SOVIET PHYSICS JETP VOLUME 36, NUMBER 3 MARCH, 1973

ELECTRIC POTENTIAL AND PARTICLE CONCENTRATION OF A PLASMA IN THE VICINITY OF A RAPIDLY MOVING CHARGE

A. P. DUBOVOĬ

Institute of Terrestrial Magnetism, the Ionosphere, and Radio Wave Propagation, U.S.S.R. Academy of Sciences

Submitted April 25, 1972

Zh. Eksp. Teor. Fiz. 63, 951–957 (September, 1972)

Results obtained in an investigation of the track structure of a rapidly moving weak point charge Q are presented. For distances r from the charge that are smaller than the Debye radius D but larger than $e|Q|/T$, analytic expressions are obtained for linearized perturbations of the particle concentration, $\delta N^{(1)}$, and for the plasma potential, $\varphi^{(1)}$. Integrals representing the solution are calculated numerically for distances from $r \sim DV_0/v_i$, i.e., in the "intermediate zone," to the distant zone ($r \gg DV_0/v_i$), where the integrals agree with the known asymptotic expressions for $\delta N^{(1)}$ and $\varphi^{(1)}$.

1. When an external charge is at rest in a plasma the concentration N and potential φ are represented correctly by Debye-Hückel equations[1] that describes how these quantities fall off exponentially with increasing distance from the charge. In the case of a moving charge both δN and φ decrease considerably more slowly. Disturbances of the plasma particle concentration and electric potential in the track of a rapidly moving charge have heretofore been investigated with sufficient thoroughness only at large distances from the charge, $r \gg DV_0/v_i$.[2-5] It is of interest, however, (particularly when examining the results of different experiments) to investigate these quantities in all regions, beginning at $r \ll D$. Accordingly, the present work is concerned with the steady flow of a two-temperature tenuous plasma past a weak electric charge in the entire linear region ($r \gg e|Q|/T_i$).

The perturbation of the plasma is small for a sufficiently weak charge ($e|Q|/T_iD \ll 1$). In this case the solution (of the kinetic problem with a self-consistent field) can take the form of an expansion in powers of the charge. In this way Fourier transforms have been obtained for the perturbations of the plasma particle concentration, $\delta N = N - N_0$, and the plasma potential.[2,4,6] At large distances these Fourier transformations were inverted analytically.[3] In the present work analytic expressions are obtained for $\delta N^{(1)}$ and $\varphi^{(1)}$ within the Debye shielding zone.[1)] These expressions are also applicable for a charge of finite radius R_0 which is sufficiently small: $R_0 \ll e|Q|/T_i \ll D$. In this case the interaction of the plasma with the surface of a body (absorption of particles) can be neglected as compared with the disturbance of the plasma by the electric field of the charge.

In the region of transition to the distant zone we have calculated numerically, linear in the charge, the plasma perturbations $\delta N^{(1)}$ and $\varphi^{(1)}$. It must be remembered that the asymptotic expressions for the linear approximation fall off with distance as $1/r^3$ and are valid only at distances satisfying the inequality

$$r \ll D\frac{V_0}{v_i}\Big/\frac{e|Q|v_i}{T_iDV_0}\ln\frac{T_iDV_0}{e|Q|v_i}.$$

At greater distances a quadratic approximation, which decreases as $1/r^2$, prevails. (Higher approximations also decrease as $1/r^2$.) This was first shown by Pitaevskiĭ,[4] and the corresponding integrals were calculated numerically by Panchenko.[5]

2. For the linearized Fourier transforms of the plasma potential and particle concentration perturbation we have the expressions[6]

$$\varphi_{\mathbf{q}}^{(1)} = 4\pi Q\left\{q^2 + \frac{1}{D^2}\frac{\sqrt{\pi}}{2i}\left[w'\left(\frac{\mathbf{V}_0\mathbf{q}}{v_e q}\right) + \frac{T_e}{T_i}w'\left(\frac{\mathbf{V}_0\mathbf{q}}{v_i q}\right)\right]\right\}^{-1},$$

$$\delta N_{a,\mathbf{q}}^{(1)} = -N_0\frac{\sqrt{\pi}}{2i}w'\left(\frac{\mathbf{V}_0\mathbf{q}}{v_a q}\right)\frac{e_a\varphi_{\mathbf{q}}^{(1)}}{T_a}, \tag{1}$$

where $D \equiv (T_e/4\pi e^2N_0)^{1/2}$ is the Debye radius, N_0 is the unperturbed particle concentration, V_0 is the velocity of the charge, $v_a \equiv (2T_a/m_a)^{1/2}$ is the thermal velocity of particles of type a (electrons or ions), and

$$w(z) \equiv e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{u^2}du\right)$$

is a Kramp function (tabulated in[8]).

Thus, assuming $v_e \gg V_0$, we obtain

$$\delta N_e^{(1)} = N_0\frac{e\varphi^{(1)}}{T_e}\left[1 + O\left(\frac{V_0}{v_e}\right)\right] \tag{2}$$

$$\delta N_i^{(1)} = -N_0\frac{eQ}{T_iDa_0}\frac{Da_0}{r}\frac{1}{2\pi^2}\int_0^\infty dq\int_{-a_0}^{a_0}dc_q\int_0^{2\pi}d\alpha_q q^2\exp\{iq(cc_q + ss_q\cos\alpha_q)\}$$

$$\cdot\frac{T_e}{T_i}\frac{\sqrt{\pi}}{2i}w'(c_q)\left\{q^2 + \left(\frac{r}{Da_0}\right)^2\left[1 + i\sqrt{\pi}\frac{v_i}{v_e}c_q + \frac{T_e}{T_i}\frac{\sqrt{\pi}}{2i}w'(c_q)\right]\right\}^{-1}. \tag{2'}$$

Here the following notation has been used:

$$a_0 \equiv V_0/v_i, \quad c \equiv \cos\angle(\mathbf{V}_0\mathbf{r}), \quad s \equiv \sin\angle(\mathbf{V}_0\mathbf{r}), \quad s_q \equiv (a_0^2 - c_q^2)^{1/2}.$$

Subject to the condition $r \ll Da_0[T_i/(T_e + T_i)]^{1/2}$ in the integral of (2′) and using the familiar formulas

$$\frac{1}{2\pi}\int_0^{2\pi}e^{iqa\cos\alpha}d\alpha = J_0(qa), \quad \int_0^\infty e^{iqb}J_0(qa)dq = \frac{1}{[a^2 - (b + i0)^2]^{1/2}} \tag{3}$$

[1)]The particle concentration inside the Debye zone around a small (point) body moving in the plasma has been determined in [7] without the limitation $r \gg e|Q|/T$, in the form of a double integral.

the integration with respect to a_q and q is now performed, yielding the result

$$\delta N_i^{(1)}(\mathbf{r}) = -N_0 \frac{eQ}{T_i D a_0} \frac{D a_0}{r} F(a_0 \sin\theta), \tag{4}$$

where $\theta \equiv \sphericalangle(\mathbf{V}_0\mathbf{r})$ and the angular function is

$$F(a_0 s) = \frac{1}{\pi}\int_{-a_0}^{a_0} dc_q \frac{\sqrt{\pi}}{2i} \frac{w'(c_q)}{[(a_0 s)^2 - (c_q + i c 0)^2]^{1/2}}. \tag{5}$$

[In (5) the branch of the root having a positive real part is chosen.]

The integral (5) can be calculated numerically. However, a simple formula for F is obtained by using an algebraic approximation for the function w′ that is given in[9]:

$$\frac{\sqrt{\pi}}{2i} w'(z) \approx -\frac{1}{4\,\mathrm{Re}(h)}\left[\frac{h^*}{(z-h)^2} + \frac{h}{(z+h^*)^2}\right],$$

$$h = 0.48 - 0.91\,i, \quad \mathrm{Im}\, z \geqslant 0. \tag{6}$$

The integration contour in (5) is extended to infinity (with a very small error of the order of $1/a_0^5$) and is closed in the lower half-plane of the complex variable c_q. Calculating the residues at the poles of w′, we have for $\theta \ge \pi/2$:

$$F(a_0 \sin\theta) \approx -2.2\,\mathrm{Im}[(a_0 \sin\theta)^2 + 0.60 + 0.87i]^{-3/2}. \tag{7}$$

We note that when the motion of the charge Q is supersonic the plasma perturbation ahead of the body ($\theta < \pi/2$) is much smaller than behind it ($\theta > \pi/2$). Therefore the structure of the perturbation is here considered only in the track of the charge.

For the potential inside the Debye shielding zone ($r \ll D[T_i/(T_e + T_i)]^{1/2}$ we obtain the Coulomb equation

$$\varphi^{(1)} = Q/r \tag{8}$$

($\varphi^{(1)}$ can be the perturbation-theory approximation only if it is much smaller than the characteristic potential of the plasma: $Q/r \ll T/e$); for the electron concentration, with accuracy of the order $V_0/v_e \ll 1$ we have the Boltzmann equation (2):

$$\delta N_e^{(1)} = N_0 \exp\left(\frac{e\varphi^{(1)}}{T_e}\right) - N_0 = N_0 \frac{e\varphi^{(1)}}{T_e}. \tag{9}$$

When calculations of $\delta N_i^{(1)}$ inside the Debye sphere using the exact equation (2′) and the approximate equation (7), respectively, were compared, it was found that the latter is a good description of the angular distribution of $\delta N_i^{(1)}$ to distances $r \sim D$. The corresponding graphs are shown in Fig. 1a; for definiteness $Q < 0$ was assumed everywhere in the calculations. Figure 1a shows that at distances $r \lesssim D$ the perturbation of the ion concentration is positive at all angles θ: $\delta N_i^{(1)}(\theta) > 0$, i.e., the ion concentration N_i exceeds its unperturbed value N_0 (the focusing effect). With increasing distance a rarefied region appears $[\delta N_i^{(1)}(\theta) < 0]$, which can be accounted for physically as follows. In the case of supersonic flow about the charge ($v_i \ll V_0$) the uniform distribution of ion density at infinity is almost undisturbed ahead of the charge Q (the charge overtakes the perturbation wave). In the direction perpendicular to V_0 the field penetrates the plasma to a distance of the order of D. At distances $z \lesssim DV_0/v_i$ along the $-V_0$ axis the thermal

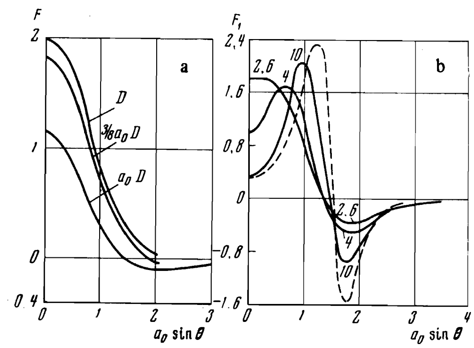

FIG. 1. Perturbation of the ion concentration. a—"near" and "intermediate" zones: the curves are labeled with distances, and

$$F \equiv -\delta N_i^{(1)} \Big/ N_0 \frac{eQ}{TDa_0}\frac{Da_0}{r};$$

b—"distant" zone: the curves are labeled with values of r/Da_0, the dashed asymptotic curve was obtained from [3], and

$$F_1 \equiv -\delta N_i^{(1)}/N_0 \frac{eQ}{TDa_0}\left(\frac{Da_0}{r}\right)^3$$

motion of the ions is unable to affect the structure of the track (to screen the charge). We can therefore consider that there is a uniform ionic background approaching with the velocity $-V_0$. Since we are considering the case $Q < 0$, the ions that move past with an impact parameter smaller than D must be deflected toward the axis, i.e., they are focused, while the ions with an impact parameter exceeding D remain practically undeflected. Since particle trajectories are lacking at distances $\sim a_0 D$ from the axis the appearance of rarefied regions is required to conserve the total number of ions.

The Coulomb expression for the potential is applicable only at distances $r \ll D[T_i/(T_e + T_i)]^{1/2}$. The results of the calculation for $r \lesssim D$ are in good agreement (for $T_e/T_i = 1$, $v_i \ll V_0 \ll v_e$) with the Debye equation

$$\varphi^{(1)} \approx (Q/r)e^{-r/D}. \tag{10}$$

For a charge at rest D is here replaced by $D[T_i/(T_e + T_i)]^{1/2} = D/\sqrt{2}$. Only electrons can screen a moving charge when $v_i \ll V_0 \ll v_0$.

At $r > D$, since the plasma space charge is comparable in magnitude with the charge of the body, the

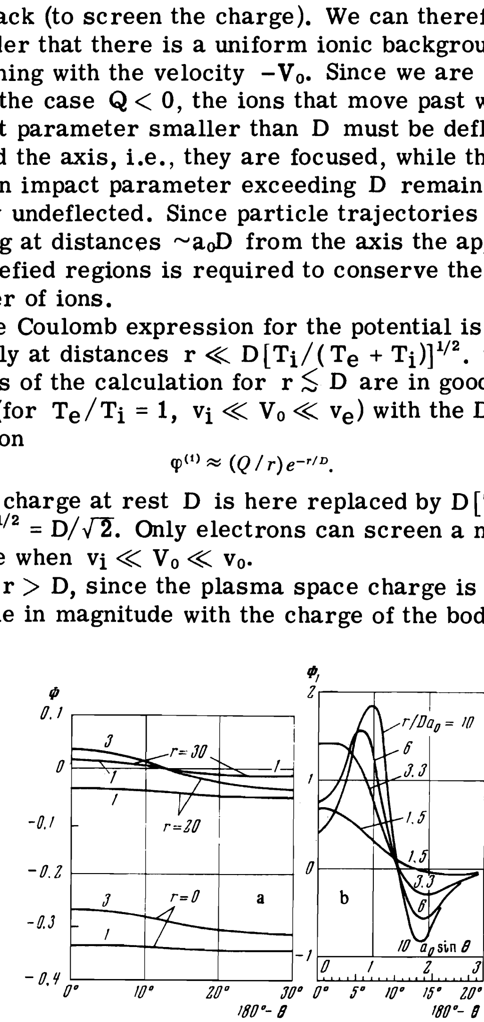

FIG. 2. Potential: a—"near" zone: the curves are labeled with the numeral 1 or 3 to designate the value of T_e/T_i ($a_0 = 8$), and $\Phi \equiv -\varphi/(Q/D)$; b—"intermediate" and "distant" zones ($a_0 = 8$):

$$\Phi_1 \equiv -\varphi \Big/ \frac{Q}{Da_0}\left(\frac{Da_0}{r}\right)^3$$

spherical symmetry of the potential is destroyed. On the axis behind the body a maximum appears which is attributable to the focusing of ions by the moving charge (Fig. 2a). This maximum increases with T_e/T_i; it follows from (4) and (9) that as T_e (i.e., v_e/V_0) increases $\delta N_e^{(1)}$ decreases, or as T_i (i.e., v_i/V_0) decreases $\delta N_i^{(1)}$ increases, so that the ionic space charge becomes the governing factor. We note that for the plasma potential in the "near zone," in contrast with $\delta N_i^{(1)}$, universal curves valid for different values of a_0 cannot be plotted: $\varphi^{(1)}(a_0, \theta, r/D)$ becomes a more complicated function.

3. An analysis of (2′) shows that for $v_i \ll V_0 \ll v_e$ the linear (with respect to the charge) approximation of the ion concentration perturbation is given correctly by an equation of the form

$$\delta N_i^{(1)} = -N_0 \frac{eQ}{T_i D a_0}\left(\frac{Da_0}{r}\right)^3 F_1\left(a_0 \sin\theta, \frac{Da_0}{r}\right). \tag{11}$$

Here the dependence of $\delta N_i^{(1)}$ on r and θ is expressed in terms of r/Da_0 and $a_0 \sin\theta$, so that F_1 can be calculated for only a single (large) value of a_0. The asymptotic form of the angular function F_1 was obtained in[3], and was shown by a numerical calculation to be sufficiently accurate for $r > 10\, a_0 D$.

The electron concentration δN_e and the potential φ at large distances are determined from δN_i through the Poisson equation $\Delta\varphi = -4\pi e(\delta N_i - \delta N_e)$, which for $r \gg a_0 D$ gives

$$\delta N_e = \delta N_i, \tag{12}$$

and from the Boltzmann equation (9), which holds true for electrons in a quasi-state of rest:

$$\varphi = (T_e/e)(\delta N_i/N_0). \tag{13}$$

Equation (2′) can be simplified as follows. For $\theta < \pi/2$ ($\cos\theta > 0$) the integration contour of the integral $\int_{-a_0}^{a_0} dc_q$ can be replaced by a semicircle of radius a_0 in the upper half-plane of the compex variable c_q, because both w' and the (dispersive) denominator of the integrand are analytic in the upper half-plane, where they possess neither zeros nor singularities. For $|c_q| = a_0 \gg 1$ we shall have

$$\frac{\sqrt{\pi}}{2i} w'(c_q) \approx -\frac{1}{2c_q^2}$$

and it can be shown that

$$\delta N_i^{(1)}(\theta < \pi/2) \sim O(1/a_0^2).$$

Furthermore, when $\theta > \pi/2$ we have

$$\delta N_i^{(1)}(\theta) = \delta N_i^{(1)}(\theta) + \delta N_i^{(1)}(\pi - \theta) + O(1/a_0^2),$$

so that in (2′) we can make the approximate substitution

$$\int_0^\infty dq \to \int_{-\infty}^\infty dq.$$

Calculating the residues in the last integral, we shall have

$$\delta N_i^{(1)} = -N_0 \frac{eQ}{T_i D a_0}\Big[\frac{Da_0}{r}\frac{4}{\pi}\operatorname{Re}\int_0^{\pi/2} d\alpha \frac{T_e}{T_i}\frac{\sqrt{\pi}}{2i} w'(a_0 s \cos\alpha)$$
$$-\frac{2}{\pi}\operatorname{Re}\int_0^{a_0} dc_q \frac{T_e}{T_i}\frac{\sqrt{\pi}}{2i} w'(c_q)\sqrt{\Delta_0(c_q)}$$
$$\times \int_0^\pi d\alpha \exp\Big\{-\frac{r}{Da_0}|cc_q + ss_q\cos\alpha|\sqrt{\Delta_0(c_q)}\Big\}\Big] \tag{14}$$

where

$$\Delta_0(c_q) \equiv 1 + i\sqrt{\pi}\frac{v_i}{v_e}c_q + \frac{T_e}{T_i}\frac{\sqrt{\pi}}{2i} w'(c_q).$$

Equation (14) was used for numerical calculations in the "intermediate zone. From the results, shown in Figs. 1b and 2b, it is seen that at $r \sim 0.4 a_0 D$ a change occurs in the behavior of both $\delta N_i^{(1)}$ and $\varphi^{(1)}$ (and therefore of $\delta N_e^{(1)}$). In the angular dependence of $\delta N_i^{(1)}$ a rarefied region appears; the potential, which is negative throughout the Debye sphere, intersects the abscissal axis at $r \sim 0.4\, a_0 D$ and its maximum on this axis becomes positive. On the whole the angular dependence of $\varphi^{(1)}$ becomes similar to the behavior of $\delta N_i^{(1)}(\theta)$. [As already mentioned, in the limit of large distances $\delta N_i^{(1)}/N_0$ and $e\varphi^{(1)}/T_e$ should coincide to within $O(V_0/v_e)$.] We can therefore consider that at $r \sim 0.4\, a_0 D$ the "near" zone ends and we find the beginning of the "intermediate" (transition) zone for the track of a rapidly moving weak point charge. In the "intermediate zone" the evolution of the plasma perturbation with distance from the charge proceeds as follows. The maximum of $\delta N_i^{(1)}$ on the axis behind the body decreases more rapidly than $1/r^3$ and is replaced by a relative minimum, which is still positive, because the aforedescribed focusing mechanism continues to be effective. A bend occurs (when $T_e = T_i$) at a distance $r \sim 2.5\, a_0 D$. It can be stated that with increasing distance from the charge the maximum of $\delta N_i^{(1)}(\theta)$ departs from the axis asymptotically at the angle $\theta \sim \arcsin(1.2/a_0)$ and becomes sharper. A bend in the angular dependence of the potential occurs at $r \sim 3.3\, a_0 D$. Obviously, in the region $(2.5-3.3)\, a_0 D$ a transition takes place from the "intermediate" to the "distant" zone. In the "distant" zone at $\theta \sim \arcsin(1.5/a_0)$ the perturbation $\delta N_i^{(1)}$ vanishes; at $\theta \sim \arcsin(1.8/a_0)$ it reaches its (negative) minimum, and for $\theta \gtrsim \arcsin(2.5/a_0)$ approaches zero rapidly. The absolute value of the minimum ($\delta N_i^{(1)} < 0$) increases to its asymptotic value. The angular behavior of the potential is similar in the "distant" zone (Fig. 2b).

In conclusion, I wish to thank Ya. L. Al'pert for suggesting this problem, for discussions of the results, and for comments; also, V. S. Nikishin and O. V. Kapustina for assistance with the numerical calculations, and A. M. Koskalenko for discussions.

[1] D. A. Frank-Kamenetskiĭ, Lektsii po fizike plasmy (Lectures on Plasma Physics), Atomizdat, 1968.

[2] L. Kraus and K. Watson, Phys. Fluids 1, 480 (1958).

[3] N. I. Bud'ko, An Investigation of Plasma Disturbance in the Distant Zone by a High-speed Body, Dissertation, ITF, AN SSSR, 1970.

[4] L. P. Pitaevskiĭ, Geomagnetizm i aéronomiya 1, 194 (1961) [Geomagnetism and Aeronomy 1, 172 (1961)].

[5] Yu. M. Panchenko, in Investigations of Cosmic Space, Trans. All-union Conference on the Physics of Cosmic Space (in Russian), Nauka, 1965, p. 254.

[6] Ya. L. Al'pert, A. V. Gurevich, and L. P. Pitaev-

skiĭ, Iskusstvennye sputniki v razrezhennoĭ plazme (Artificial Satellites in a Tenuous Plasma), Nauka, 1964, Sec. 17.
[7] A. M. Moskalenko, Geomagnetizm i aéronomiya 4, 260 (1964) [Geomagnetism and Aeronomy 4, 203 (1964)].
[8] V. N. Faddeeva and N. M. Terent'ev, Tablitsy znacheniĭ integrals veroyatnosteĭ ot kompleksnogo arguments (Tables of the Probability Integral for Complex Arguments), Gostekhizdat, 1954.
[9] B. D. Fried, C. L. Hedrick, and J. McCune, Phys. Fluids 11, 249 (1968).

Translated by I. Emin
100