MACROSCOPIC THEORY OF FAST PROCESSES IN TRANSLATION-INVARIANT SYSTEMS

G. E. SKVORTSOV

Leningrad State University

Submitted January 27, 1971

Zh. Eksp. Teor. Fiz. 63, 502–515 (August, 1972)

A macroscopic theory of fast linear processes in general Boltzmann systems is discussed. The effect of external forces having a large constant component is taken into account. The basic closing relation defining the nonlocal relationship between the fluxes and the forces is obtained by means of the projection technique. Fourier-Laplace transforms of the kinetic kernels—the nonequilibrium kinetic coefficients—are studied. The connection between the latter and the correlation functions is established. A perturbation theory including both slow and superfast regimes is presented. The general theory is applied to a relaxing impurity, a simple gas, a charged impurity, and radiation. Macroscopic equations for them are presented which are valid at high gradients. The case of diffusion and the hydrodynamics of fast processes are studied in greater detail. The propagation constant for hypersound in a gas is calculated by means of it. The effects of space-time dispersion of the coefficients for the systems under consideration are discussed.

1. INTRODUCTION

In connection with the improved experimental possibilities for investigating processes occurring with high gradients and in strong fields the development becomes quite timely of a theory of processes the scale of which is comparable to the characteristics of the motion of the particles. In this connection a method appears promising which is based on equations for the determining macroscopic quantities (DMQ). However at high gradients such an approach encounters serious difficulties (the choice of the system of DMQ, the possibility of a closed formulation, its effectiveness, etc.).

In this article we discuss a macroscopic theory of fast processes in systems described by a general linear translation—invariant Boltzmann equation. The analysis includes the effect of external forces with a large constant component; in the course of this discussion deviations of quantities from their values in a constant field are considered. The proposed theory is based on closing the equations for the DMQ by means of operator relations (of the type of a convolution) between fluxes and forces. In contrast to the approximate method utilized in a previous article[1] these relations are obtained in an exact manner by means of the projection technique[2,3]. The physical content and the derivation of these relations have similarities with the Leontovich-Mandel'shtam theory[4]. The space-time nonlocality of the basic relations is due to the relaxation of the "internal parameters" and is the determining consideration for fast regimes. We note systems (with "vanishing collision frequency") for which the nonlocality is essential at low gradients and velocities. Expressions are given for kinetic kernels (convolutions) in terms of correlation functions; for one type of force they coincide with formulas from[1].

In the case of conserved DMQ and small forces the equations of section 2 agree with the results of the general theory of irreversible processes (Mori[3], Richardson[5]) applied to the systems being studied. We note that a consideration of only the conserved DMQ excludes from the analysis systems "without conservation" (neutrons in an absorbing medium, radiation), and also very fast regimes. In the study of specific problems transport relations were given previously which take into account the space-time dispersion (cf.,[6,1,7]). The theory of Sec. 2 includes similar results, makes them more precise and enables us to extend them to large gradients.

The evaluation of kinetic kernels is difficult and is simplified by using the perturbation theory contained in Sec. 3 for slow and superfast regimes. General expressions are given for the kernels in these cases. They show, in particular, that at high gradients the **k**-components of the kernels fall off as 1/k. These expressions are useful for a semiphenomenological utilization of the theory. The analysis of Sec. 3 includes the linear variant of the Chapman-Enskog method.

The general theory is applied in section 4 to certain systems. For them the form of the transport relations is made more precise and equations are given for fast processes using the traditional choice of DMQ. The case of diffusion is studied in the greatest detail; for it we give a simple expression for the diffusion kernel which is applicable to very high gradients. The hydrodynamics of fast processes in a simple gas is discussed; in this case the results of[1] are corrected. It is applied to the study of hypersound and leads to a value in agreement with experiment.

2. EQUATIONS OF THE MACROSCOPIC THEORY

The dynamics of the systems under discussion is described (in the Fourier representation) by the equation

$$\partial\varphi/\partial t = S\varphi + q, \quad S \equiv S_a + J, \tag{1}$$

where φ is the deviation of the distribution function

$$f(\mathbf{p}, \mathbf{k}, t) = f_s[1 + \varphi(\mathbf{p}, \mathbf{k}, t)] \tag{2}$$

from the stationary value $f_s(\mathbf{p}, \mathbf{F})$ in a field of constant forces **F**. The evolving operator $S(\mathbf{k}, \mathbf{F})$ consists of a time reversible flux part

$$S_d = -i\mathbf{k}\cdot\mathbf{V}(\mathbf{k}) - \mathbf{F}\cdot\frac{\partial_s}{\partial\mathbf{p}}, \quad \frac{\partial_s}{\partial\mathbf{p}} \equiv \left(f_s^{-1}\frac{\partial}{\partial\mathbf{p}}f_s\right) \tag{3}$$

and the collision operator $\mathbf{J}(\mathbf{k}, \mathbf{F})$ which includes a convolution with respect to time[1)]. The nonhomogeneous term $-i\mathbf{k}\cdot\mathbf{V}\varphi$ contains a correction for its nonideal nature, and for charged particles it contains the contribution of the induced fields. The source term $q(\mathbf{p}, \mathbf{k}, t)$ is of a different nature in specific problems; in particular, it can take into account the nonstationary part of the external forces, $q = \mathbf{q}_e(\mathbf{p})\delta\mathbf{F}_e(t)$.

Basing ourselves on the content of the problem we choose a system of defining macroscopic quantities (DMQ). We consider their small dynamic perturbations $c'_\alpha(\mathbf{k}, t)$, $\alpha = 1, \ldots, n$ (by a prime we denote the dimensional quantities):

$$c_\alpha'(\mathbf{k}, t) = \frac{1}{n_0}\int \psi_\alpha'^{*}\varphi f_s\, d\mathbf{p} \equiv \langle\psi_\alpha', \varphi\rangle, \quad n_0 = \int f_s\, d\mathbf{p}. \tag{4}$$

The microscopic quantities $\psi'_\alpha(\mathbf{p}, \mathbf{F})$ are independent, and they can be treated as orthogonal. Further it is convenient to deal with the normalized quantities ψ_α, $\langle\psi_\alpha, \psi_\gamma\rangle = \delta_{\alpha\gamma}$ and with the quantities c_α corresponding to them.

In order to obtain a macroscopic description it is convenient to utilize the projection scheme due to Zwanzig[2]. With the aid of the projection operators $P = \psi_\alpha\langle\psi_\alpha, \psi\ \ , \cdot\rangle$ onto the subspace of the quantities DMQ and $Q = 1 - P$ onto the orthogonal complement we represent the dynamic part of the distribution in the form $\varphi = P\varphi + Q\varphi$. Such a decomposition separates out the part of interest to us and the "internal" part $Q\varphi$; for the transition to the abbreviated description the latter must be expressed in terms of $P\varphi$. Applying to (1) the projection operators P and Q in sequence we obtain the "vector" equation for the perturbations of DMQ:

$$\frac{\partial}{\partial t}P\varphi - PSP\varphi - Pq = PSQ\varphi \tag{5}$$

and the equation determining the component $Q\varphi$:

$$\left[\frac{\partial}{\partial t} - \hat{S}\right]\hat{\varphi} = \hat{S}P\varphi + \hat{q}; \tag{6}$$

$\hat{S} \equiv QS$ is the "reduced" evolving operator, $\hat{\varphi} = Q\varphi$.

We obtain a closed macroscopic formulation by starting with the simplest dynamic problem with an initial condition[2)]. According to the formal solution of (6)

$$\hat{\varphi} = \hat{R}\left(\frac{\partial}{\partial t}, \mathbf{k}, \mathbf{F}\right)[S\psi_\alpha c_\alpha(\mathbf{k}, t) + \hat{q}] + e^{\hat{S}t}\hat{\varphi}_{t=0}, \quad \hat{R} \equiv \left[\frac{\partial}{\partial t} - \hat{S}\right]^{-1} \tag{7}$$

the component $\hat{\varphi}$ is completely determined by the right hand side of (6), is expressed in terms of the DMQ and the known source term, if it is equal to zero at the initial instant, i.e., the initial condition has the form

$$\varphi_{t=0} = c_\alpha(\mathbf{k})\psi_\alpha. \tag{8}$$

An initial perturbation of this type can be realized by means of external forces which are switched off at the instant $t = 0$[3)] and can either be an acceptable approximation to the known function $\varphi(\mathbf{p}, \mathbf{k}, 0)$, or can be regarded as the result of the evolution of an arbitrary initial distribution after the initial stage. Substituting expression (7) under the condition (8) into (5) we arrive at a system of equations closed with respect to the selected DMQ:

$$\left[\frac{\partial}{\partial t} - \langle\psi_\alpha, S_d\psi_\beta\rangle\right]c_\beta - \langle\psi_\alpha, q\rangle = \dot{c}_\alpha,$$
$$\dot{c}_\alpha \equiv [\langle\psi_\alpha, J\psi_\beta\rangle + \langle S^+\psi_\alpha, \hat{R}\hat{S}\psi_\beta\rangle]c_\beta + \langle S^+\psi_\alpha, \hat{R}q\rangle. \tag{9}$$

On the right hand side of (9) we have separated out the rate of the irreversible variation in DMQ, the "derivation" of the value of c_α. We express it in analogy with the basic relationship in the thermodynamics of irreversible processes in terms of the "forces" $i\mathbf{k}c_\alpha$, c_α (internal) and $\delta\mathbf{F}_e$ (external):

$$\begin{aligned}\dot{c}_\alpha = {} & i\mathbf{k}\cdot\langle\mathbf{g}_\alpha^+, \hat{R}\mathbf{g}_\beta\rangle\cdot i\mathbf{k}c_\beta - [\langle h_\alpha^+, \hat{R}\mathbf{g}_\beta\rangle + \langle\mathbf{g}_\alpha^+, \hat{R}h_\beta\rangle]\cdot i\mathbf{k}c_\beta \\ & + [J_{\alpha\beta} + \langle h_\alpha^+, \hat{R}h_\beta\rangle]c_\beta + [\langle h_\alpha^+, \hat{R}\mathbf{q}_e\rangle - i\mathbf{k}\cdot\langle\mathbf{g}_\alpha^+, \hat{R}\hat{\mathbf{q}}_e\rangle]\cdot\delta\mathbf{F}_e;\end{aligned} \tag{10}$$

here

$$\mathbf{g}_\gamma = \hat{\mathbf{V}}(\mathbf{k})\psi_\gamma, \quad \mathbf{g}_\gamma^+ = Q\mathbf{V}^+\psi_\gamma, \quad h_\gamma = \hat{S}_F\psi_\gamma, \quad S_F = -\mathbf{F}\cdot\frac{\partial_s}{\partial\mathbf{p}} + J. \tag{11}$$

In accordance with relations (10) the thermodynamic fluxes are related to the forces in a manner nonlocal in space and in time by means of operators of a convolution type. The nonlocality which also occurs in the case of a local evolving operator is due to a transition to an abbreviated description. It is essential in the case of appreciable gradients and must be taken into account when the value of kv_T, or $\partial\ln\varphi/\partial t$ will be comparable with the inverse relaxation time of "internal parameters" (kinetic kernels) $\nu_I = 1/\tau_I$. We note that the last term in (10) vanishes for homogeneous forces and is small in the general case.

The Laplace transforms of the kinetic kernels (convolutions) in (10)

$$K_{MN}(p, \mathbf{k}, \mathbf{F}) = \langle\varphi_M^+, R(p, \mathbf{k}, \mathbf{F})\varphi_N\rangle, \quad \varphi_M \equiv g_{\alpha i}, h_\alpha \tag{12}$$

can be naturally called the nonequilibrium kinetic coefficients (NKC). One should distinguish between the NKC of the gradient type, $K_{i\alpha, j\beta}$ $(i, j = 1, 2, 3)$, the relaxation type, $K_{\alpha,\beta}$, the force type, $\langle h^+_\alpha, R\hat{\mathbf{q}}_e\rangle$, and the mixed type. It is evident that in the case of strong inhomogeneity the principal ones are the terms with the gradient type of NKC, and in the case of weak inhomogeneity the principal terms are the relaxation and the force terms. The space-time dispersion of the coefficients which is essential in the case of large gradients and velocities must be taken into account also in the case of small k, p in those cases when the collision frequency of the particle becomes very small for certain values of the energy. (Of nonlocal nature must be the macroscopic theory of such processes as the "running away" of electrons, relaxation in case of a soft interaction[8,1], diffusion of resonance radiation, the Ramsauer transport of electrons in a gas.) We note that the nonequilibrium kinetic tensors (NK tensors) (12) do not have a simple symmetry with respect to the indices, since the operator $S(\mathbf{k}, \mathbf{F})$ is not hermitean. If it is related to S^+ in a parametric manner, there exists a generalized symmetry (cf., below).

We produce the expressions for the NKC in terms of the correlation functions (CF) $\langle\psi, R(p, \mathbf{k})\varphi\rangle$. They can be obtained from (12) by using the relation

1)The dependence of k and the presence of a convolution (nonlocality and the duration of a collision) have the nature of small corrections for the systems under discussion.

2)The equations obtained by this method reflect the internal dynamics of the system and will be useful for other problems.

3)As an example we can taken the removal of a shell which guarantees the difference in the parameters of the subsystems (density, velocity, etc.).

$\hat{R} = [I + RPS^{-1}]R$. However, it is more convenient by starting with Eq. (6) to obtain a system for the quantities $\langle \psi_\alpha, S\hat{\varphi} \rangle$ which determines the fluxes; its solution gives us the expressions for the NKC in terms of the correlation functions:

$$K_{\alpha\beta}(p, \mathbf{k}, \mathbf{F}) = \frac{\Delta^{\gamma\alpha}}{p\Delta} \langle \varphi_\gamma'^{+}, R\varphi_\beta \rangle, \quad \varphi_\gamma' \equiv V_i\psi_\gamma, \quad S_F\psi_\gamma, \tag{13}$$

where $\Delta = \|\langle \psi_\gamma, R\psi_\delta \rangle\|$, $\Delta^{\gamma\delta}$ is the algebraic complement. Formulas (13) are the general fluctuation-dissipation relations taking into account the **k**, p-dispersion and a strong field in the case of the systems under study. In the case of a single scalar DMQ (usually the density or the energy) the NKC in accordance with (13) are expressed in terms of the ratio of the corresponding CF; the form of the NKC in this case can be made to agree with formulas of a simple method (cf.,[1]). In the case of "conserved" DMQ, $J^{+}\psi_\alpha = 0$, for small gradients and fields the relations (13) coincide with the results of the Green-Kubo-Mori theory; for example, for isotropic systems they will have the form $K_{MN}(p, 0, 0) = \langle \psi^{+}_M, R\varphi_N \rangle$. In the case of a small number of forces formulas (13) can be utilized for an experimental determination of the NKC from measurements of the spectral functions (the Fourier-components of the CF). We note that the expressions for the hydrodynamic NK tensors in accordance with (13) correct formulas (50) and (51) of[1] and show that the neglect of the nondiagonal components is justified only for small k.

Of interest is the appearance of the spectrum of the macroscopic theory (9). The spectrum is defined by the dispersion function $D_{MT}(p, \mathbf{k}, \mathbf{F})$ which, if all the quantities $\langle \psi_\alpha, S\hat{\varphi} \rangle$ differ from zero, is equal to

$$D_{MT}(p, \mathbf{k}) = \|\langle \psi_\gamma, R(p, \mathbf{k}, \mathbf{F})\psi_\delta \rangle\|^{-1}. \tag{14}$$

It is clear that the eigenvalues of (9)—the zeros D_{MT}, $p = p(\mathbf{k}, \mathbf{F})$—are the poles of the correlators $\langle \psi_\alpha, R\psi_\beta \rangle$. Thus, as is shown by the spectral decomposition of the CF[1] the discrete spectrum (9) forms a part of the kinetic spectrum. The domain of nonanalyticity of function (14) defines the continuous spectrum of equations (9); as a rule it will coincide with the kinetic spectrum[4]. The presence of a continuum in the macroscopic theory is unusual and is due to the excitation of an infinite number of degrees of freedom of the medium. To it are related the anomalous values of the transport coefficients (the limits of the NKC for $p, \mathbf{k} \to 0$) for systems with a "vanishing frequency" indicated earlier[10,1][5], and also the nonanalyticity of the dielectric permittivity of the Vlasov-Landau plasma. The continuum is essential for fast regimes, and the dependence of NKC on p and **k** for large values of the arguments will not be regular.

The proposed formulation of (9) has obvious advantages in comparison with the initial one (1) and with the well-known methods (the moment method, the Chapman-Enskog method, etc.) Giving an abbreviated description it is simpler than the initial one and can serve as the basis for semiphenomenological theories of fast processes in systems under study. In the method of moments in the case of high gradients one must take into account many moments; Eqs. (9) enable one for arbitrary gradients to utilize the minimal selection of DMQ. The method of moments does not take the continuum into account[6]: its dispersion function is regular. We note that it has only a small effect in a strong field, since the stationary distribution differs appreciably from the equilibrium one which is utilized in it as a weighting factor. Equations (9) are free from these defects.

In choosing the system of DMQ together with general considerations one can also utilize formulas (11). They show that if for ψ_α one chooses the eigenfunctions of the operators S_F, S_F^{+} (first of all, the "conservations," $S_F^{+}\psi_\gamma = 0$), then $h_\alpha = h_\alpha^{+} = 0$ and all the NKC vanish with the exception of the gradient ones. Formulas (11) indicate the possibility of extending the system of DMQ by means of adding to the quantities ψ_α the quantities g_α and h_α. It should be noted that the choice of DMQ defines in agreement with relation (10) the system of "thermodynamic" forces.

3. PERTURBATION THEORY

The evaluation of NKC is associated with great difficulties, and of great importance is the analysis of different limiting cases. Within the framework of the proposed formulation along with slow processes one can consider with the aid of perturbation theory also the opposite case of very high gradients, and also the limit of a strong field. One of the variants of the general perturbation theory given below is the linear version of the well-known Chapman-Enskog (CE) method.

Using perturbation theory we obtain approximate expressions for the NKC for different limiting cases. The starting point is the expansion of the operator R in the expression for $\hat{\varphi}$ in powers of small parameters. In order to shorten the calculations we consider the local evolution operator and omit the source term; additions required in the general case are obvious.

1. In the case of low gradients, $kl_I \ll 1$ ($l_I(F) = v_I\tau_I$, $\hat{S}_F^{-1} \sim \tau_I$) which includes processes fast with respect to time we shall have

$$\hat{\varphi}(\mathbf{k}, p) = \sum_{s=0} (i\mathbf{k})^s \cdot \hat{R}^{(s)} (-i\mathbf{k}\cdot\mathbf{g}_\alpha + h_\alpha) c_\alpha \equiv \sum \Phi^{(s)}, \tag{15}$$

$$\hat{R}^{(s)} \equiv R_F[-\hat{V}\hat{R}_F]^s, \quad \hat{R}_F \equiv [p - \hat{S}_F]^{-1} \tag{16}$$

(ordered in powers of **k**). Here and below a period indicates contraction of tensors (its order is determined by the rank of the tensor and by the equation), the notation a^s is utilized to indicate the tensor product of s vectors **a**. As a first step in the determination of $\Phi^{(0)}$ it is necessary to solve the equation

$$(p - \hat{S}_F)\Phi^{(0)} = \hat{S}_F P\varphi. \tag{17$_0$}$$

Then, in first order with respect to k it is necessary to solve the equation

$$(p - \hat{S}_F)\Phi^{(1)} = -i\mathbf{k}\cdot\hat{V}[\Phi^{(0)} + P\varphi] \tag{17$_1$}$$

---

[4]The appearance of the continuum of (1) for a number of systems is given in [9,8,1]. It is preserved when the induced and selfconsistent forces are taken into account and is obvious in the case of an Enskog gas.

[5]The NKC for the relaxation $\tau(p)$ [1] for a soft interaction is complex for small real $p < 0$.

[6]It is just because of this that with its aid one cannot obtain the Landau damping.

etc. In considering Eqs. (17) and those analogous to them, just as in the CE theory, it is necessary to discuss the problem of whether they are soluble[7]. Without discussing it in detail we point out that if p does not belong to the spectrum of S_F (the usual assumption in the first stage of the Laplace method), then the conditions for it to be soluble reduce to the initial requirement $P\Phi^{(n)} = 0$.

Solving equations $(17)_{0\ldots n}$, we obtain for the NKC the expressions

$$K_{MN} \approx \sum_{s}^{n} (i\mathbf{k})^{s} \cdot \langle \varphi_M^{+}, \hat{R}^{(s)}(p, \mathbf{F}) \varphi_N \rangle. \tag{18}$$

Thus, in the case under discussion we arrive at a sequence of spatially local equations for the DMQ of increasing order with respect to the gradients with retardation. The latter can be neglected in the case of small values of the time $t \ll \tau_I$ ($K_{MN}(p) \to 0$ as $p \to \infty$) and for large values[8] $t \gg \tau_I$. In the absence of gradients, $k = 0$, we shall have a generalized relaxation theory in which in accordance with (10) the relaxation NKC play a role as well as those associated with forces.

For the study of inhomogeneous processes the theory of the second order in terms of the gradients is the most convenient one. Being considerably simpler than the third order theory it includes the principal effects of inhomogeneity, and if a somewhat broader system of the DMQ is utilized compared to the minimal one it is useful as an interpolation device up to very high gradients. The principal second order terms appearing in it have the form

$$i\mathbf{k} \cdot \left\langle \mathbf{g}_\alpha, \frac{\mathbf{g}_\beta}{p - \hat{S}_F} \right\rangle \cdot i\mathbf{k} c_\beta. \tag{19}$$

It contains terms which take into account the spatial dispersion of the NKC associated with relaxation and with forces. The possibility of utilizing it at high gradients depends on the form of the terms (19), $\sim i\mathbf{k}c_\beta$ as $k \to \infty$, and by the consideration of the finite value of the velocity of short wave perturbations[9]. For a gas such a theory will be the thirteen-moment relaxation theory (cf., below).

Perturbation theory and the expressions for the NKC are considerably simplified if for the functions ψ_α one adopts the eigenfunctions of S_F. In this case $\hat{S}_F\psi_\alpha = 0$, $\Phi^{(0)} = 0$ and the k^s-component of the quantity $\dot{c}_\alpha$ is according to (10) and (15) equal to

$$\dot{c}_\alpha^{(s)} = (i\mathbf{k})^{s-1} \cdot [\langle \mathbf{g}_\alpha^{+}, \hat{R}^{(s-2)} \mathbf{g}_\beta \rangle + \langle h_\alpha^{+}, \hat{R}^{(s-1)} \mathbf{g}_\beta \rangle] \cdot i\mathbf{k} c_\beta. \tag{20}$$

In particular, if it turns out that $\hat{S}_F g_\beta = \lambda_{\beta'} g_\beta$, then $(20)_2$ assumes the form

$$\dot{c}_\alpha^{(2)} = i\mathbf{k} \cdot [\langle \mathbf{g}_\alpha^{+}, \mathbf{g}_\beta \rangle + \langle h_\alpha^{+}, \hat{R}_F \mathbf{g}_\beta \rangle] \cdot \frac{i\mathbf{k} c_\beta}{p - \lambda_{\beta'}}. \tag{21}$$

Such expressions are characteristic of the second order relaxation theory. The case of the "eigen" quantities for a Boltzmann gas without forces was considered by Grad[11]. In this case, just as in the CE theory, it turns out to be possible to exclude retardation[10].

2. We consider a variant of perturbation theory analogous to the linear version of the CE method. We choose for the required functions the zeros of the operator S_F, and for the small operator we take $(\partial_t + i\mathbf{k} \cdot \mathbf{V})/\hat{S}_F$[11]. The initial expansion in this case has the form

$$\hat{\varphi} = \sum_{n} \hat{R}^{(n)} \left(\frac{\partial}{\partial t}, \mathbf{k}, \mathbf{F} \right) i\mathbf{k} \cdot \mathbf{g}_\alpha c_\alpha,$$
$$\hat{R}^{(n)} = \hat{S}_F^{-1} \left[\left(\frac{\partial}{\partial t} + i\mathbf{k} \cdot \hat{\mathbf{V}} \right) \hat{S}_F^{-1} \right]^{n}. \tag{22}$$

In virtue of Eqs. (9), the time derivatives of the DMQ can be expressed in the present case in terms of the spatial derivatives, and this enables us to eliminate them from the expansion (22). The elimination of the derivatives with respect to t together with the simultaneous ordering of the series (22) in powers of **k** is accomplished with the aid of equations of appropriate order:

$$\frac{\partial_{(0)}}{\partial t} c_\alpha = - i\mathbf{k} \cdot \langle \psi_\alpha, \mathbf{V} P \varphi \rangle, \quad \frac{\partial_{(n)}}{\partial t} c_\alpha = - i\mathbf{k} \cdot \langle \mathbf{g}_\alpha, \Phi^{(n)} \rangle, \quad n \neq 0. \tag{23}$$

The functions $\Phi^{(n)}$ corresponding to the order k^n are determined by the equations

$$\Phi^{(0)} = 0,$$
$$\hat{S}_F \Phi^{(1)} = i\mathbf{k} \cdot \mathbf{g}_\alpha c_\alpha, \quad \hat{S}_F \Phi^{(2)} = \left(\frac{\partial_{(0)}}{\partial t} + i\mathbf{k} \cdot \hat{\mathbf{V}} \right) \Phi^{(1)}, \ldots \tag{24}$$

For a simple gas without forces we arrive (by a simpler and more natural manner) at the linear CE theory. The NK tensors (of rank s) in the case under discussion have the form of series in terms of **k**:

$$K_{MN}^{(s)}(p, \mathbf{k}, \mathbf{F}) = \sum_{n} (i\mathbf{k})^{n} \cdot K_{MN}^{(s+n)}(\mathbf{F}). \tag{25}$$

Such expansions enable us to bring out the effects of dispersion; but they do not extend via the leading terms the range of gradients under investigation. We note that in the case of "vanishing frequency" for energies for which $\hat{S}_F \to 0$ the expansion (22) loses its meaning, and (25) does not hold (as, for example, for the diffusion coefficient for resonance radiation).

If the constant external forces are small it is useful to utilize representations of quantities (and of operators) in powers of **F**. A stationary distribution will have the form

$$f_s = f_0 \left[1 + \sum \varphi^{(n)} \right], \quad \varphi^{(n)} = \left(J_0^{-1} \mathbf{F} \cdot \frac{\partial_0}{\partial \mathbf{p}} \right)^{n} 1. \tag{26}$$

Here the operators are defined with the aid of the equilibrium distribution f_0; the effect of the field on a collision is neglected. Obviously the functions ψ_α, the resolvent $\hat{R}$, and the NKC will be expressed as series in powers of the parameter $F\tau_I/mv_T$. Such expressions for the NKC enable us to elucidate the effect of the field on transport phenomena.

Equations obtained with the aid of the perturbation theory considered above can be applied to the study of moderately fast and slow processes. The superfast and high gradient regimes correspond to the opposite case of perturbation theory.

---

[7] The condition that the equation $Ax = g$ should be soluble is the orthogonality of g to the solutions of the equation $A^{+}y = 0$.

[8] This can not be accomplished in the case of "vanishing frequency" when dispersion is essential for small values of p.

[9] The experimental basis for this is given by the study of hypersound in a gas [11].

[10] The method of Grad differs from the one used here in its starting point, but the final equations coincide.

[11] In the usual CE method the force F is regarded as small and for the appropriate functions one takes the zeros of the collision integral.

3. We consider the case of very large values of the parameters kl_I and $p\tau_I$. In this case for moderate forces we have

$$\hat{\varphi} = \frac{1}{p + i\mathbf{k}\cdot\hat{\mathbf{v}}} \sum_n \left(\hat{S}_F \frac{1}{p + i\mathbf{k}\cdot\hat{\mathbf{v}}} \right)^n (-i\mathbf{k}\cdot\hat{\mathbf{v}} + \hat{S}_F) P\varphi, \quad (27)$$

$$(p + i\mathbf{k}\cdot\hat{\mathbf{v}})\Phi^{(1)} = -i\mathbf{k}\cdot\hat{\mathbf{v}}\, P\varphi, \ldots \quad (28)$$

In the case under discussion the interaction between particles is not very essential, and in first order one obtains universal results. Evaluating with the aid of (27) the NKC we have for $k \to \infty$

$$K_{MN}(p, \mathbf{k}) = \frac{1}{ik} k_{MN}\left(\frac{p}{ik}, \frac{\mathbf{k}}{k} \right), \quad (29)$$

where $\mathbf{k}_{MN}$ are bounded functions. The vanishing of the NKC as the gradients increase qualitatively differentiate this case from the weakly inhomogeneous one. Equations (9) as $k \to \infty$ in accordance with (29) turn out to be, in essence, of the first order in the gradients and must give rise to a finite velocity of perturbations. The high frequency limit is utilized further for the analysis of hypersound. We note that the fact that Eqs. (28) are soluble is established by means of linear algebra.

The perturbation theory can be made more precise when specific systems are analyzed. Utilizing the form of the NKC for the opposite limiting cases one can obtain interpolation expressions useful over the whole range of gradients. The latter can serve as the basis of a semiphenomenological theory.

4. INVESTIGATION OF SPECIFIC SYSTEMS

We discuss certain systems by utilizing the general theory.

1. We consider a relaxing impurity in an equilibrium medium; the particles of the impurity and of the medium have masses of the same order, $m/M \sim 1$ (neutrons in a medium containing hydrogen, ions in a gas of the same material). In accordance with the picture of the spectrum of the evolution operator[1] the density mode has a much greater lifetime up to the values of $k' \equiv kv_T/\nu_0 \approx 1$. In this case for the DMQ it is sufficient to take the density $n = n_0c_1$, $\psi_1 = 1$. The equation of continuity (5) subject to an appropriate boundary condition $\varphi_{t=0} = c_1(\mathbf{k})$ reduces to the equation of fast diffusion

$$\frac{\partial n}{\partial t} = i\mathbf{k}\cdot \int_0^t d^{(2)}(t-\tau, \mathbf{k})\cdot i\mathbf{k}n(\tau)\,d\tau. \quad (30)$$

The NK diffusion tensor according to (13) is equal to

$$d^{(2)}(p, \mathbf{k}) = \frac{\langle \mathbf{v}, R\mathbf{v}\rangle}{p\langle 1, R1\rangle}. \quad (31)$$

The spectrum of Eq. (30) is determined by the dispersion function $\langle 1, R(p, \mathbf{k})1\rangle^{-1}$ (cf., the discussion of (14) and[1]).

We exhibit the approximate expressions for the NKC for diffusion in different regimes. For small gradients we have the case of the CE perturbation theory, and $d^{(2)}$ is represented by the series (25) in which as a result of the isotropic nature of J odd terms are absent.

From the spectral decomposition of CF[1] and (31) it is evident that the density branch of the evolution operator and the NK diffusion tensor are related by the equation

$$p_1(\mathbf{k}) = -\mathbf{k}\cdot d^{(2)}(p_1(\mathbf{k}), \mathbf{k})\cdot\mathbf{k} \quad (32)$$

(the expansion $d^{(2)}(p_1, \mathbf{k})$ is the one provided by the CE theory). The quantity $p_1(\mathbf{k})$ is calculated more simply than $d^{(2)}$ and, moreover, it is determined experimentally. It is convenient to utilize formula (32) to determine the NKC of diffusion in a one-dimensional problem up to gradients $k' \approx 1$:

$$d_{zz}(p, k) \approx d(k) = -p_1(k)/k^2 \quad (k \equiv k_z). \quad (33)$$

According to the results of calculations of $p_1(k)$ for the self-diffusion of spheres[1] $d(k)$ can be represented by means of an interpolation formula:

$$d(k) \doteq \frac{d_0}{1 + ck^2}, \quad (34)$$

$d_0 = 0.54v_T^2/\nu_0$, $c = 0.1v_T^2/\nu_0^2$ which is valid up to the limiting value $k' \simeq 1.7$. Starting from (34) we obtain the dispersion law for density waves of frequency ω:

$$k(\omega) \doteq \left[\frac{i\omega}{d_0 - i\omega c} \right]^{1/2}. \quad (35)$$

Formulas of the form (34) can be utilized for constructing a semiphenomenological theory of fast diffusion[12)].

In the case of superhigh gradients in accordance with the perturbation theory of (28) the NKC for diffusion is equal to

$$d(p, k) \approx \frac{1}{ik}\left[\frac{p}{ik} - \left\langle \left(v_z + \frac{p}{ik} \right)^{-1} \right\rangle^{-1} \right]. \quad (36)$$

In this limit the spectrum is concentrated on the imaginary axis. We point out that the analysis of a shift flow of a simple gas with the velocity $u_x(z, t)$ is completely analogous to the preceding one.

2. We consider a Boltzmann gas. For it as a result of the equation $S^+(\mathbf{k}) = S(-\mathbf{k})$ which also holds for the operator $\hat{S}(\mathbf{k})$ in the subspace Q, the NK tensors have the symmetry properties

$$K_{MN}(p, \mathbf{k}) = K_{NM}(p, \mathbf{k}), \quad K_{MN}^*(p, \mathbf{k}) = \varepsilon_M\varepsilon_N K_{MN}(p^*, \mathbf{k}), \quad (37)$$

where ϵ_M is the parity sign with respect to $\mathbf{p}$ of the function $\varphi_M(\mathbf{p})$. Formulas (37) are generalized Onsager relations for the case under discussion.

We discuss the hydrodynamics of fast processes (HFP). In this case $c_0 = n/n_0$, $c_i = \sqrt{2}u_i/v_T$, $c_4 = \sqrt{3/2}\,\delta T/T_0$ are the relative perturbations of density, velocity and temperature. Equations (9) assume the form

$$\frac{\partial c_\alpha}{\partial t} + i\mathbf{k}\cdot\langle\psi_\alpha, \mathbf{v}\psi_\beta\rangle c_\beta = \int_0^t d\tau\, i\mathbf{k}\cdot\langle g_\alpha, \hat{R}(t-\tau, \mathbf{k})g_\beta\rangle\cdot i\mathbf{k}c_\beta(\tau). \quad (38)$$

Here $\sqrt{1/2}\rho_0 v_T g_{ij}$ and $\sqrt{3/8}\rho_0 v_T^2 g_{4i}$ are quantities characteristic of the tensor of viscous stresses and of the heat flux. The forces are the temperature gradient and the tensor of the velocities of deformation. The kinetic kernels in (38) correspond to the four NK tensors: $\rho_0\mathbf{K}_{ij,i'j'}(p, \mathbf{k})$—viscosity, $3n_0k_B\mathbf{K}_{4i,4j}/2$—heat conductivity, $(\sqrt{3}\rho_0 v_T/2)\mathbf{K}_{4i,i'j}$—the "viscous heat conductivity" and $(\sqrt{2}n_0k_B/v_T)\mathbf{K}_{ij,4i'}$—heat viscosity (the latter are related by formula (37)). The cross terms which are absent in the usual hydrodynamics point out the relation between heat flux and the viscous stresses. The

12)With their aid it turns out to be possible to describe a whole series of experiments on the diffusion of neutrons in small assemblies.

values of NKC for small k in the linear approximation (cf. (25)) play a role in Barnett's theory.

We apply the HFP to an analysis of hypersound. This problem has been studied by different methods[11,13,9], and different results have been obtained. A new approach is provided for us by the theory being developed and, in particular, by the HFP. We shall start with the dispersion equation of the one-dimensional problem (38):

$$\begin{Vmatrix} p - k^2v_T^2/2p + k^2K_{33} & ikv_T/\sqrt{3} + k^2K_{34} \\ ikv_T/\sqrt{3} + k^2K_{43} & p + k^2K_{44} \end{Vmatrix} = 0, \qquad (39)$$

$p = i\omega$, $k = k_r + ik_i$, which determines the constant for sound propagation $k = k(\omega)$. For $\omega\tau_0 \gg 1$ we evaluate the NKC by means of perturbation theory utilizing (28) or the principal term of (13). The dispersion function for $\omega \to \infty$ has the form

$$D_{\text{HFP}} = \Delta_{as}[z + \langle v_3^2/(v_3 - z)\rangle], \quad z \equiv \omega/k, \qquad (40)$$

where Δ_{as} is the principal term of the determinant of the system which determines the fluxes. We consider the analytic continuation of the function (40) into the lower half plane. Its first zero (of the second factor) yields for the complex phase velocity the value $V_\infty = V_0/(0.53 + i0.22)$, V_0 is the acoustic sound velocity. The latter coincides with the experimental result of[11]. The use of the analytic continuation is associated with the absence of trajectories $k(\omega)$ for $\omega > \omega_{\text{cont}}$ on the spectral sheet[9].

For the study of different problems it is convenient to take an extended system of DMQ and at the same time to utilize the simpler expressions for the NKC. Thus, including among the number of DMQ the heat flux and the tensor of viscous stresses, one can utilize relaxation formulas for the NKC (19) and $(20)_2$. For example, for Maxwellian molecules we have the case (21), and can utilize formulas

$$\dot{c}_N \approx \lambda_{N'}c_N + ik \cdot \sum_{M,M'} \langle g_N, \chi_M\rangle\langle\chi_M, g_{M'}\rangle \cdot \frac{ikc_{M'}}{p - \lambda_M}, \qquad (41)$$

where χ_M are the eigenfunctions of this model, with M′ = 11; 02 and M = 12.20; 03 respectively for the heat flux and for the viscous stresses. Such a thirteen-moment relaxation theory holds up to quite high gradients, $k' \sim 3$; in the problem of sound it describes experiment well for frequencies $\omega \lesssim |\lambda_{02}|$. We note that the relaxation approximation in the hydrodynamics of a simple gas is not satisfactory (it does not include the Barnett effect; according to it $V_\infty \approx 1.4V_0$).

3. We consider a relaxing admixture of charged particles in an external electromagnetic field (ions, electron gas). The unperturbed distribution is found from the equation)

$$\mathbf{F} \cdot \frac{\partial f_s}{\partial \mathbf{p}} = If_s, \quad \mathbf{F} = e\mathbf{E} + \frac{e}{mc}[\mathbf{p}, \mathbf{H}]. \qquad (42)^*$$

We regard the collision integral I(F) to be local and conserving the number of particles, i.e., $J^+1 = 0$. The source term in Eq. (1) for the given case is determined by the nonstationary part of the external forces, $q = -\delta\mathbf{F}\cdot\partial \ln f_s/\partial\mathbf{p}$. The induced forces $\delta\mathbf{F}_i[\varphi]$ are obtained with the aid of the Maxwell equations.

In the case of a strong field the solution of Eq. (42) presents considerable difficulties and is possible only under a number of simplifying assumptions (the case of Davydov[14], the τ-approximation). In the presence of only an electric field the form of f_s for the N-term ν-model of the collision operator is obtained from the formulas of[1]. Thus, in the case of constant frequency we have

$$f_s = \frac{1}{V} f_0(v_\perp) \int_{-\infty}^{v_z} du\, f_0(u) \exp\left\{\frac{u - v_z}{V}\right\} \times \sum_{M=0}^{N} c_M \left(1 + \frac{\lambda_M}{\nu}\right) \chi_M(v_\perp, u),$$
$$c_M = \langle\chi_M\rangle, \quad V = eE/m\nu > 0; \qquad (43)$$

the first eigenvalues λ_M are given in[15]. In accordance with (43) we have

$$\langle v_z\rangle = V\tau_{01}', \quad \langle v^2\rangle = {}^3/_2 v_T^2 + 2V^2\tau_{10}'\tau_{01}', \quad \tau_M' = \nu/|\lambda_M|; \qquad (44)$$

such expressions are utilized for the construction of the system $\{\psi_\alpha\}$. Under the action of only a magnetic field which does not affect the collisions, the solution of (42) is the equilibrium distribution, and the analysis is simplified. In particular, as a result of detailed balance, the NKC have the symmetry

$$K_{MN}(p, \mathbf{k}, \mathbf{H}) = K_{MN}(p, \mathbf{k}, -\mathbf{H}). \qquad (45)$$

In the case when the masses of the particles of the impurity and of the medium are of the same order it is sufficient for inhomogeneous problems to choose density as the DMQ. Here the relations (10) reduce to an expression for the deviation of the flux of the particles from the stationary value $\mathbf{j}_s = n_0\langle\mathbf{v}\rangle$:

$$\delta\mathbf{j}(p, \mathbf{k}) = -d^{(2)}\cdot i\mathbf{k}n + n_0\mu_E^{(2)}\cdot\delta\mathbf{E} - n_0\mu_H^{(2)}\cdot\delta\mathbf{H}, \qquad (46)$$

$$d^{(2)} = \langle\mathbf{v}, \hat{R}\mathbf{v}\rangle, \quad \mu_E^{(2)} = -e\left\langle\mathbf{v}, \hat{R}\frac{\partial \ln f_s}{\partial\mathbf{p}}\right\rangle,$$
$$\mu_H^{(2)} = \frac{e}{c}\left\langle\mathbf{v}, \hat{R}\left[\mathbf{v}\cdot\frac{\partial \ln f_s}{\partial\mathbf{p}}\right]\right\rangle. \qquad (47)$$

Here, along with $d^{(2)}$ the NK tensors of the mobility $\mu_E^{(2)}$ and $\mu_H^{(2)}$ play a role. In the limit $F \to 0$ (in the absence of degeneracy) the generalized Einstein relation holds

$$d^{(2)}(p, \mathbf{k}, 0) = \frac{k_B T_0}{e}\mu_E^{(2)}(p, \mathbf{k}, 0) \qquad (48)$$

(already in the next order with respect to the field it is not satisfied). In the presence of only a magnetic field which does not affect the collisions the last term in (46) vanishes. In the case under discussion, just as in the case without a field, the single-mode approximation to NKC is a convenient one which is valid up to $kl_0(F) \approx 1$.

In the case of carriers of low mass in view of the slow exchange of energy with the medium it is necessary to include among the DMQ the internal energy per particle:

$$\vartheta = \vartheta_s + \delta\vartheta, \quad \delta\vartheta = \langle(\mathbf{p} - \langle\mathbf{p}\rangle)^2/2m - \vartheta_s, \varphi\rangle \equiv \langle\psi_2', \varphi\rangle.$$

The basic relations reduce to expressions for the perturbation of the flux of particles and for the rate of dissipation of energy:

$$\delta\mathbf{j}(p, \mathbf{k}) = \delta\mathbf{j}^* - K_{12}^{(2)}\cdot i\mathbf{k}\delta\vartheta + K_{14}\cdot\delta\vartheta, \qquad (49)$$

$$\delta\dot{\vartheta} = i\mathbf{k}\cdot\delta\mathbf{W} + (J_{22} + K_{24})\delta\vartheta + K_{2E}\cdot\delta\mathbf{E}, \qquad (50)$$

where

$$\delta\mathbf{W} = -K_{22}^{(2)}\cdot i\mathbf{k}\delta\vartheta - K_{21}^{(2)}\, i\mathbf{k}n - K_{2E}^{(2)}\cdot\delta\mathbf{E} + K_{23}n + K_{23}\delta\vartheta \qquad (51)$$

*$[\mathbf{p}, \mathbf{H}] \equiv \mathbf{p}\times\mathbf{H}$.

is the perturbation of the energy flux. Here it was assumed that $\delta\mathbf{H} = 0$; $\delta\mathbf{j}^*$ has the form (46)[13], $\delta\mathbf{E}$ includes the induced field. In expressions (49)—(51), along with the nonequilibrium generalizations of known terms, new terms occur which were not considered previously: the ultimate and the penultimate in (50). They should be taken into account in the case of strong fields and in the case of "vanishing" frequency. The use of relations (49)—(51) enables us to study various high-gradient electronic processes taking into account heating in a strong field.

We consider the case when there is no stationary state in a field of force. In so doing we can consider the period from the instant of switching on the forces as long as the conditions of linearity are satisfied; the equilibrium distribution should be taken as the unperturbed one. For example, we consider the problem of a current in a homogeneous system under the action of constant electric and magnetic fields. Linearity is guaranteed here for all t. The system (5), (6) reduces in this case to the equation

$$(p - S_F)\Phi = -(e\mathbf{E}/p)\cdot(\partial \ln f_0/\partial \mathbf{p}). \quad (52)$$

The tensor of nonequilibrium conductivity is equal to

$$\sigma^{(2)}(p, \mathbf{E}, \mathbf{H}) = -n_0 e^2 \left\langle \mathbf{v}, R\frac{\partial \ln f_0}{\partial \mathbf{p}} \right\rangle. \quad (53)$$

This expression also follows from the analysis of[1].

4. The theory developed here enables us to study the transport of radiation in an equilibrium medium. The deviation of the distribution of photons f(q, k, t), $|\mathbf{q}| = h\nu/c$, where ν is the frequency, from the equilibrium value is determined from Eq. (1) with the evolution operator

$$S = c\left[-i\mathbf{k}\cdot\frac{\mathbf{q}}{q} - \sigma_a(\nu) + J_c\right], \quad (54)$$

where σ_a is the absorption coefficient. The scattering operator J_c conserves the number of particles and in the absence of retardation can be symmetrized as a result of the existence of detailed balance.

For DMQ we take the density and the average internal energy per photon ϑ, $\delta\vartheta = \langle hcq - \vartheta_0, \varphi\rangle$. Equations (10) in the case under discussion contain all their terms.

The expressions for $\dot{n}$ and $\dot{\vartheta}$ are considerably simplified in the case of frequently used approximations: negligibly small scattering, $J_c = 0$, pure scattering, scattering by lines. As a result of the fact that the average number of photons in equilibrium is determined by the temperature usually only the average energy is considered. In this case there are two NKC—the "diffusion of radiation" and the relaxation ones. Explicit expressions which follow from the analysis of Van Kampen[16] of a simple model without scattering can serve as an example. In discussing the case of pure scattering it is appropriate to utilize[17]. The spectrum of relaxation times obtained in it determines the choice of DMQ. In the problem of the diffusion of radiation with line scattering the dispersion of NKC is essential even for small **k** and **p** as a result of the determining role of the large photon mean free paths. In accordance with the paper of Holstein[18] the NKC for the diffusion of radiation for k → 0 (and p = 0) increases as 1/k, for the Doppler contour. The time dispersion of NKC should be taken into account in the case when the interval between the absorption and the emission of a photon is comparable in magnitude with the time corresponding to a mean free path. The simplest model taking retardation into account (cf.,[19]) leads to the following form for $\dot{\vartheta}$:

$$\delta\dot{\vartheta} = i\mathbf{k}\cdot\frac{\langle \mathbf{q}, R\mathbf{q}\rangle}{p\langle q, Rq\rangle}\cdot i\mathbf{k}\delta\vartheta - c\sigma\frac{p + \alpha(1-\lambda)}{p+\alpha}\delta\vartheta. \quad (55)$$

Here λ is the probability of a photon surviving, α is the inverse collision time.

5. CONCLUSION

1. In this paper we have obtained macroscopic equations describing high-gradient processes in translation-invariant Boltzmann systems. The equations are nonlocal in space and in time, and this is due to a transition to an abbreviated description. The nonlocality or the **r**, t-dispersion of the transport coefficients is essential for large gradients and velocities, and also in the case of a "vanishing" collision frequency. The latter occurs for a number of systems (including a plasma and quasiparticles at low temperature), and the macroscopic description of processes in them must be nonlocal.

2. The theory proposed here serves as a suitable basis for a semiphenomenological analysis of fast processes in translation-invariant systems. It indicates a system of forces and fluxes, the relationship between them (by means of convolutions), and also the general form of transport kernels. Suitable expressions for the kernels can be obtained by means of perturbation theory given in the paper. Experimental results can be taken into account in different ways, and, in particular, one can utilize the expressions given here for the Fourier-components of the kernels in terms of measured (for example, by means of light scattering) spectral correlation functions.

3. In this article are given macroscopic equations for fast processes for a number of systems. They are valid for moderately high gradients and, as is shown by an investigation of hypersound, also give a satisfactory description for superfast regimes. For the discussion of fast diffusion a simple equation is proposed nonlocal with respect to r which can be utilized in a semiphenomenological manner. For a simple gas the hydrodynamics of fast processes is discussed, and with its aid the dispersion law for hypersound is evaluated. The application of the general theory to an electron gas gives a number of new terms in the fluxes of particles and of energy. One can, for example, apply the resultant equations to the problem of sound amplification by drift of carriers. The transport of radiation is discussed; for a simple model an equation nonlocal with respect to **r** and t is given. For the systems being considered the effects of **r**, t-dispersion are discussed.

4. The macroscopic formulation of the paper can be regarded as a method of solving the (linear) Boltzmann equation in the case of large gradients. It includes well-known methods and possesses obvious advantages in comparison with them. It leads to a convenient perturbation theory which enables us to study opposite limiting cases.

[13]An approximate expression for this part of the flux for small (finite) k for the Davydov case was obtained in [7].

For intermediate regimes in this case we can use interpolation expressions for transport kernels. The method being developed enables us to obtain such theories as the thirteen-moment relaxation theory indicated in the paper. It is only a little more complicated than the well-known theory due to Grad and in terms of time dispersion gives a good description of the experiment involving sound (in Grad's theory $V_\infty = 1.65V_0$; taking dispersion into account the velocity of hypersound is equal to the observed one, $2V_0$).

[1] G. E. Skvortsov, Zh. Eksp. Teor. Fiz. **57**, 2056 (1969) [Sov. Phys.-JETP **30**, 1114 (1970)].
[2] R. W. Zwanzig, Lectures in Theoret. Phys. **3** (1961) p. 106.
[3] H. Mori, Progr. Theoret. Phys. (Kyoto) **33**, 423 (1965).
[4] L. I. Mandel'shtam and M. A. Leontovich, Zh. Eksp. Teor. Fiz. **7**, 438 (1937).
[5] M. Richardson, Math. Anal. Appl. **1**, 12 (1960).
[6] R. A. Guyer and J. A. Krumshansl, Phys. Rev. **148**, 778 (1966).
[7] A. Ya. Shul'man and Sh. M. Kogan (Zh. Eksp. Teor. Fiz. **57**, 2112 (1969) [Sov. Phys.-JETP **30**, 1146 (1970)].
[8] G. E. Skvortsov, Zh. Eksp. Teor. Fiz. **52**, 1283 (1967) [Sov. Phys.-JETP **25**, 853 (1967)].
[9] G. E. Skvortsov, Zh. Eksp. Teor. Fiz. **49**, 1248 (1965) [Sov. Phys.-JETP **22**, 864 (1966)].
[10] L. M. Biberman, Zh. Eksp. Teor. Fiz. **17**, 416 (1947).
[11] H. Grad, Phys. Fluids **6**, 147 (1963).
[12] E. Meyer and G. Sessler, Z. Physik **149**, 15 (1957).
[13] G. Pekeris, L. Alterman, L. Finkelstein and K. Frankowski, Phys. Fluids **5**, 1608 (1962). L. Sirovich and J. Thurber, JASA **37**, 329 (1965).
[14] B. I. Davydov, Zh. Eksp. Teor. Fiz. **7**, 1069 (1937).
[15] T. Kichara, Rev. Mod. Phys. **25**, 844 (1953).
[16] N. G. Van Kampen, Physica **46**, 315 (1970).
[17] G.C. Pomraning, Astrophys. J. **152**, 809 (1968).
[18] T. Holstein, Phys. Rev. **83**, 1159 (1951).
[19] Teoriya zvezdnykh spektrov (Theory of Stellar Spectra), 1966.

Translated by G. Volkoff
53