Transient Processes in Parametrically Unstable Nonlinear Media

L. M. Gorbunov
P. N. Lebedev Physics Institute, USSR Academy of Sciences
Submitted November 30, 1971
Zh. Eksp.Teor. Fiz. 62, 2141–2146 (June, 1972)

The space-time development of a parametric instability in a nonlinear medium is discussed with allowance for the escape of the growing waves from the region of localization of the pump wave. It is shown that the use of the approximation of weak parametric wave coupling and the assumption that the group velocities of the unstable waves are essentially different leads to a stationary state at the end of the transient process. While this state is being reached, the wave amplitudes may exceed the stationary values.

1. There is considerable evidence that the interaction between a strong electromagnetic wave and a nonlinear medium leads to an unstable state of the latter. Perturbations existing in the medium (for example, thermal perturbations) begin to grow in time, and if the region in which the electromagnetic field is localized is restricted (for example, in the focus of a laser) the instability region is also restricted in space. The growing perturbations eventually leave this region, and this affects the character of the instability development, and frequently determines the consequences to which the instability is expected to lead. The transient process may thus terminate in a stationary distribution of perturbations in space. Another possible situation is that the transient process does not lead to a stationary state and the growth of the initial perturbations is not balanced by their escape from the instability region.[1)] Unfortunately, there has been no discussion in the literature of the problem of parametric instability development with allowance for the spatial localization of perturbation growth.

Transient processes in the case of weak parametric coupling of the perturbations are the simplest from the mathematical standpoint. Here, the instability can be interpreted either as the decay of the pump wave into two other waves (in plasma theory this is referred to as decay instability[2]) or as the stimulated Raman scattering if one of the resulting waves is a transverse electromagnetic wave (we have in mind both SRS and SMBS). Previous papers[3,4] (see also[5]) have been concerned with transient processes in stimulated Raman scattering. However, the assumptions adopted in these papers (see below) were such that the authors could not consider the instability of initial perturbations and, therefore, the final results did not describe the transient process in a complete fashion.

In this paper we report the solution of the problem when the transient process involved in the development of the instability both in space and in time can be considered in full detail. We shall assume weak parametric coupling, an unlimited medium, and a pump-wave amplitude which is nonzero and constant in a restricted region of space. We shall restrict our attention to the one-dimensional case and will take into account the escape of only one (the fastest) of the two growing coupled waves from the instability region. The resulting solution shows that a stationary state is reached after a certain definite interval of time. However, in the course of the transient process, the unstable-wave amplitudes may exceed substantially the stationary value, and this is connected with the development of the initial perturbations in time.

2. The equations describing the slow variation in time and in space of the complex amplitudes E_1 and E_2 of the two waves which are parametrically coupled through the pump wave (given amplitude E_0) are of the form[6]

$$\frac{\partial E_1}{\partial t}+\gamma_1E_1+\mathbf{v}_1\frac{\partial E_1}{\partial \mathbf{r}}=\alpha E_2,\qquad(1)$$
$$\frac{\partial E_2}{\partial t}+\gamma_2E_2+\mathbf{v}_2\frac{\partial E_2}{\partial \mathbf{r}}=\beta E_1,$$

where $\mathbf{v}_1$, $\mathbf{v}_2$, γ_1, γ_2 are, respectively, the group velocities and growth rates. The coefficients α and β are proportional to E_0 and characterize the nonlinear coupling of the waves E_1 and E_2 to the pump wave.

We shall discuss Eq. (1) in the one-dimensional case, where E_1 and E_2 are functions of x only. Moreover, we shall suppose that the velocity of one of the waves is much greater than that of the other[2)] ($v_2 \gg v_1$) and the wave E_2 propagates from right to left. Neglecting the displacement of the slow wave, the above set of equations then assumes the form

$$\frac{\partial E_1}{\partial t}+\gamma_1E_1=\alpha E_2,\qquad(2)$$
$$\frac{\partial E_2}{\partial t}+\gamma_2E_2-v_2\frac{\partial E_2}{\partial x}=\beta E_1.\qquad(3)$$

3. We shall begin the analysis of Eqs. (2) and (3) by considering the two limiting cases which are usually discussed in the literature.

In an unbounded medium (see[2,7-9]) no point in space is distinguished in any way from any other. It is, therefore, natural to omit the term containing the derivatives with respect to the coordinates and seek the solution in the form $E_1 \sim E_2 \sim e^{pt}$. If we then suppose that at t = 0 we have

$$E_1(t=0)=C_1,\quad E_2(t=0)=C_2,\qquad(4)$$

we obtain

$$E_1(t)=A_1e^{p_1t}+B_1e^{p_2t},\quad E_2(t)=A_2e^{p_1t}+B_2e^{p_2t},\qquad(5)$$

where

$$p_{1,2}=-{}^1/_2(\gamma_1+\gamma_2)\pm{}^1/_2[(\gamma_1-\gamma_2)^2+4\alpha\beta]^{1/2},\qquad(6)$$

$$A_1=\frac{C_1(p_2+\gamma_1)-\alpha C_2}{p_2-p_1},\qquad B_1=\frac{\alpha C_2-C_1(p_1+\gamma_1)}{p_2-p_1},\qquad(7)$$

---

1) In the Landau and Lifshitz terminology[1] we have the drift or convective instability in the first case and absolute instability in the second.

2) For example, the decay results in an acoustic or ion-acoustic wave (slow wave) and a transverse electromagnetic or longitudinal electron wave (fast wave).

$$A_2 = \frac{C_2(p_2+\gamma_2)-\beta C_1}{p_2-p_1}, \quad B_2 = \frac{\beta C_1 - C_2(p_1+\gamma_2)}{p_2-p_1}. \tag{8}$$

When $\alpha\beta < \gamma_1\gamma_2$, the values of $p_{1,2}$ in Eq. (6) are negative and the initial perturbations decrease. If, on the other hand, $\alpha\beta > \gamma_1\gamma_2$, then $p_1 > 0$ and the initial perturbations will grow. The onset of the instability is therefore determined by the condition

$$\alpha\beta = \gamma_1\gamma_2 \tag{9}$$

We emphasize that the instability of the initial perturbations is a consequence of Eqs. (2) and (3) only when the time derivatives of the amplitudes of both waves are taken into account.

Let us now consider Eqs. (2) and (3) in the stationary case ($\partial E_{1,2}/\partial t = 0$) (see[10-12]). We note that since the wave E_2 propagates from right to left, the condition that it will grow in space is $\partial E_2/\partial x < 0$. It then follows from Eq. (3) that $\gamma_2 E_2 - \beta E_1 < 0$ and if we use Eq. (2) we find that the condition for the growth of the field E_2 in space is the same as the condition for growing initial perturbations in an unbounded medium.

We shall find the solution of Eqs. (2) and (3) in the stationary case, subject to the following assumptions. We shall suppose that the pump-wave amplitude is nonzero and constant for $0 < x < l$. Since the wave E_2 propagates from right to left, we can then suppose that E_2 at the point where the pump wave enters the interaction region ($x = l$) is given and is equal to the initial value:

$$E_2(x=l) = C_2. \tag{10}$$

The solution is then of the form [here and henceforth $t_0 = (l - x)/v_2$]

$$E_1 = \frac{\alpha}{\gamma_1} C_2 \exp\left\{t_0\left(\frac{\alpha\beta}{\gamma_1}-\gamma_2\right)\right\}, \quad E_2 = C_2 \exp\left\{t_0\left(\frac{\alpha\beta}{\gamma_1}-\gamma_2\right)\right\}. \tag{11}$$

4. We shall now consider the transient solution of Eqs. (2) and (3), subject to the initial and boundary conditions given by Eqs. (4) and (10). We then apply to Eqs. (2) and (3) the Laplace transform with respect to time, eliminate E_1, and solve the resulting equation in x. The final result will be the expression for $E_2(p, x)$, and if we apply the reverse Laplace transform we obtain

$$\begin{aligned} E_2(x,t) = {} & C_2\theta(t-t_0)\gamma_1 \int_0^{t-t_0} d\tau\, e^{-\gamma_1\tau} I_0(2\sqrt{\alpha\beta t_0\tau})\, e^{-t_0\gamma_2} \\ & + A_2 e^{p_1 t}\left\{1 - e^{-t_0(p_1+\gamma_2)}\theta(t-t_0)(p_1+\gamma_1)\int_0^{t-t_0} d\tau\, e^{-\tau(p_1+\gamma_1)} I_0(2\sqrt{\alpha\beta t_0\tau})\right\} \\ & + B_2 e^{p_2 t}\left\{1 - e^{-t_0(p_2+\gamma_2)}\theta(t-t_0)(p_2+\gamma_1)\int_0^{t-t_0} d\tau\, e^{-\tau(p_2+\gamma_1)} I_0(2\sqrt{\alpha\beta t_0\tau})\right\}. \end{aligned} \tag{12}$$

Next, using Eq. (3), we find that

$$\begin{aligned} E_1(x,t) = {} & C_2\alpha e^{-t_0\gamma_2}\theta(t-t_0)\int_0^{t-t_0} d\tau\, e^{-\gamma_1\tau} I_0(2[\alpha\beta t_0\tau]^{1/2}) \\ & + A_1 e^{p_1 t}\left\{1 - \theta(t-t_0)e^{-t_0(p_1+\gamma_2)}(p_1+\gamma_1)\int_0^{t-t_0} d\tau\, e^{-(p_1+\gamma_1)\tau} I_0(2[\alpha\beta t_0\tau]^{1/2})\right\} \\ & + B_1 e^{p_2 t}\left\{1 - \theta(t-t_0)e^{-t_0(p_2+\gamma_2)}(p_2+\gamma_1)\int_0^{t-t_0} d\tau\, e^{-(p_2+\gamma_1)\tau} I_0(2[\alpha\beta t_0\tau]^{1/2})\right\}. \end{aligned} \tag{13}$$

The parameters A_1, A_2, B_1, B_2, p_1, and p_2 in Eqs. (12) and (13) are given by Eqs. (6)—(8) and $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$. As $t \to \infty$, the fast terms in Eqs. (12) and (13) become identical with the two equations in Eq. (11), respectively. The remaining terms are proportional to $e^{p_1 t}$ and $e^{p_2 t}$ and vanish as $t \to \infty$. This is readily demonstrated as follows. In the terms proportional to A_2 and A_1 the argument of the exponential in the integrand is positive ($p_1 + \gamma_1 > 0$), and there are well-known expressions for these integrals for $t \to \infty$ (see[13]). If we use these expressions, we find that the terms in the curly brackets cancel out. Therefore, if $p_1 < 0$ and the initial perturbations do not grow, both factors in the terms proportional to A_1 and A_2 will tend to zero as $t \to \infty$. On the other hand, if $p_1 > 0$, we have $e^{p_1 t} \to \infty$, but the expression in the curly brackets tends to zero. By using the rules for removing the indeterminacy, we find that as $t \to \infty$ the terms proportional to A_1 and A_2 will also vanish. As regards the terms containing B_1 and B_2, here we always have $p_2 < 0$ and $p_2 + \gamma < 0$. Therefore, as $t \to \infty$ we have $e^{p_2 t} \to 0$, but the expression in the curly brackets tends to infinity. If we again use the rules for removing the indeterminacy, we find that these terms will also vanish as $t \to \infty$.

Let us consider Eqs. (12) and (13) in greater detail. When $t < t_0$, the boundaries have no effect on the development of the initial perturbations. It is clear from these formulas that, in this case, the perturbations develop in the same way as in an unbounded medium. When $p_1 > 0$ they will grow exponentially and their amplitudes will be independent of the coordinates for $x < l - v_2 t$.

At $t = t_0$ the perturbation amplitudes are given by

$$E_1(x, t_0) = A_1 e^{p_1 t_0} + B_1 e^{p_2 t_0}, \tag{14}$$

$$E_2(x. t_0) = A_2 e^{p_1 t_0} + B_2 e^{p_2 t_0}. \tag{15}$$

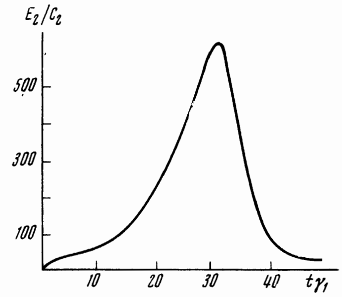
When $\gamma_2 = 0$ and $\alpha\beta < \gamma_1^2$, we find from Eq. (6) that $p \approx \alpha\beta/\gamma_1$, $p_2 \approx -\gamma_1$. Equations (14) and (15) then assume the form

$$E_1(x,t_0) = \frac{\alpha}{\gamma_1}\left(C_2 + \frac{\beta}{\gamma_1}C_1\right)e^{\alpha\beta t_0/\gamma_1} + \left(C_1 - \frac{\alpha}{\gamma_1}C_2\right)e^{-t_0\gamma_1},$$

$$E_2(x,t_0) = \left(C_2 + \frac{\beta}{\gamma_1}C_1\right)e^{\alpha\beta t_0/\gamma_1} - \frac{\beta}{\gamma_1}\left(C_1 - \frac{\alpha}{\gamma_1}C_2\right)e^{-t_0\gamma_1}.$$

Comparing these expressions with Eq. (11) (for $\gamma_2 = 0$), we find that if $|C_2 + \beta C_1/\gamma_1| > |C_2|$, the amplitudes which are approached by the initial perturbations in the time $t = t_0$ exceed the stationary value at the end of the transient process. This happens because, in accordance with Eq. (11), the stationary values of E_1 and E_2 are determined only by the boundary value of E_2 which, according to Eq. (10), is equal to C_2. On the other hand, during the transient process E_2 depends not only on the initial value $E_2(t = 0)$, which is equal to C_2, but also on the initial value $E_1(t = 0)$ which is equal to C_1 [see Eqs. (7) and (8)]. Therefore, if C_1 is large enough, the value of E_2 during the transient process may exceed the stationary amplitude. In particular, when $C_2 = 0$, we have $E_2(t = 0) = E_2(t = \infty) = 0$. However, E_2 is different from zero during the transient process.

For $t > t_0$ the perturbation development begins to be affected by the presence of boundaries. According to Eq. (3), this phenomenon is determined by the magnitude and sign of $\partial E_2/\partial x$. If the amplitude E_2 given by Eq. (15) exceeds the amplitude E_2 at $x = l$, which is equal to C_2, then $\partial E_2/\partial x < 0$. The growth of the initial



perturbations is then slowed down and is replaced by decay. If, on the other hand, $E_2(x, t_0) < C_2$, then $\partial E_2/\partial x > 0$, and the growth of the initial perturbation is accelerated.

The figure shows a plot of E_2 as a function of time, which is based on Eq. (12) and uses the following values of the parameters: $\gamma_2 = 0$, $\alpha\beta t_0/\gamma_1 = 3$, $\alpha\beta/\gamma_1^2 = 0, 1$; $\beta C_1/\gamma_1 = 28$, $C_2 = 1$. It is clear from this figure that when $t\gamma_1 < 30$ the perturbations will grow. This is followed by a fall, and a stationary state is reached for $t\gamma_1 \sim 40$. During the transient process the amplitude will exceed the stationary value by a factor of $(1 + \beta C_1/\gamma_1 C_2) \sim 30$.

As already noted, transient processes during stimulated Raman scattering were investigated in[3,4]. The basic set of equations which was used to carry out these calculations is similar to Eq. (1). However, in obtaining their solutions, the authors of[3,4] assume that $\gamma_2 = 0$ and $|\partial E_2/\partial t| \ll v_2|\partial E_2/\partial x|$ and these restrictions have prevented them from obtaining a correct description of the development of the initial perturbations. In particular, this has prevented them from considering the instability of the initial perturbations. The results reported in[4] follow from Eqs. (12) and (13) when $t \gg t_0$ and $t_0 \ll \mathrm{Max}(\sqrt{\alpha\beta}; \gamma_1)$. These restrictions reflect the fact that the development of the initial perturbations is by then practically complete.

5. Let us consider the validity of the above conclusions. On the one hand, we have neglected the displacement of the slow wave, and this means that $l - x > tv_1$. On the other hand, transition to the limit $t \to \infty$ in Eqs. (12) and (13) is, in fact, valid only under the following conditions:

$$\text{if} \quad |p_2+\gamma_1|,\ |p_1+\gamma_1| > \frac{\alpha\beta}{v_2}(l-x), \text{then } t > \frac{1}{|p_2+\gamma_1|} + \frac{l-x}{v_2};$$

$$\text{if} \quad |p_2+\gamma_1|,\ |p_1+\gamma_1| < \frac{\alpha\beta}{v_2}(l-x), \text{then } > \frac{v_2}{\alpha\beta(l-x)} + \frac{l-x}{v_2}.$$

Consider the case $\gamma_2 = 0$ and $\alpha\beta < \gamma_1^2$. The above conditions are simultaneously satisfied if $t > \gamma_1/\alpha\beta$ and $v_2 > \alpha\beta(l - x)/\gamma_1 > v_1$. If, on the other hand, $\alpha\beta(l - x)/v_2 > \gamma_1$, $v_1/(l - x)$, then we can neglect the motion of the wave E_1 and can pass to the limit as $t \to \infty$ for $(l - x)/v_1 > t > v_2/\alpha\beta(l - x)$. It is clear from these expressions that our solutions must be re-examined for sufficiently small $(l - x)$. Moreover, v_1 cannot be neglected when t is large enough.

In conclusion, we should like to thank N. E. Andreev for carrying out the numerical calculations and V. P. Silin for his interest in this work.

[1]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Mechanics of Continuous Media), Gostekhizdat, 1954.
[2]V. N. Oraevskiĭ and R. Z. Sagdeev, Zh. Tekh. Fiz. **32**, 1291 (1962) [Sov. Phys.-Tech. Phys. **7**, 955 (1963)].
[3]N. M. Kroll, J. Appl. Phys. **36**, 34 (1965).
[4]Chen-Show Wang, Phys. Rev. **182**, 482 (1969).
[5]V. S. Starunov and I. L. Fabelinskiĭ, Usp. Fiz. Nauk **98**, 441 (1969) [Sov. Phys.-Usp. **12**, 463 (1970)].
[6]Yu. L. Klimontovich, Statisticheskaya teoriya neravnovesnykh protsessov v plazme (Statistical Theory of Nonequilibrium Processes in Plasma), MGU, 1964.
[7]R. Z. Sagdeev, Voprosy teorii plazmy (Problems in Plasma Theory), 1964, Chap. 4, p. 55.
[8]B. B. Kadomtsev, Voprosy teorii plazmy (Problems in Plasma Theory), 1964, Chap. 4, p. 226.
[9]V. N. Oraevskiĭ, Dissertatsiya (Dissertation), Kiev, 1969.
[10]N. Bloembergen, Nonlinear Optics. [Benjamin, 1961].
[11]S. A. Akhmanov and R. V. Khokhlov, Problemy nelineĭnoĭ optiki (Problems in Nonlinear Optics), VINITI, 1964.
[12]I. L. Fabelinskiĭ, Molekulyarnoe raseyanie sveta (Molecular Scattering of Light), Nauka, 1965, p. 423.
[13]I. S. Gradshteĭn and I. M. Ryzhik, Tablitsy integralov, summ, ryadov i proizvedeniĭ (Tables of Integrals, Sums, Series, and Products), Fizmatgiz, 1963.

Translated by S. Chomet
248