Baryon Number Variables for the Description of a System of Particles and Antiparticles

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Submitted January 14, 1972
Zh. Eksp. Teor. Fiz. 62, 2013–2018 (June, 1972)

A method is proposed for the construction of a properly symmetrized wave function for a system containing several nonrelativistic nucleons and antinucleons. For a many-particle system with its baryon number equal to zero, a formula is derived which expresses the G-parity in terms of the properties of the Young pattern associated with the baryon number part of the wave function.

In addition to the well-known case of positrionium, other systems consisting of particles and antiparticles are being intensively investigated at the present time. In particular, in connection with the development of sufficiently intense antiproton beams it has become possible to investigate systems containing antinucleons, e.g., atoms with an antiproton in the atomic orbit. The quasinuclear mesons predicted theoretically in[1-4] may serve as another example of such systems. Quasinuclear mesons are a nonrelativistic system of nucleon and antinucleon, bound together by the strong interaction. The existence of such mesons is apparently confirmed by a number of recent experiments.[5-7] The investigation of the bound states of two nucleons and a single antinucleon[8] and of two nucleons and two antinucleons is of great interest. The former should appear experimentally as baryon resonances, and the latter should appear as heavy meson resonances in the so-called "X-region" of the mass.[9] It is impossible to use the wave functions customarily used to describe nucleons in nuclear physics in order to describe such systems, since in the present case the system contains two types of particles.

In the present article it is shown that it is convenient to introduce baryon number variables in order to facilitate the construction of the wave function for a system consisting of nonrelativistic particles and antiparticles. In particular, the introduction of baryon number variables makes it possible to determine the G-parity of a system containing an equal number of nucleons and antinucleons. It turns out that the G-parity of such a many-particle system is not related to any kinematic quantities, i.e., it is not related to the orbital momenta, spin, and isospin (a relation of this type does hold for the case of two particles), but instead the G-parity is an independent, exact quantum number which is determined solely by the permutation symmetry of the wave function with respect to the baryon number variables.

1. THE BARYON NUMBER WAVE FUNCTIONS

The nucleon and antinucleon differ from each other by the value of the baryon number: The nucleon has baryon number B = +1, and the antinucleon has B = −1. Let us include the baryon number in the total set of coordinates used to describe a particle, together with the spatial coordinates, the spin, and the isospin. Then the total wave function of a system containing a total number n of nucleons and antinucleons can be written in the form of the following direct product:

$$\Psi(x,\sigma,\tau,q)=\psi(x)\times\chi(\sigma)\times\eta(\tau)\times\omega(q),$$
$$x=\{x_1,x_2,\ldots,x_n\},\quad \sigma=\{\sigma_1,\sigma_2,\ldots,\sigma_n\},\quad \tau=\{\tau_1,\tau_2,\ldots,\tau_n\},\quad q=\{q_1,q_2,\ldots,q_n\}. \tag{1}$$

Here x_i, σ_i, τ_i and q_i denote respectively the spatial, spin, isospin, and baryon number variables of the i-th particle. According to the generalized Pauli principle, the wave function (1) must change sign upon permuting the coordinates of any two particles, that is

$$\Pi(ij)\Psi=-\Psi. \tag{2}$$

where $\Pi(ij)$ is the operator which permutes the coordinates of the i-th and j-th particles. One can represent the operator $\Pi(ij)$ in the form of a product of operators which permute separately the spatial, spin, isospin, and baryon number variables:

$$\Pi(ij)=\Pi^x(ij)\Pi^\sigma(ij)\Pi^\tau(ij)\Pi^q(ij). \tag{3}$$

Let the system under consideration consist of n particles, n_1 of them nucleons and the remaining $(n-n_1)$ particles antinucleons. Let us denote the spatial, spin, and isospin coordinates of the i-th particle by the single symbol x_i. For simplicity we consider a Hamiltonian containing only pair interactions:

$$H=\sum_{i>j=1}^{n}U(x_i,x_j). \tag{4}$$

In accordance with our notation

$$U(x_k,x_l)=V(x_k,x_l),\quad k,l=1,2,\ldots,n_1;$$
$$U(x_m,x_r)=\bar V(x_m,x_r),\quad m,r=n_1+1,n_1+2,\ldots,n; \tag{4a}$$
$$U(x_k,x_m)=W(x_k,x_m),$$

where V, $\overline{V}$, and W are respectively the nucleon-nucleon, antinucleon-antinucleon, and nucleon-antinucleon potentials. Each of these potentials is invariant under isospin rotations. In addition

$$V=\bar V \tag{4b}$$

owing to the invariance of the strong interactions under G-conjugation. On this basis it is not difficult to verify that the Hamiltonian (4) is invariant under the following three types of permutations:

$$x_k\rightleftarrows x_l,\quad x_m\rightleftarrows x_r,\quad x_k\rightleftarrows x_m,$$
$$k,l=1,2,\ldots,n_1;\quad m,r=n_1+1,n_1+2,\ldots,n. \tag{5}$$

Invariance of the Hamiltonian under permutations of the third type is the basis for the introduction of baryon number variables. With their aid one can construct wave functions which transform irreducibly under all of the permutations (5). The choice between functions which transform according to different irreducible representations of the permutation group is made with the aid of the generalized Pauli principle, which implies that the total wave function ψ must be antisymmetric with respect to the interchange of any two particles. The representation of the wave function in the form (1) reduces to expanding it in the Clebsch-Gordan series for the symmetric group S_n.[10] The inner product of the irreducible representations obeys the associative law; therefore Eq. (1) can be written in the form

$$\Psi(x, \sigma, \tau, q) = \varphi(x, \sigma, \tau) \times \omega(q), \tag{6}$$

where

$$\varphi(x, \sigma, \tau) = \psi(x) \times \chi(\sigma) \times \eta(\tau). \tag{7}$$

In order for the function ψ to be antisymmetric, the functions φ and ω must transform according to conjugate representations of the group S_n. The baryon number variable q takes two values; therefore the Young pattern corresponding to the function ω cannot contain more than two rows. Consequently the conjugate Young pattern for the function φ does not contain more than two columns.

Let us denote the baryon number function of the i-th nucleon by α(i) and the baryon number function of the j-th antinucleon by β(j), that is

$$B\alpha(i) = \alpha(i), \qquad B\beta(j) = -\beta(j), \tag{8}$$

where B is the baryon number operator.

The baryon number function ω of the entire system, which transforms according to a definite irreducible representation of the group S_n, is constructed from the baryon number functions α(i) and β(j) of the individual particles. In principle such a construction can be achieved with the aid of the Young projection operators.[10] It is easy, however, to notice that the problem of constructing the baryon number function of a system with a definite baryon number and a definite permutation symmetry is completely equivalent to the problem of constructing the spin function for a system with spin component $S_3 = B/2$ and a definite permutation symmetry. Such spin functions are treated in detail in[11] for the case of a three-particle system, and these functions are written down, for example, in[12] for the case of a four-particle system.

After introducing the baryon number variables it is now impossible to indicate which particles are nucleons and which are antinucleons, that is, it is impossible to say, for example, that the particle with label i is a nucleon and the particle with label j is an antinucleon. Therefore it is necessary to write the Hamiltonian as an operator in the space of the baryon number functions. The Hamiltonian describing the interaction of two particles i and j may be written as follows:

$$U(ij) = V(ij)P(ij) + \bar{V}(ij)\bar{P}(ij) + W(ij)Q(ij). \tag{9}$$

Here P(ij), $\bar{P}$(ij), and Q(ij) are projection operators w which are symmetric with respect to i and j and which operate on the baryon number basis functions according to the following laws:

$$\begin{gathered} P(ij)\alpha(i)\alpha(j) = \alpha(i)\alpha(j), \quad P(ij)\alpha(i)\beta(j) = P(ij)\beta(i)\beta(j) = 0, \\ \bar{P}(ij)\beta(i)\beta(j) = \beta(i)\beta(j), \quad \bar{P}(ij)\alpha(i)\beta(j) = \bar{P}(ij)\alpha(i)\alpha(j) = 0, \\ Q(ij)\alpha(i)\beta(j) = \alpha(i)\beta(j), \quad Q(ij)\alpha(i)\alpha(j) = Q(ij)\beta(i)\beta(j) = 0. \end{gathered} \tag{10}$$

For specific problems, for example, in order to calculate transition amplitudes or in order to calculate bound states, we need to know the matrix elements $\langle \omega | U(ij) | \omega' \rangle$ of the Hamiltonian (9) between the baryon number functions. Here it is sufficient to know only the matrix elements of the Hamiltonian U(12) since the matrix elements of the other pair potentials can be reduced to matrix elements of U(12) by using the antisymmetry of the total wave function. The nonvanishing matrix elements for a system (2N$\bar{\text{N}}$) consisting of two nucleons and one antinucleon and for a system (2N2$\bar{\text{N}}$) consisting of two nucleons and two antinucleons are given below. The Yamanouchi symbols[10] are used in order to indicate the symmetry of the baryon number functions (the numbers in the symbol indicate in which row of the Young diagram the elements with numbers n, n − 1, ..., 1 respectively stand). The 2N$\bar{\text{N}}$ system has the following nonvanishing matrix elements:

$$\begin{gathered} \langle [111] | U(12) | [111] \rangle = {}^1/_3 V(12) + {}^2/_3 W(12), \\ \langle [211] | U(12) | [211] \rangle = {}^2/_3 V(12) + {}^1/_3 W(12), \\ \langle [121] | U(12) | [121] \rangle = W(12), \\ \langle [111] | U(12) | [211] \rangle = {}^1/_3 \sqrt{2} [V(12) - W(12)]. \end{gathered} \tag{11}$$

The 2N2$\bar{\text{N}}$ system has the following matrix elements:

$$\begin{gathered} \langle a | U(12) | a \rangle = {}^1/_3 V(12) + {}^2/_3 W(12), \quad a = [1111],\ [2111], \\ \langle b | U(12) | b \rangle = {}^2/_3 V(12) + {}^1/_3 W(12), \quad b = [1211],\ [2211], \\ \langle c | U(12) | c \rangle = W(12), \quad c = [1121],\ [2121], \\ \langle [1111] | U(12) | [2211] \rangle = \langle [2111] | U(12) | [1211] \rangle \\ = {}^1/_3 \sqrt{2} [V(12) - W(12)]. \end{gathered} \tag{12}$$

2. THE G-PARITY OF A MANY-PARTICLE SYSTEM

Now let us consider the question of the G-parity of a system of nucleons and antinucleons whose total baryon number is equal to zero (the system kNk$\bar{\text{N}}$, where k = 1, 2, ...). It is well-known that the G-parity of the two-particle (k = 1) nucleon-antinucleon system is expressed by the formula

$$G = (-1)^{l+s+T}. \tag{13}$$

It will be shown below that, in contrast to the two-particle case, for the kNk$\bar{\text{N}}$ system with $k > 1$ the G-parity does not depend on the orbital momenta, spin or isospin characterizing the many-particle system, but it is solely determined by the symmetry of the baryon number function.

Let us introduce isospin functions γ(i) and δ(j) of the individual particles, where γ(i) corresponds to the 3-component of the isospin of the i-th particle being equal to +1/2, and δ(j) corresponds to the 3-component of the isospin of the j-th particle being equal to −1/2. Then one can represent the nucleon and antinucleon states in the form

$$|p\rangle = \alpha\gamma, \quad |n\rangle = \alpha\delta, \quad |\bar{n}\rangle = \beta\gamma, \quad |\bar{p}\rangle = -\beta\delta. \tag{14}$$

It is well-known[13] that under the operation of G-conjugation the basis vectors (14) transform according to the law

$$G|p\rangle = -|\bar{n}\rangle, \quad G|n\rangle = |\bar{p}\rangle, \quad G|\bar{n}\rangle = |p\rangle, \quad G|\bar{p}\rangle = -|n\rangle. \tag{15}$$

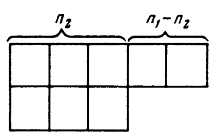
As is clear from Eqs. (14) and (15), the effect of the operator G acting on the baryon number functions is given by the formulas

$$G\alpha = -\beta, \quad G\beta = \alpha. \tag{16}$$

The spatial, spin, and isospin variables do not undergo any changes under G-conjugation. Thus, the effect of the operator G on the wave function, written in the form (6), is given by the formula

$$G\Psi = G\varphi(x, \sigma, \tau)\,\omega(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_k}, \beta_{i_{k+1}}, \beta_{i_{k+2}}, \ldots, \beta_{i_n}) = \varphi(x, \sigma, \tau)\,\omega(-\beta_{i_1}, -\beta_{i_2}, \ldots, -\beta_{i_k}, \alpha_{i_{k+1}}, \alpha_{i_{k+2}}, \ldots, \alpha_{i_n}). \quad (17)$$

As already indicated above, in general the Young pattern associated with a given baryon number function contains two rows. Let the number of squares in the f first row be n_1, let the number of squares in the second row be n_2, let $n_1 + n_2 = n$, and let us denote this partition by (n_1, n_2). Then n_2 is the number of pairs of the variables α and β with respect to which the function ω is antisymmetrized, and the difference $(n_1 - n_2)$ is equal to the number of symmetrized (with respect to interchanges among themselves) pairs α and β. According to Eq. (16) the operation of G-conjugation can be regarded as the replacement of all α by β with a subsequent change in the sign of all β. It is easy to see that such a transformation does not change the sign of the ensemble of variables α and β, which are located in the n_2 columns of the Young pattern—because in each pair of these variables a change of sign occurs twice: once due to antisymmetrization, and the second time—due to the change in the sign of β. The remaining set of $(n_1 - n_2)/2$ pairs which are symmetric with respect to the variables α and β gives a factor $(-1)^{(n_1 - n_2)/2}$ upon G-conjugation.



Thus, the G-parity of a system of n nucleons and antinucleons, whose baryon number function is described by the Young pattern with n_1 squares in the first row and n_2 squares in the second row $(n_1 + n_2 = n)$ is given by

$$G = (-1)^{(n_1 - n_2)/2}. \quad (18)$$

The functions $\varphi(x, \sigma, \tau)$ and $\omega(q)$ are described by conjugate Young patterns. Therefore, for the Young pattern associated with the function $\varphi(x, \sigma, \tau)$ the difference $(n_1 - n_2)$ is equal to the number of rows which contain a single square. In particular, for the nucleon-antinucleon system the symmetric function $\varphi(x, \sigma, \tau)$ is described by the Young pattern with $n_1 - n_2 = 0$ and according to formula (18) $G = +1$ for this pattern, but the antisymmetric function $\varphi(x, \sigma, \tau)$ has the Young pattern with $n_1 - n_2 = 2$ and, consequently, $G = -1$ for it. The possibility of writing down formula (13) for the G-parity of the nucleon-antinucleon system is due to the fact that in this simplest case, by knowing l, S, and T one can uniquely reconstruct the Young pattern of the function $\varphi(x, \sigma, \tau)$.

Let us cite an example which indicates that this cannot be done in general. Let us consider the $2N2\bar{N}$ system with L = 0, S = 1, and T = 1. Let the coordinate function of this system be symmetric, i.e., let it be described by the partition (4). The values of S and T give partitions of the form (3, 1) for the spin and isospin functions. But the product (4) × (3, 1) × (3, 1) of the partitions contains both the partition (2, 1, 1) for which G = −1 as well as the partition (2, 2) with G = +1.

Thus, our main conclusion is the following: For the $kNk\bar{N}$ system with $k > 1$ the G-parity is solely determined by the formula (18) and it is a quantum number which does not depend on the orbital momenta, spin, or isospin of the particles or of groups of particles.

It follows from the conservation of G-parity that the matrix elements of the potential between baryon number functions of different G-parity vanish (formulas (12) may serve as illustrations of this fact). If such matrix elements are calculated without assuming beforehand that the potentials V and $\bar{V}$ are equal, then they turn out to be proportional to the difference $(V - \bar{V})$.

In conclusion we note that since G-conjugation is a generalization of C-conjugation, then it is trivial to transfer everything said above to a system possessing a C-invariant interaction.

The authors express their gratitude to V. A. Karmanov, L. A. Kondratyuk, and M. S. Marinov for a discussion and for helpful comments.

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Translated by H. H. Nickle
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