LIMITING CURRENTS IN COMPENSATED ELECTRON BEAMS WITH A RELATIVISTIC ENERGY SPREAD

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The stability of a compensated electron beam with a relativistic energy spread has been studied in a cylindrical drift space in a strong longitudinal magnetic field. It is shown that in this system for $u \ll c$, where u is the directed beam velocity, development is possible of a weakly increasing kinetic instability similar to the well known ion-sound instability of a current plasma. The frequency and increment of the unstable oscillations in this case are appreciably less than ω_{Li}—the Langmuir ion frequency. For $u \to c$ the kinetic instability becomes aperiodic and its increment already exceeds ω_{Li}, gradually changing to a hydrodynamic instability, which becomes dominant only for ultrarelativistic velocities when $\ln 2\gamma \gg 1$, where $\gamma = (1 - u^2/c^2)^{-1/2}$, the increment of the hydrodynamic instability being of the order of the increment of the aperiodic kinetic instability. Proceeding from the condition of development of electrostatic instabilities, in this paper we determine the longitudinal currents, which turn out to be T/mc^2 times larger than the corresponding limiting currents of monoenergetic electron beams[1].

1. The stability of a monoenergetic compensated electron beam has been investigated previously[1] (see also ref. 2) for the purpose of determining the limiting currents in bounded systems. Under actual conditions the electrons of a beam, as a rule, have an appreciable energy spread, and for ultrarelativistic electron beams ($\gamma \gg 1$) this spread can exceed the electron rest energy. Therefore we investigate below the stability of a beam with the following electron momentum distribution:

$$f = \frac{n_e}{8\pi}\left(\frac{c}{T}\right)^3 \exp\left[-\frac{cp - \mathbf{up}}{T}\right]. \qquad (1)$$

Here n_e is the electron density ($n = n_e = n_i$) and $T \gg mc^2$ is the effective temperature characterizing the electron energy spread. As in refs. 1 and 2, the stability is considered of an electron beam of radius r_0 traveling along the axis of a metallic waveguide with radius $R \geq r_0$. The system is assumed to be in an external strong longitudinal magnetic field which satisfies the conditions

$$B_0 \gg \frac{T}{er_0}, \quad \left(8\pi nT\frac{T}{mc^2\gamma^3}\right)^{1/2}, \quad (8\pi nT\gamma)^{1/2} \qquad (2)$$

The first two of these conditions assure magnetization of the beam electrons, and the third permits the discussion to be limited to potential oscillations in study of the stability of the system.

Solving the kinetic equation for the electrons with inclusion of the conditions (2) and limiting ourselves to terms of order $1/B_0$, we obtain for the potential of small electrostatic oscillations the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\varepsilon_\perp r\frac{\partial\Phi}{\partial r}\right) + \left(\frac{l}{r}\frac{\partial g}{\partial r} - \varepsilon_\perp\frac{l^2}{r^2} - \varepsilon_\parallel k_z^2\right)\Phi = 0, \qquad (3)$$

where

$$\varepsilon_\perp = 1 - \omega_{Li}^2/\omega^2, \quad g = -\omega_{Le}^2/\omega'\Omega_e,$$

$$\varepsilon_\parallel = 1 - \frac{\omega_{Li}^2}{\omega^2} + \frac{\omega_{Le}^2}{k_z'^2c^2}\frac{mc^2}{T}\left\{1 + \frac{\omega'}{2k_z'c}\left[\ln\left|\frac{\omega' - k_z'c}{\omega' + k_z'c}\right| + i\pi\eta\right]\right\},$$

$$\eta = \begin{cases} 1 \text{ for } \omega' < ck_z' \\ 0 \text{ for } \omega' > ck_z' \end{cases}. \qquad (4)$$

Here ω_{Le} and ω_{Li} are the Langmuir frequencies of electrons and ions in the laboratory system and Ω_e and Ω_i are their Larmor frequencies, ω is the frequency of the oscillations, l and k_z are the azimuthal and longitudinal wave numbers, and $\omega' = (\omega - k_z u)\gamma$, $k_z' = (k_z - \omega u/c^2)\gamma$. Equation (3) must be supplemented by the boundary conditions[2]

$$\{\Phi\}_{r=r_0} = 0, \quad \Phi|_{r=R} = 0,$$

$$\left\{\varepsilon_\perp\frac{\partial\Phi}{\partial r} + \frac{l}{r}g\Phi\right\}_{r=r_0} = 0. \qquad (5)$$

The boundary-value problem formulated leads to the following dispersion equation for electrostatic oscillations of the system:

$$\varepsilon_\perp\frac{1}{J_l(i\alpha r_0)}\frac{dJ_l(i\alpha r_0)}{dr_0} + \frac{l}{r_0}g + f_l = 0, \qquad (6)$$

where $\alpha = k_z^2\epsilon_\parallel/\epsilon_\perp$; J_l, I_l, and K_l are Bessel functions, and

$$f_l = |k_z|\frac{I_l(|k_z|R)K_l'(|k_z|r_0) - K_l(|k_z|R)I_l'(|k_z|r_0)}{I_l(|k_z|r_0)K_l(|k_z|R) - I_l(|k_z|R)K_l(|k_z|r_0)}. \qquad (7)$$

2. Under the conditions that the electron beam completely fills the waveguide, when $r_0 = R$, the quantity $f_l \to \infty$, and therefore we obtain from Eq. (6)

$$J_l(i\alpha R) = 0 \qquad (8)$$

or, what is the same thing,

$$k_z^2\varepsilon_\parallel + \frac{\mu_{ls}^2}{R^2}\varepsilon_\perp = 0, \qquad (9)$$

where μ_{ls} are the roots of the Bessel function, $J_l(\mu_{ls}) = 0$. This equation has been studied previously[3], where it was noted that for $m/M \ll mc^2/T\gamma^3$ unstable oscillations are possible only in the frequency region $\omega \ll k_z u$. In this limit we obtain from Eq. (9)

$$1 - \frac{\omega_{Li}^2}{\omega^2} + \frac{\omega_{Le}^2R^2m}{T\gamma^3(k_z^2R^2+\mu_{ls}^2)}\left\{1 - \frac{u}{2c}\ln\frac{c+u}{c-u} - i\frac{\pi}{2}\frac{u}{c}\right.$$

$$+\frac{\omega}{k_z c}\left[\frac{3u}{c}+\frac{1}{2}\left(1-\frac{3u^2}{c^2}\right)\ln\frac{c+u}{c-u}\right]\Bigg\}=0. \qquad (10)$$

It is easy to show that the small terms in this equation, which are proportional to $\omega/k_z c$, are important only in the limit of ultrarelativistic electron energies, when $\ln 2\gamma \gg 1$ (more accurately, $(m/M)\ln^2 4\gamma^2 > \pi^3 mc^2/8T\gamma^3$). If $\ln 2\gamma \lesssim \pi/2$, these terms can be neglected and we find from Eq. (10)

$$\omega^2=\omega_{Li}^2\left\{1+\frac{\omega_{Le}^2R^2m}{T\gamma^3(k_z^2R^2+\mu_{ls}^2)}\left(1-\frac{u}{2c}\ln\frac{c+u}{c-u}-i\frac{\pi}{2}\frac{u}{c}\right)\right\}^{-1}. \qquad (11)$$

In this case both hydrodynamic and kinetic instabilities are possible. For small directed electron velocities, when $u \ll c$, only the kinetic instability is possible and it is due to the jump in ion-sound oscillations. The frequency spectrum and increment in this case are determined by the formulas ($\omega \gg \gamma$)

$$\omega^2=\omega_{Li}^2\left(1+\frac{\omega_{Le}^2 m}{T\gamma^3k^2}\right)^{-1}, \quad \gamma=\frac{\pi}{4}\frac{u}{c}\omega\left(1+\frac{\gamma^3Tk^2}{m\omega_{Le}^2}\right)^{-1}, \qquad (12)$$

where $k^2 = k_z^2 + \mu_{ls}^2/R^2$. The maximal increment occurs for oscillations with $\omega_{Le}^2 m \approx 2\gamma^3 Tk^2$, where

$$\gamma_{max}\approx\frac{\pi}{2}\frac{u}{c}\frac{\omega_{Li}}{3\sqrt{3}},$$

and the frequency corresponding to this increment is $\omega_{max} \approx \omega_{Li}/\sqrt{3}$.

With increasing electron velocity the increment of the kinetic instability increases and for $u \to c$ we have $\gamma \to \omega_{Li}$. The increment reaches its maximal value for the condition

$$1+\frac{\omega_{Le}^2 m}{\gamma^3TR^2}(1-\ln 2\gamma)=0, \qquad (13)$$

where $\omega_{max} \approx \gamma_{max} \approx \pi^{-1/2}\omega_{Li}(\ln 2\gamma - 1)^{1/2}$. For further increase in electron energy the hydrodynamic instability sets in and for $\ln 2\gamma \gg 1$ this instability becomes dominant. Modes for which Eq. (13) is satisfied have the maximal increment. Here the oscillations, in essence, increase aperiodically and

$$\omega=\frac{1+i\sqrt{3}}{2}\left(\frac{\omega_{Li}^2k_zc}{2}\right)^{1/3}. \qquad (14)$$

The maximal increment of the oscillations is greater than ω_{Li}. Actually

$$\gamma_{max}\approx\omega_{Li}\frac{\sqrt{3}}{2}\left(\frac{M}{m}\frac{mc^2}{\gamma^3T}\ln 2\gamma\right)^{1/6}\approx\omega_{Li}\frac{\sqrt{3}}{2}\left(\frac{M}{m}\frac{mc^2}{\gamma^3T}\right)^{1/4}>\omega_{Li}.$$

Thus, in a compensated electron beam with a relativistic energy spread for small directed electron velocities where $\ln 2\gamma \lesssim 1$, development is possible only of a kinetic instability similar to the ion-sound instability of a nonisothermal current plasma. The frequency of the oscillations excited in this case substantially exceeds the increment of the oscillations, $\omega_{Li} > \omega \gg \gamma$. Therefore this instability cannot lead to formation of a virtual cathode and breakup of the current in the beam, not to mention that in pulsed systems with a duration $\tau \lesssim 2\pi/\omega_{Li}$ this instability in general cannot develop. With increasing electron energy the kinetic instability increment increases and for $\ln 2\gamma > 1$ (i.e., $\gamma > 1.5$) the instability becomes aperiodic, gradually changing to a hydrodynamic instability for $\ln 2\gamma \gg 1$. Here the increment already is substantially greater than ω_{Li} and, consequently, the instability can develop in pulsed systems. In addition, as a result of the aperiodic nature of the instability it can lead to formation of a virtual cathode and suppression of the current in the beam. Therefore the condition for its development, Eq. (13), minimized with respect to the wave numbers l and k_z, determines the limiting current in a compensating beam completely filling a metallic waveguide[3]1):

$$J_{\text{lim}}=\frac{(2{,}4)^2mc^3\gamma^3}{4e(\ln 2\gamma-1)}\frac{T}{mc^2}. \qquad (15)$$

According to Eq. (15), with increasing electron energy spread the limiting current increases in comparison with the limiting current of a monoenergetic beam by approximately T/mc^2 times. Furthermore, it must be emphasized that the limiting current in electron beams with a relativistic energy spread exists only for the condition $\ln 2\gamma > 1$, or $\gamma > 1.5$. For lower energies only a weakly increasing kinetic instability can develop which cannot lead to limitation of the current in the beam.

3. We will now discuss the case in which a beam of small radius passes along the axis of a waveguide and fills only a small part of it, i.e., $R \gg r_0$. Following ref. 1, we will limit ourselves to study of long-wave oscillations where $\alpha r_0 \ll 1$ and $k_z R \ll 1$. As the result we obtain from Eq. (6)

$$\varepsilon_\perp\left[l+\frac{\alpha^2r_0^2}{2(l+1)}\right]+lg+r_0f_l=0, \qquad (16)$$

where

$$f_l=\begin{cases}\dfrac{1}{r_0}\dfrac{1}{\ln(R/r_0)} & \text{for } l=0\\ l/r_0 & \text{for } l\neq 0\end{cases} \qquad (17)$$

For axially symmetric modes with $l = 0$, Eq. (16) reduces to the form

$$k_z^2\varepsilon_\parallel+\frac{2}{\ln(R/r_0)}\frac{1}{r_0^2}=0. \qquad (18)$$

Substituting here Eq. (4) for $\epsilon_\parallel$ and expanding in powers of $\omega/k_z u$, as was done above, we finally obtain the equation

$$1+\frac{2}{k_z^2r_0^2\ln(R/r_0)}-\frac{\omega_{Li}^2}{\omega^2}+\frac{\omega_{Le}^2m}{k_z^2T\gamma^3}\left\{1-\frac{u}{2c}\ln\frac{c+u}{c-u}\right.$$
$$\left.-i\frac{\pi}{2}\frac{u}{c}+\frac{\omega}{k_zc}\left[3\frac{u}{c}+\frac{1}{2}\left(1-\frac{3u^2}{c^2}\right)\ln\frac{c+u}{c-u}\right]\right\}=0. \qquad (19)$$

This equation is similar to (10), and therefore its entire analysis is also similar to that carried out above. For $\ln 2\gamma \lesssim \pi/2$ we can neglect terms $\sim \omega/k_z c$, and the solution of Eq. (19) is written in the form (compare with Eq. (11))

$$\omega^2=\omega_{Li}^2\left[1+\frac{2}{k_z^2r_0^2\ln(R/r_0)}+\frac{\omega_{Le}^2m}{k_z^2T\gamma^3}\left(1-\frac{u}{2c}\ln\frac{c+u}{c-u}-i\frac{\pi}{2}\frac{u}{c}\right)\right]^{-1}. \qquad (20)$$

In the case of nonrelativistic beams with $u \ll c$, the solution (20) corresponds to weakly increasing oscillations of the ion-sound type, the frequency and increment being appreciably less than ω_{Li}. With increasing directed velocity of the beam, the increment in development of the kinetic instability increases and in the region of

1)It should be noted that for $\ln 2\gamma \gtrsim 1$, when the instability has a kinetic nature, Eq. (15) (and also Eqs. (24) and (28)) gives a reduced value of the limiting current in the system. For accurate determination of the limiting current in this it is necessary to draw on quasilinear theory.

relativistic beam energies ($u \approx c$, for $\ln 2\gamma \lesssim \pi/2$) it reaches a maximal value for the condition (compare with Eq. (13))

$$1+\frac{2}{k_z^2r_0^2\ln(R/r_0)}+\frac{\omega_{Le}^2 m}{k_z^2T\gamma^3}(1-\ln 2\gamma)=0, \qquad (21)$$

where

$$\omega_{max}\approx\gamma_{max}\approx\frac{\omega_{Li}}{\sqrt{\pi}}\left(\frac{\ln 2\gamma-1}{1+2/k_z^2r_0^2\ln(R/r_0)}\right)^{1/2}. \qquad (22)$$

With further increase of electron energy, when $\ln 2\gamma \gg 1$, the hydrodynamic stability sets in and becomes dominant. At this stage the modes for which condition (21) is satisfied have the maximal increment, and

$$\omega_{max}\approx\gamma_{max}\approx\frac{\sqrt{3}}{2}\left[\frac{\omega_{Li}^2k_zc}{2+4/k_z^2r_0^2\ln(R/r_0)}\right]^{1/3}\gtrsim\omega_{Li}. \qquad (23)$$

Thus, for excitation of axially symmetric modes of oscillations ($l = 0$) with large increments comparable or even greater than ω_{Li}, it is necessary that condition (21) be satisfied. From this condition we find the limiting current for excitation of axially symmetric modes of oscillations:

$$J_0=\frac{mc^3\gamma^3}{4e(\ln 2\gamma-1)}\frac{2}{\ln(R/r_0)}\frac{T}{mc^2}. \qquad (24)$$

In this case also the limiting current is T/mc^2 times greater than the limiting current of a monoenergetic beam under the corresponding conditions (compare with ref. 1).

For asymmetric oscillation modes with $l \neq 0$, we have from Eq. (16)

$$1-\frac{\omega_{Li}^2}{\omega^2}+\frac{\omega_{Le}^2}{\Omega_e k_z u}\left(1+\frac{\omega}{k_z u}\right)+\frac{k_z^2r_0^2}{2l(l+1)}\left\{1-\frac{\omega_{Li}^2}{\omega^2}+\frac{\omega_{Le}^2 m}{k_z^2T\gamma^3}\right. \qquad (25)$$
$$\times\left[1-\frac{u}{2c}\ln\frac{c+u}{c-u}-i\frac{\pi}{2}\frac{u}{c}+\frac{\omega}{k_zc}\right.$$
$$\left.\left.\times\left[3\frac{u}{c}+\frac{1}{2}\left(1-\frac{3u^2}{c^2}\right)\ln\frac{c+u}{c-u}\right]\right]\right\}+\frac{r_0}{l}f_l=0.$$

The presence of the third term in this equation for $l \neq 0$ has an important effect on the nature of the oscillations and the stability of the beam. Under the conditions of a strong magnetic field, when

$$\frac{4u\gamma^3}{|k_z|\Omega_e r_0^2}\frac{T}{mu^2}\approx\frac{4u\gamma^3L}{\pi\Omega_e r_0^2}\frac{T}{mu^2}\ll 1 \qquad (26)$$

(here L is the longitudinal length of the system and Ω_e is the cyclotron frequency of the electrons), this term can be neglected, and the entire analysis of Eq. (25) is similar to that carried out above. The current necessary for excitation of a mode with $l \neq 0$ turns out in this case to be greater than for a symmetric mode, and therefore the limiting current is determined by Eq. (24).

When the inequality inverse to Eq. (26) is satisfied, the hydrodynamic instability can become important even for nonrelativistic beams when $u \ll c$, and up to electron energies satisfying the condition $\ln 2\gamma < \pi/2$ the hydrodynamic and kinetic instabilities are related. For $\ln 2\gamma \gg 1$, as in the case of excitation of axially symmetric modes, the hydrodynamic instability is dominant. The condition for maximal increment is the same in all stages and has the form

$$2+\frac{\omega_{Le}^2}{\Omega_e k_z u}+\frac{\omega_{Le}^2 m r_0^2}{2l(l+1)T\gamma^3}\left[1-\frac{u}{2c}\ln\frac{c+u}{c-u}\right]=0. \qquad (27)$$

From this we find the expression for the limiting current for excitation of a mode with $l \neq 0$:

$$J_l=\frac{mu^3}{e}\frac{l(l+1)T\gamma^3}{mu^2}\left[\frac{u}{2c}\ln\frac{c+u}{c-u}-1+\frac{2l(l+1)u\gamma^3TL}{\pi\Omega_e r_0^2mu^2}\right]^{-1}. \qquad (28)$$

For $u \ll c$, Eq. (28) has meaning only when the inequality inverse to (26) is satisfied. If $\ln 2\gamma > 1$, then the expression for J_l is valid also under the conditions of a strong magnetic field, where the inequality (26) is satisfied.

It is evident that the limiting current for excitation of axially asymmetric modes corresponds to the minimum of expressions (24) and (28), i.e.,

$$J_{lim}=\min(J_0,\ J_l). \qquad (29)$$

From the analysis carried out above it follows that in bounded electron beams with a relativistic energy spread the instability limiting the beam current turns out to be the aperiodic kinetic instability over a wide range of energies, while the hydrodynamic instability becomes dominant only in ultrarelativistic beams for which $\ln 2\gamma \gg 1$.

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