic collisions or in connection with an investigation of multiphoton ionization and the excitation of atoms associated with channeling in single crystals). An extension of the range of applicability of the "imaginary-time" method would be extremely desirable.

I wish to take this opportunity to thank M. S. Marinov, A. M. Perelomov, and M. V. Terent'ev for a discussion of this work.

APPENDIX

PROPERTIES OF THE FUNCTIONS $\psi(u)$ AND $\tau(u)$

As is shown in Secs. 2–4, the calculation of the probability for ionization of a level in the presence of an electric field reduces to a one-dimensional integral which depends on the specific form of the applied field only through the function $\psi(u)$. Let us summarize the formulas relating $\psi(u)$ to the field $\varphi(s)$, where $s = \omega t$ and $\varphi(s) = F(t)/F(0)$.

For the extremal trajectory one has

$$p(t) = e\int_0^t E(t')\,dt' = i\varkappa u.$$

By changing to "imaginary time" $\tau = -i\omega t = -is$ and introducing the notation

$$\varphi(i\tau) = \tilde{\varphi}(\tau), \quad \gamma = \omega\varkappa / eF, \tag{A.1}$$

we find the velocity of the subbarrier motion in the form

$$u = -\frac{ip}{\varkappa} = \frac{1}{\gamma}\int_0^\tau \tilde{\varphi}(\tau')\,d\tau'. \tag{A.2}$$

We note that the function $\tilde{\varphi}(\tau)$ is usually real (for $\tau < \tau_1$, where $s_1 = i\tau_1$ is the position of the nearest singular point of $\varphi(s)$ on the imaginary axis). Because of this, the change from t to τ allows us to explicitly separate out the imaginary part. Let $\tau = \tau(x)$ be the function which is the inverse to

$$x = x(\tau) = \int_0^\tau \tilde{\varphi}(\tau')\,d\tau'.$$

Then, from Eq. (A.2) we obtain $d\tau = \gamma\psi(\gamma u)du$, where

$$\psi(u) = \tau'(u). \tag{A.3}$$

by definition.

Sometimes it is convenient to use another definition of $\psi(u)$, namely,

$$\psi(u) = \frac{1}{\varphi(s)} \quad \text{for} \quad u = -i\int_0^s \varphi(s')\,ds'. \tag{A.4}$$

It is not difficult to see that (A.3) and (A.4) are equivalent to each other. The initial moment $t = t_0$ of the subbarrier motion corresponds to u = 1; therefore

$$t_0 = \frac{i}{\omega}\tau(\gamma) = i\frac{\varkappa}{eF}\frac{\tau(\gamma)}{\gamma}. \tag{A.5}$$

Now let us consider the integral $\int \xi(p)dt$, where $\xi(p)$ is any arbitrary analytic function of the momentum p. By changing to an integration over u, we obtain the relation

$$\int_0^t \xi(p(t))\,dt = \frac{i\varkappa}{eF}\int_0^u \xi(i\varkappa u)\psi(\gamma u)\,du, \tag{A.6}$$

whose application gives formulas (12), (15), and (22).

In the adiabatic case $\gamma \ll 1$, the function $\psi(u)$ enters for small values of u. By using the expansion (17) for F(t) near the maximum point, we find

$$\begin{aligned}\psi(u) &= 1 - a_1u^2 + (5/3\,a_1^2 - a_2)u^4 + \ldots,\\ \tau(u) &= u - 1/3\,a_1u^3 + \ldots\ .\end{aligned} \tag{A.7}$$

Substituting this series into (12) and (22), we arrive at the adiabatic expansions for the functions $g(\gamma)$, $c_1(\gamma)$, and $c_2(\gamma)$.

The behavior of $\psi(u)$ in the region $u \gg 1$ is determined by the singularities of $\varphi(s)$ associated with complex values of s. Let the nearest singularity be located at the point $s = i\tau_1$, where $\varphi(s) = c(\tau_1 - \tau)^{-\alpha}$ as $\tau \to \tau_1$. Then, for $u \gg 1$ we find

$$\psi(u) = \begin{cases} C_1u^{-\alpha/(\alpha-1)}, & \alpha > 1 \\ C_2\exp(-u/c), & \alpha = 1 \end{cases} \tag{A.8}$$

where, for example, $C_1 = [c/(\alpha-1)^\alpha]^{1/(\alpha-1)}$. Thus, $\psi(u)$ decreases with increasing values of u, which leads to a corresponding increase of the ionization probability w in the adiabatic region $\gamma \gg 1$.

In order to determine the momentum spectrum of the emitted electrons and the pre-exponential factor in the formula for w, it is not only necessary to take the contribution from the extremal trajectory into account, but it is also necessary to take the whole beam of classical trajectories, which are sufficiently close to the extremal trajectory, into account. In the case under consideration this amounts to the introduction of a nonvanishing momentum **p** at emergence from the barrier. Here the initial moment $t_0 = i\tau_0/\omega$ of the subbarrier motion is determined by Eq. (5):

$$p_\perp^2 + \left(p_\parallel + \frac{i\varkappa}{\gamma}\int_0^{\tau_0}\varphi(\tau)\,d\tau\right)^2 = -\varkappa^2,$$

which gives

$$\tau_0 = \tau[\gamma(1 + p_\perp^2/\varkappa^2)^{1/2} + i\gamma p_\parallel/\varkappa]. \quad \text{(A.9)}$$

In the quasiclassical regime p is always $\ll \kappa$, which is clear from the answer. Expanding in powers of the parameter p/κ, we obtain the following result correct to terms of second order:

$$\tau_0(\mathbf{p}) = \tau(\gamma) + i\gamma\psi(\gamma)p_\parallel/\varkappa + {}^1/_2[\gamma\psi(\gamma)p_\perp^2/\varkappa^2 - \gamma^2\psi'(\gamma)p_\parallel^2/\varkappa^2]. \quad \text{(A.10)}$$

The displacement of the initial point t_0 from the imaginary axis turns out to be small. Therefore, the "imaginary-time" method also works for nonextremal trajectories.

Formulas (A.3) and (A.4) enable us to explicitly determine the function $\psi(u)$. Without dwelling on the elementary examples which are collected in Table I, let us consider several more complicated cases.

I. For a Gaussian pulse $\varphi(s) = \exp(-s^2)$ we have

$$u = \exp(\tau^2)w(\tau), \quad \psi(u) = \exp(-\tau^2), \quad \text{(A.11)}$$

where

$$w(\tau) = \exp(-\tau^2)\int_0^\tau \exp(x^2)\,dx$$

is a tabulated function. By eliminating τ, we can determine the relation between ψ and u:

$$u = \frac{1}{2}\int_0^{1/\psi}\frac{dt}{\ln^{1/2} t}. \quad \text{(A.12)}$$

It follows from here that

$$\psi(u) = \begin{cases} 1 - u^2 + {}^7/_6 u^4 + \ldots, & u \to 0 \\ (2u\ln^{1/2} u)^{-1} & u \to \infty \end{cases}$$

II. Let us consider a field of the form $\varphi(s) = (1 + s^2)^{-\nu}$. Here

$$\psi(u) = 1 - \nu u^2 + {}^1/_2({}^7/_3\nu^2 - \nu)u^4 + \ldots$$

as $u \to 0$, and as $u \to \infty$ we have

$$\psi(u) = \begin{cases} 4e^{-2u} & \text{for } \nu = 1 \\ [2(\nu-1)u]^{-\nu/(\nu-1)} & \text{for } \nu > 1 \end{cases}$$

III. In order to estimate the role of the higher harmonics in the spectrum of the incident wave, let us set

$$\varphi = \mathrm{cn}(s, \varkappa) = 1 - {}^1/_2 s^2 + {}^1/_{24}(1 + 4\varkappa^2)s^4 + \ldots. \quad \text{(A.13)}$$

Besides $s = \omega t$, this function still depends on the parameter κ and is periodic in t. The period is given by

$$T = \frac{4K}{\omega} = \frac{1}{\omega}\int_0^{2\pi}\frac{dx}{(1 - \varkappa^2\sin^2 x)^{1/2}} \quad (0 \leqslant \varkappa < 1).$$

With the aid of Eq. (A.4) we find

$$\psi(u) = \left[1 + \left(\frac{\mathrm{sh}\,\varkappa u}{\varkappa}\right)^2\right]^{-1/2}. \quad \text{(A.14)}$$

IV. Another example related to elliptic functions is as follows:

$$\varphi(s) = \mathrm{sn}(s + K, \varkappa) = \mathrm{cn}(\omega t, \varkappa)/\mathrm{dn}(\omega t, \varkappa). \quad \text{(A.15)}$$

In this case

$$\psi(u) = \left[\cos^2\varkappa u + \left(\frac{\sin\varkappa u}{\varkappa}\right)^2\right]^{-1/2}. \quad \text{(A.16)}$$

It is curious that here (in contrast to all of the examples considered earlier) $\psi(u)$ is a periodic function of u. Therefore, $g(\gamma)$ does not decrease monotonically with increasing γ, and in the limit $\gamma \to \infty$ the ionization probability w has the same order of magnitude as for the case of a constant field. Qualitatively one can explain this by the fact that the function (A.15) approaches a square wave as $\kappa \to 1$ (whereas the elliptic cosine (A.13), regarded as a function of the normalized argument s/K, takes the shape of a sharp pulse).

[1] A. M. Perelomov, V. S. Popov, and M. V. Terent'ev, Zh. Eksp. Teor. Fiz. **51**, 309 (1966) [Sov. Phys.-JETP **24**, 207 (1967)].

[2] V. S. Popov, V. P. Kuznetsov, and A. M. Perelomov, Zh. Eksp. Teor. Fiz. **53**, 331 (1967) [Sov. Phys.-JETP **26**, 222 (1968)].

[3] L. P. Kotova, A. M. Perelomov, and V. S. Popov, Zh. Eksp. Teor. Fiz. **54**, 1151 (1968) [Sov. Phys.-JETP **27**, 616 (1968)].

[4] V. S. Popov, Zh. Eksp. Teor. Fiz. **61**, 1334 (1971) [Sov. Phys.-JETP **34**, 709 (1971)].

[5] M. S. Marinov and V. S. Popov, Yad. Fiz. **15**, 1271 (1972) [Sov. J. Nucl. Phys. **15**, 702 (1972)].

[6] B. Ya. Zel'dovich and T. I. Kuznetsova, Usp. Fiz. Nauk **106**, 47 (1972) [Sov. Phys.-Uspekhi **15**, 25 (1972)].

[7] Julian Schwinger, Phys. Rev. **82**, 664 (1951).

[8] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1945 (1964) [Sov. Phys.-JETP **20**, 1307 (1965)].

[9] H. Barry Bebb and Albert Gold, Phys. Rev. **143**, 1 (1966).

[10] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **50**, 255 (1966) [Sov. Phys.-JETP **23**, 168 (1966)].

[11] A. M. Perelomov, V. S. Popov, and M. V. Terent'ev, Zh. Eksp. Teor. Fiz. **50**, 1393 (1966) [Sov. Phys.-JETP **23**, 924 (1966)].

[12] B. A. Zon, N. L. Manakov, and L. P. Rapoport, Zh. Eksp. Teor. Fiz. **60**, 1264 (1971); **61**, 968 (1971) [Sov. Phys.-JETP **33**, 683 (1971); **34**, 515 (1972)].

[13] E. Brezin and C. Itzykson, Phys. Rev. **D2**, 1191 (1970).

[14] N. B. Narozhnyĭ and A. I. Nikishov, Yad. Fiz. **11**, 1072 (1970) [Sov. J. Nucl. Phys. **11**, 596 (1970)].

[15] V. S. Popov, Zh. Eksp. Teor. Fiz. **62**, 1248 (1972) [Sov. Phys.-JETP **35**, 659 (1972)].

[16] Ya. B. Zel'dovich and V. S. Popov, Usp. Fiz. Nauk **105**, 403 (1971) [Sov. Phys.-Uspekhi **14**, 673 (1972)].

[17] A. M. Perelomov and V. S. Popov, Zh. Eksp. Teor. Fiz. **52**, 514 (1967) [Sov. Phys.-JETP **25**, 336 (1967)].

[18] A. M. Perelomov, V. S. Popov, and V. P. Kuznetsov, Zh. Eksp. Teor. Fiz. **54**, 841 (1968) [Sov. Phys.-JETP **27**, 451 (1968)].

Translated by H. H. Nickle
174