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Ions and Ring Vortices in Superfluid Helium

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Two possible models of formation of an "ion-vortex ring" observed in experiments with charged particles in superfluid helium are considered. It is shown that an axisymmetric location of the ion with respect to the ring is unstable for large-size rings. A model Lagrangian is proposed for a model in which the ion is assumed to be fixed to the vortex filament.

1. It can be regarded as established[1,2] that quantized ring vortices with unit circulation are produced when charged particles move in helium. It follows from these experiments that an ion with a vortex can form a stable bound state. Cade[3] has determined experimentally the energy necessary to detach the ion from the vortex. It amounts to at least several times ten degrees. It is well known[4] that the ion in liquid helium is a complicated formation. A positive ion drags with it, via electrostatic forces, a volume of liquid surrounding it. The effective radius of such a formation is ~7 Å and the effective mass is of the order of one hundred times the mass of the helium atom m_4. A negative ion complex is formed by an electron penetrating into the liquid and a spherical cavity formed around it. The cavity radius is ~15 Å and the effective mass is ~250 m_4. The dimensions of the vortex rings capturing the ions vary from ten to 10^5 Å.

In classical hydrodynamics, the momentum, energy, and velocity of a vortex ring of radius R and circulation Γ are given by[5]

$$P=\pi\rho\Gamma R^2,\quad E={}^1/_2\rho\Gamma^2R(\eta-{}^7/_4),$$
$$v_E=\Gamma(\eta-{}^1/_4)/4\pi R,\quad \eta=\ln(8R/a_0). \tag{1}$$

If we assume that these relations hold for quantum vortices, then the arbitrary radius of the core of the vortex is $a_0 = 1.3$ Å, and $\Gamma = 10^{-3}$ cm²/sec is the unit quantum of circulation.

Two possibilities arise in natural fashion with respect to the structure of the bound state "ion + vortex ring" in helium[1,2,6–8]. The first possibility is an axially-symmetrical placement of the ion relative to the vortex ring. This recalls the situation in classical hydrodynamics[9]. In the second possible model the ion lies on the vortex filament (according to Reif,[8], the model of a snake that has swallowed a grapefruit). The purpose of the present paper is to analyze quantitatively the two models.

2. We start the calculation with the axially-symmetrical model, and leave aside questions connected with the dynamics of the formation of the vortex ring. The hydrodynamic problem, in which the ion complex is represented in the form of a sphere of radius a, can be solved accurately in terms of elliptic functions, but the corresponding calculations are quite cumbersome. Usually one deals with the case (not pertaining to the problem of vortex formation) in which the dimension of the vortex ring greatly exceeds the radius of the ion complex. In this case the calculation becomes much simpler.

We consider thus a vortex ring with an ion moving along the z axis (we place the vortex ring at the origin and the ion at the point $z = l$). The ion moves with velocity u and the ring with velocity v_E (expression (3)). At the point $z = l$, the vortex ring produces on its axis a velocity $u_1 = {}^1/_2\Gamma R^2(l^2 + R^2)^{-3/2}$. Since the dimensions of the ion are small compared with R and $H = (R^2 + l^2)^{1/2}$, we can neglect the change of the ring-induced velocity in the volume of the ion, and assume that the ion moves relative to the liquid with a velocity $u_{rel} = u - u_1$ and produces a velocity potential

$$\varphi={}^1/_2u_{rel}a^3(z-l)/r^3,$$

where a is the radius of the ion and r is the distance from the center of the ion to the observation point. This leads to a change in the velocity of the elements of the vortex ring. Denoting the coordinate of the ring by Z, we obtain

$$\frac{dZ}{dt}=v_E+u_{rel}\frac{a^3}{2H^3}\left(\frac{3l^2}{H^2}-1\right), \tag{2}$$

$$\frac{dR}{dt}=-\frac{3}{2}u_{rel}a^3\frac{Rl}{H^5}. \tag{3}$$

The presence of friction forces leads in the case of a quantum filament to a change[10] in Eq. (3). The friction force F_{fr} depends on the temperature T and on the ring radius[1], $F_{fr} = \alpha(T)\eta$. Taking the friction force into account, we obtain

$$dR/dt=-{}^3/_2u_{rel}a^3Rl/H^5-\alpha\eta/2\pi\rho\Gamma R. \tag{3'}$$

Using the momentum conservation law, we write down the equation of motion for the "ion + vortex ring" system:

$$M\frac{du}{dt}+{}^1/_2M\mu\frac{du_{rel}}{dt}+\frac{d\tilde{P}}{dt}=eE-F_{fr}. \tag{4}$$

Here M is the mass of the ion complex, $\mu M/2$ is the dynamically attached mass, $\mu \approx \rho/\rho_{sol}$, E is the electric field directed along z, $\tilde{P} = P - \mu uM$, where P and $\tilde{P}$ are respectively the momentum of the free vortex (1) and the momentum of the vortex in the presence of the ion.

Equations (2)–(4) determine completely the dynamics of the bound formation "ion + vortex ring." The mutual positions of the ring and of the ion and their joint velocity (under the condition that the bound state exists) are determined by the radius of the ring, the value of which is obtained by solving the equation

$$eE=F_{fr}(R).$$

Let us consider the free motion of this bound formation. To this end we change over, first, to dimensionless quantities, introducing the following units: the distance R_0 (which is the root of the equation $\eta(R_0) = 2\pi$), the velocity $v_0 = \Gamma/2R_0$, and accordingly the time $t_0 = R_0/v_0$. Second, we introduce the small parameter $\epsilon = a^3/R_0^3$, which we have in fact already used in the derivation of Eqs. (2) and (3). We write the equations of motion in terms of the dimensionless units, retaining the principal terms with respect to ϵ; recognizing that the ring radius varies little, we represent it in the form $R = L + \xi$, where L is a certain equilibrium value of the dimensionless radius, and $\xi \sim \epsilon$. The equations take the form

$$\frac{d\xi}{dt} = -\frac{3}{2}u_{\text{rel}}\frac{Rl}{H_0^5}, \quad \frac{dZ}{dt} = \frac{\chi}{L}, \qquad (5)$$
$$\frac{d}{dt}\left(\varkappa u - \frac{L^2}{H_0^3}\right) = -2L\frac{d\xi}{dt}.$$

Here $H_0^2 = l^2 + L^2$, $\chi = \eta/2\pi$, $\kappa = 2/3\,\mu + 1/3 = 1.05$.

Introducing the ion coordinate z, we recast the system (5) in the form of a Lagrange equation with respect to the coordinates ξ, z, and Z. The Lagrangian of the problem is given by

$$\mathscr{L} = 2L\xi(\dot Z - \chi/L) + {}^1/_2(\dot z - L^2/H_0^3)^2 + {}^1/_2(\varkappa - 1)\dot z^2.$$

We change over to a new variable $l = z - Z$. Then $\dot l = \dot z - \chi/L$, and the part of the Lagrangian connected with the variable l takes the form

$$\mathscr{L}_l = \frac{\varkappa}{2}\dot l^2 + \frac{1}{2}\left[\frac{\chi}{L} + \frac{L^2}{(l^2+L^2)^{3/2}}\right]^2.$$

At $l = 0$ the system is in a state of stable equilibrium, with respect to which it can execute small oscillations with frequency $\omega L^2 = [3(1-\chi)/\kappa]^{1/2}$. We see that the motion becomes absolutely stable at $\chi \to 1$, $R \to R_0 \approx 90$ Å.

The presence of jointly-acting friction forces and an electric field leads to a shift of the point of stable equilibrium from the center of the ring farther on its axis. At a certain value of the field (and of the velocity) the joint motion becomes impossible, owing to the instability. Numerical calculations show that the presence of friction forces and of an electric field hardly alter the value of the critical ring dimension down to 1°K. It follows from the foregoing that a model with an axially symmetrical placement of the ion can hardly account for the experimental data, since it does not agree with the fact that rings of much larger size exist.

3. We proceed now to the structure of the bound state in a model in which the ion is assumed to be located on the core of the vortex filament. At T = 0 the ionic negative complex captured by the vortex can form two states. First, the usual spherical "bubble" can land on a vortex, slightly altering its own shape and the shape of the adjacent section of the vortex filament. Second, the shape of the "bubble" formed by the electron may become toroidal, and the axis of this torus coincides with the core of the vortex ring. Calculations show, however, that at a fixed momentum the energy of the second state is $(R/r_0)^{2/3}$ times larger than that of the first, where R is the radius of the ring and r_0 is the radius of the cavity of the "free" negative complex. (The comparison was made at a constant momentum or, equivalently, at a constant ring radius, since the momentum of the ion complex and the ion-induced change of the momentum of the ring can be neglected.) The first situation is therefore realized, and the particle mass in the effective Schrödinger equation describing the behavior of this ion should be taken to be of the order of the effective mass of the "free" negative ion complex, and the potential of the toroidal-symmetry field in which the effective particle moves should be taken to be the energy of the interaction of the ionic spherical "bubble" with the vortex; this energy is apparently of hydrodynamic origin. In spectrum of such a quasiparticle, the groups of allowed states are separated from one another by an energy on the order of 0.01 eV, corresponding to optical transitions of the electron in a potential well. Each such group constitutes a set of bands (which intersect in the general case), characterized by "magnetic" and "orbital" numbers.

The uncertainty of the ion momentum in the ground state, $\hbar/R$, is negligibly small in comparison with the uncertainty due to the thermal motion, $(MkT)^{1/2}$, starting with temperatures on the order of $T_0 \sim \hbar^2/kMR^2 \sim T_\lambda q^{-1}(l_0/R)^2$ (l_0 is the interatomic distance and $q = M/m_4$). For example, for rings of radius $R = 10^3$ Å this temperature amounts to 10^{-8} °K. Thus, the classical delocalization practically always prevails over the quantum delocalization. On the other hand, there exists a temperature region (from 10^{-8} to 10^{-3}°K for $R = 10^3$ Å), where the thermal uncertainty of the momentum is small in comparison with the momentum of the translational motion of the ion. One can then speak of an ion that is immobile and pinned to the vortex filament. At higher temperatures, the ion can be regarded as smeared over the core of the filament and the ring can be regarded as charged. We note once more that even in this case the resonant characteristics of such a formation are close to the characteristics of a "free" ionic complex. All the statements made above in this paragraph pertain also to a positive ion captured by a vortex ring.

To calculate the model of the immobile ion, we consider first a closed filament of arbitrary shape. The velocity of the section of the filament under the influence of its remaining parts is given by the expression

$$\mathbf{v}(\mathbf{R}_s) = \frac{\partial \mathbf{R}_s}{\partial t} = \frac{\Gamma}{4\pi}\oint\frac{ds'}{|\mathbf{R}_s - \mathbf{R}_{s'}|^3}\left[\frac{d\mathbf{R}_{s'}}{ds'}(\mathbf{R}_s - \mathbf{R}_{s'})\right]. \qquad (6)$$

Here R_S specifies the equation of the filament in a natural parametric form. Expression (6) diverges logarithmically at small and large values of $|s - s'|$. The divergence at the upper limit is lifted because the filament is closed, and the lower limit in the integral of (6) is chosen to be the arbitrary radius a_0 of the core, and the integral is calculated with logarithmic accuracy. Since the value of the integral is determined by the behavior of the numerator at close values of s and s′, we can therefore expand the function $\mathbf{R}_S'$, which is slowly varying in this sense, in terms of $s' - s$. As a result we obtain the local relation

$$\left[\frac{\partial \mathbf{R}_s}{\partial t}\frac{\partial \mathbf{R}_s}{\partial s}\right] = \eta\frac{\Gamma}{4\pi}\frac{\partial^2\mathbf{R}_s}{\partial s^2}. \qquad (7)$$

From this equation we get, for example, an expression for the velocity of the circular vortex (1), the dispersion law of the oscillations of a linear vortex[11], etc. Equation (7) can be obtained from the variational principle

$$\delta S_0 = 0, \quad S_0 = \int dt(T-U),$$

$$T = \frac{\Gamma\rho}{3}\oint ds \mathbf{R}_s\left[\frac{\partial \mathbf{R}_s}{\partial s}\frac{\partial \mathbf{R}_s}{\partial t}\right], \quad U = \frac{\Gamma^2\rho\eta}{4\pi}\oint ds.$$

As already noted, the radius of the vortex ring greatly exceeds the dimensions of the ion, and the radius of the ion is very slightly larger than the correlation radius. It is therefore reasonable to consider a model in which a pointlike ion is secured stationarily on the vortex filament. We assume first that the dynamics of the "ion + vortex ring" system can be analyzed jointly, by imposing the constraint conditions

$$S = S_0 + S_i + S_{int}, \quad \delta S = 0, \tag{8}$$

S_i is the Lagrangian of the "free" ion complex, and S_{int} takes the foregoing constraints into account:

$$S_{int} = \int dt \oint ds\delta(s)\mathbf{F}(\mathbf{R}_s - \mathbf{r}_i),$$

where r_i is the radius vector of the ion; the length is reckoned from the point where the ion is secured. The Lagrange factor $\mathbf{F}$ has the meaning of the reaction force.

From the equations obtained on the basis of the variational principle we determine the local frequency of oscillations of an ion fastened on a linear vortex filament. The equations of motion of the filament

$$\rho\Gamma\frac{\partial u_y}{\partial t} = \frac{\rho\Gamma^2\eta}{4\pi}\frac{\partial^2 u_x}{\partial z^2} - F_x\delta(z),$$

$$-\rho\Gamma\frac{\partial u_x}{\partial t} = \frac{\rho\Gamma^2\eta}{4\pi}\frac{\partial^2 u_y}{\partial z^2} - F_y\delta(z),$$

where u is the vector of displacement in a plane perpendicular to the vortex axis at the given point, and the equation of motion of the ion, $M d^2\mathbf{r}_i/dt^2 = \mathbf{F}$, should be solved simultaneously subject to the additional condition $\mathbf{u} = \mathbf{r}_i$ at $z = 0$. The secular equation for the local frequency

$$\int_{-\infty}^{\infty}\frac{dk}{\omega + k^2\eta\Gamma/4\pi} = \frac{2\pi\rho\Gamma}{M\omega^2}$$

leads to the result

$$\omega = \frac{\Gamma}{\pi a^2}\left(\frac{\eta}{4\mu^2}\right)^{2/3}, \tag{9}$$

which is obviously valid also for ring of sufficiently large dimensions. The frequency is of the order of 10^5 MHz.

[1)]In this case the upper limit in the integral of (6) is the wavelength.

Let us examine the motion of the "ion + ring" configuration under the influence of the electric-field and friction forces. We change over to a reference frame that moves with the ring (the origin is at the center of the ring), and represent $\mathbf{R}_s$ in the form

$$\mathbf{R}_s = R_0\mathbf{e} + \mathbf{x}, \quad \mathbf{x} = a\mathbf{e} + b\frac{\partial \mathbf{e}}{\partial\varphi} + c\left[\mathbf{e}\frac{\partial \mathbf{e}}{\partial\varphi}\right]. \tag{10}$$

The vector e has the components $\cos\varphi$, $\sin\varphi$, and 0, where φ is the polar angle. Assuming x to be small, we linearize Eqs. (7)

$$\begin{aligned} -\rho\Gamma\frac{\partial a}{\partial t} + \frac{\rho\Gamma^2\eta}{4\pi R^2}\frac{\partial^2 c}{\partial\varphi^2} &= \frac{\alpha\eta}{2\pi R} + \frac{\delta(\varphi)F}{R}, \\ \rho\Gamma\frac{\partial c}{\partial t} + \frac{\rho\Gamma^2\eta}{4\pi R^2}\left(\frac{\partial^2 a}{\partial\varphi^2} + a\right) &= 0. \end{aligned} \tag{11}$$

The equation of motion of the ion is

$$Md^2c/dt^2|_{\varphi=0} = eE + F - f.$$

Here f is the resistance force exerted on the ion by the normal component and by the impurities. Equations (11) allow us to determine the dispersion law of the oscillations of the free vortex ring

$$\omega = \frac{\Gamma\eta}{4\pi R^2}n(n^2-1)^{1/2}. \tag{12}$$

We note that the spectrum of the oscillations of the "ion + vortex ring" system (9) and (12) can be investigated by the usual resonance methods. In the stationary case we can determine the shape of the deformed ring:

$$a = b = 0, \quad c = \alpha R(\varphi - \pi)^2/\rho\Gamma^2.$$

The appearance of a discontinuity in the derivative is due to the neglect of the dimensions of the ion.

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