Diffusion of Particle Spins in Storage Rings

YA. S. DERBENEV AND A. M. KONDRATENKO

Nuclear Physics Institute, Siberian Division, USSR Academy of Sciences

Submitted May 13, 1971

Zh. Éksp. Teor. Fiz. 62, 430–443 (February, 1972)

Depolarization effects due to scattering of the trajectories of particles moving in an electromagnetic field are investigated. The source of stochastic perturbation of the orbital motion of the particles may be quantum fluctuations in radiation, collisions with the residual gas, and so forth. Formulas are obtained for the rate of diffusion of spins of particles in a storage ring with an arbitrary field.

1. INTRODUCTION

It is well known[1-4] that in the classical motion of a particle the spin $\zeta(t)$ satisfies the equation

$$\dot{\zeta}=[W_L\zeta],$$

$$W_L=\left(1+\gamma\frac{q'}{q_0}\right)\frac{[v\dot{v}]}{v^2}-\frac{q}{\gamma}\frac{(Hv)v}{v^2}-\frac{g}{\gamma^2v^2}[vE], \quad (1.1)^*$$

where $q = q_0 + q' = e/m + q'$ is the gyromagnetic ratio, q' is its anomalous part, $\gamma = (1 - v^2)^{-1/2}$ ($c = 1$), and v and $\dot{v}$ are the velocity and acceleration of the particle in an electromagnetic field **E**, **H**.

In the absence of diffusion of the particle trajectory, W_L is a regular function of time determined by the momentum and coordinate values of the particle at the initial moment. In this case a beam of particles moving near an equilibrium orbit can be depolarized only near spin resonances, when the spin motion is particularly sensitive to the parameters of the particle trajectory.[5,6]

When scattering of the particles by external sources is taken into account, W_L undergoes a chaotic variation. Directly in collision events the spin $\zeta(t)$ remains continuous, and the deflection $\delta\zeta(t)$ is chosen "integrally" in the following moments as the result of change in the particle trajectory. The chaotic nature of the jumps leads to diffusion of the spins and a decrease in the initial degree of polarization of the beam.

This depolarization mechanism was studied for the first time by Baĭer and Orlov,[7] who showed that, on deviation of the equilibrium orbit from a plane, the energy jumps arising as the result of quantum fluctuations in the synchrotron radiation can lead to depolarization of the electron (positron) beam.

In this article we have developed a general approach to the problem of spin diffusion, this approach which does not depend on the specific nature of the source of momentum fluctuations and the structure of the electromagnetic field of the storage ring. Formulas have been obtained for the depolarization time far from spin resonances, and estimates are given of the rate of diffusion in a resonance situation. As an illustration we discuss the important case of motion close to ideal (by ideal motion we mean motion with separated vertical and radial oscillations about a plane closed orbit). In the ideal approximation the spin diffusion is due to vertical fluctuations of momentum. If they are small in comparison with the longitudinal fluctuations, the latter can become predominant as the result of departures from an ideal situation. Depolarization effects due to energy fluctuations depend strongly on the specific form of the perturbations and are not listed to those indicated in ref. 7.

It should be noted that far from spin resonances the depolarization mechanism being considered can be important only for systems with friction (damping) in the orbital motion (more accurately, under conditions where the lifetime of the beam is large in comparison with the damping times); in systems without "friction", scattering processes cannot depolarize the beam during the lifetime.

In the present work we do not take into account the direct interaction of the spin with the "scatterer." Effects associated with this are sensitive to a considerably smaller degree to the properties of the dynamic spin motion in an external field than the effects studied in the present work. In order to construct the kinetics of polarization in the region of dominance of various effects, it is sufficient to consider the effects independently without taking into account possible correlations between them. In refs. 8–10 and 3 the effect of radiative polarization of light particles has been studied for motion in a magnetic field. It is shown that in the case of a plane orbit in a time

$$\tau_p=\left(\frac{15\sqrt{3}}{16}\gamma^2\frac{\lambda\!\!\!^{-}}{R}\delta_{\text{rad}}\right)^{-1} \quad (1.2)$$

(here $\lambda\!\!\!^{-}$ is the Compton wavelength, R^{-1} is the radius of curvature of the orbit, and $\delta_{\text{rad}} = -\mathcal{E}^{-1}d\mathcal{E}/dt$ is the decrement of radiation loss) an equilibrium degree of polarization $8/(5\sqrt{3}) \approx 92\%$ is established.

This conclusion is valid only for $\tau_d \gg \tau_p$ (τ_d is the depolarization time obtained without inclusion of the interaction of the spin with the scatterer). In the opposite case the beam is completely depolarized in a time τ_d.

2. THE MAIN DIFFUSION EQUATION

The kinetics of polarization for the diffusion mechanism considered obviously depend substantially on the properties of the spin motion in an external field. The general nature of the dynamic motion of the spins of an ensemble of particles (without inclusion of diffusion and frictional processes) can be represented in the following way. The spin precession frequency W_L can

$^*[W_L\zeta] \equiv W_L \times \zeta$.

be written as a function of the dynamic variables of the particle in the form

$$\mathbf{W}_L(I_i, \Psi_i, \theta) = \mathbf{W}_L(I_i, \Psi_i + 2\pi, \theta) = \mathbf{W}_L(I_i, \Psi_i, \theta + 2\pi), \quad (2.1)$$

where I_i and Ψ_i are the action variables and the phases of synchrotron and betatron motions, and θ is the generalized azimuth of the particle in the storage ring.

We will assume that the motion of the spin in the equilibrium trajectory where $\mathbf{W}_L = \mathbf{W}_S(\theta)$ is known. Following ref. 4 we will introduce a periodic system of unit vectors

$$\mathbf{n}, \ \mathbf{e} = \mathbf{e}_1 + i\mathbf{e}_2 = \boldsymbol{\eta} e^{i\nu\theta}, \quad (2.2)$$

where $\mathbf{n}$ and η are orthogonal solutions of the equation

$$\boldsymbol{\zeta}' \equiv d\boldsymbol{\zeta}/d\theta = \omega_s^{-1}[\mathbf{W}_s\boldsymbol{\zeta}] \quad (2.3)$$

(ω_S is the equilibrium frequency of rotation of the particle), which have properties

$$\mathbf{n}(\theta) = \mathbf{n}(\theta + 2\pi), \ \boldsymbol{\eta}(\theta + 2\pi) = e^{-2\pi i\nu}\boldsymbol{\eta}(\theta), \quad \mathbf{n}^2 = 1, \ \boldsymbol{\eta}\boldsymbol{\eta}^* = 2, \ \nu = \text{const}, \quad (2.4)$$

and the quantity $2\pi\nu$ has the meaning of the angle of rotation of the spin about $\mathbf{n}$ during the period of motion of the particle. In the system of the unit vectors (2.2) the spin of a non-equilibrium particle satisfies the equation

$$\boldsymbol{\zeta}' = [\mathbf{W}\boldsymbol{\zeta}], \quad \mathbf{W} = \nu\mathbf{n} + \mathbf{w} = \nu\mathbf{n} + \mathbf{W}_L/\dot{\theta} - \mathbf{W}_s/\omega_s, \quad (2.5)$$

$\mathbf{w}(I_i, \Psi_i, \theta)$ is due to deviation of the particle trajectory from the equilibrium trajectory.

In analogy to the case of a singly periodic dependence, in a fixed non-equilibrium trajectory there is a solution $\mathbf{m}(I_i, \Psi_i, \theta)$ of Eq. (2.5) having the properties (2.1):

$$\mathbf{m}(I_i, \Psi_i, \theta) = \mathbf{m}(I_i, \Psi_i + 2\pi, \theta) = \mathbf{m}(I_i, \Psi_i, \theta + 2\pi). \quad (2.6)$$

For our purpose there is no need of proving this statement here. All remaining solutions rotate about $\mathbf{m}$ with a single angular velocity. Thus, the general solution can be written in the form

$$\boldsymbol{\zeta} = \zeta_m \mathbf{m}(I_i, \Psi_i, \theta) + \sqrt{1-\zeta_m^2}\,\mathrm{Re}\, e^{-i\psi}\mathbf{l}(I_i, \Psi_i, \theta), \quad (2.7)$$

where $\zeta_m = \boldsymbol{\zeta}\cdot\mathbf{m} = \text{const}$; ψ is the phase of spin precession about $\mathbf{m}$; $\mathbf{l} = \mathbf{l}_1 + i\mathbf{l}_2$ is the complex unit vector transverse to $\mathbf{m}$, satisfying the property (2.6).

The choice of $\mathbf{l}$ uniquely determines the angular velocity ψ'. Taking into account that $\mathbf{m}$ is the solution of (2.5), we obtain from (2.7)

$$\psi' = \mathbf{W}\mathbf{m} - {}^1\!/_2 i\mathbf{l}'\mathbf{l}^*,$$

and we will choose $\mathbf{l}$ so that ψ' does not depend on the phases Ψ_i, θ. In the system of the unit vectors $\mathbf{l}$ and $\mathbf{m}$ the spin moves in a constant "field" $\boldsymbol{\psi}' = \psi'\mathbf{m}$; in the equilibrium trajectory we have

$$\mathbf{m}_s = \mathbf{n}(\theta), \quad \mathbf{l}_s = \mathbf{e}(\theta), \quad \psi_s' = \nu.$$

The sensitivity of $\mathbf{m}$ to the trajectory parameters depends substantially on the closeness to spin resonances.

For a small variation of $\mathbf{w}$ ($\mathbf{w} \to \mathbf{w} + \delta\mathbf{w}$), when the property (2.6) is taken into account we obtain in the first approximation

$$\delta\mathbf{m} = \mathrm{Re}\,\mathbf{l}\sum_k \frac{(\mathbf{l}^*\delta\mathbf{w})_k}{\psi' - \nu_k}\exp(-i\Psi_k),$$

$$\mathbf{l}^*\delta\mathbf{w} = \sum_k (\mathbf{l}^*\delta\mathbf{w})_k \exp(-i\Psi_k), \quad \Psi_k' = \nu_k. \quad (2.8)$$

Let ψ' not be equal to any combination of frequencies of the orbital motion:

$$\psi' - k - \sum_{i=1}^{3} k_i\Psi_i' \equiv \psi' - \nu_k \equiv \Phi' \neq 0 \quad (2.9)$$

(k, k_i are integers). Then the particle spin value averaged for time $t > |\dot{\Phi}|^{-1}$ at a given azimuth θ will be

$$\langle\boldsymbol{\zeta}\rangle_0 = \zeta_m\langle\mathbf{m}(I_i, \Psi_i, \theta)\rangle_{\Psi_i}, \quad (2.10)$$

where $\langle\ldots\rangle_{\Psi_i}$ indicates averaging over the phases Ψ_i. If there is a spread $\Delta\Phi'$ in the frequencies Φ', the average spin for a group of particles with close-together values of I_i and ζ_m will be determined by a similar formula after mixing in the phases ψ and ψ_i for a time $t > |\omega_S\Delta\Phi'|^{-1}$.

At the moment of the collision the spin projection ζ_m and the phase ψ change discontinuously. In those cases in which in a time $|\dot{\Phi}|^{-1}$ the quantities $\mathbf{m}$ and Φ' change only slightly as a result of collisions, it is sufficient for description of the depolarization process to find the average rate of change of ζ_m:

$$\dot{\zeta}_m = \overline{\delta\zeta_m}, \quad (2.11)$$

where $\overline{\delta\zeta_m}$ is the increase per unit time, averaged over the collisions. Using the continuity of $\boldsymbol{\zeta}$ in the collision event, we obtain for the increment $\delta\zeta_m$ from Eq. (2.7)

$$\delta\zeta_m = \boldsymbol{\zeta}\delta\mathbf{m} = -{}^1\!/_2\zeta_m\delta\mathbf{m}^2 + \sqrt{1-\zeta_m^2}\,\mathrm{Re}\,\mathbf{l}\delta\mathbf{m}\exp(-i\psi). \quad (2.12)$$

The right-hand part of (2.11) can be averaged over all phases ψ, Ψ_i, and θ if the following conditions are satisfied:

1. The relaxation time τ_m of the distribution in $\mathbf{m}$ (the characteristic time for change of $\mathbf{m}$ as the result of diffusion and damping processes) is large in comparison with the times of the dynamical motion of the spin:

$$|\dot{\Phi}| \gg \tau_m^{-1}. \quad (2.13a)$$

2. The change of the "field" $\dot{\Phi}\mathbf{m}$ as a result of collisions during a time $|\dot{\Phi}|^{-1}$ is small:

$$|\dot{\Phi}|^3 \gg \overline{\delta\dot{\boldsymbol{\Phi}}^2} = \overline{\delta\dot{\Phi}^2} + \dot{\Phi}^2\overline{\delta\mathbf{m}^2}. \quad (2.13b)$$

On averaging over the phases, the second term in (2.12), which takes into account to first order in the momentum jumps δp the action of dissipative forces (frictional forces), goes to zero. As a result we obtain

$$\dot{\zeta}_m = -{}^1\!/_2\langle\delta\mathbf{m}^2\rangle_I\zeta_m, \quad (2.14)$$

where $\langle\delta\mathbf{m}^2\rangle_I$ designates the variation, averaged over the phases Ψ_i and θ, of the square of the scattering angle $\mathbf{m}$ per unit time $\overline{\delta\mathbf{m}^2}$. Considering $\mathbf{m}$ as a function of coordinates and momenta $(\mathbf{r}, \mathbf{p})$, we can express $\overline{\delta\mathbf{m}^2}$ in terms of the momentum scattering tensor

$$d_{\alpha\beta} = {}^1\!/_2\,\overline{\delta p_\alpha\,\delta p_\beta}, \quad \frac{1}{2}\overline{\delta\mathbf{m}^2} = \frac{\partial\mathbf{m}}{\partial p_\alpha}\frac{\partial\mathbf{m}}{\partial p_\beta}d_{\alpha\beta}. \quad (2.15)$$

As should be the case, in the region of applicability of Eq. (2.14) $\langle \delta m^2 \rangle_I \ll |\dot{\Phi}|$. The value $\langle \delta m^2 \rangle_I$ increases as spin resonances are approached (see Eq. (2.8)). Here the maximum rate of depolarization is limited by the conditions (2.13). Far from resonances the spread in m is small ($|\Delta m| \ll 1$) and the dominant condition is (2.13a) ($|\dot{\Phi}| \gg |\Delta\dot{\Phi}|$), where

$$\tau_m^{-1} \sim \langle \delta m^2 \rangle |\Delta m|^{-2}, \tag{2.16}$$

$\langle \ldots \rangle$ includes averaging also over I_i.

Near a resonance, where rotation of m is possible as the result of diffusion, the dominant condition is (2.13b).

For a complete description of the kinetic process it is necessary, strictly speaking, to include the equation describing the diffusion (or mixing) of the phase ψ. When the conditions (2.13) are satisfied there is no practical need of this, since the mixing time $\zeta_\perp = \sqrt{1-\zeta_m^2}\,\mathrm{Re}\,1\exp(-i\psi)$ cannot be greater than the damping time of ζ_m: even with neglect of the spread in frequencies $\dot{\Phi}$, as we can satisfy ourselves, a uniform distribution in $\zeta_\perp$ is established in a time less than $\langle \delta m^2 \rangle_I^{-1}$.

The conditions (2.13) can be violated near resonances. Here the use of Eq. (2.11) becomes ineffective, since, as the result of the rapid change in m due to diffusion and damping processes, ζ_m is not an integral of the motion in times $\sim |\dot{\Phi}|^{-1}$.

When condition (2.13) is violated, we can estimate the depolarization time from simple physical considerations (see Sec. 5 and 6).

3. DEPOLARIZATION TIME IN THE NONRESONANT CASE

Let us find the depolarization time far from spin resonances where the deviation of m from n is small ($|m - n| \ll 1$).

It is easy to see that in this case in system without friction in the orbital motion, $\tau_m^{-1} \sim \langle \delta m^2 \rangle$ and during the lifetime of the beam τ_{max} the initial degree of polarization is preserved. Actually, from (2.14)

$$|\Delta\zeta_m|_{max} \sim \langle \delta m^2 \rangle \tau_{max} \sim |\Delta m|^2_{max} \ll 1. \tag{3.1}$$

In systems with damping,

$$\tau_m^{-1} \sim \langle \delta m^2 \rangle |\Delta m|^{-2} \gg \langle \delta m^2 \rangle, \tag{3.2}$$

where $|\Delta m|$ is the equilibrium spread in m in the beam. From (3.2) it follows that during the time of a small change in polarization, repeated mixing of the particle trajectories occurs. Averaging (2.14) over the equilibrium distribution of I_i, we obtain

$$\dot{\zeta}_m = -\tau_d^{-1}\zeta_m, \quad \tau_d^{-1} = {}^1/_2\langle \delta m^2 \rangle. \tag{3.3}$$

As we can see, the average value (at a given azimuth) of the spin $\langle \zeta \rangle_\theta$ is damped exponentially with a decrement τ_d^{-1}:

$$\langle \zeta \rangle_\theta = \langle \zeta_m \mathbf{m} \rangle_\theta \approx \langle \zeta_m \rangle \mathbf{n}(\theta) = \langle \zeta \mathbf{n} \rangle_{t=0} \mathbf{n}(\theta) e^{-t/\tau_d}. \tag{3.4}$$

Using the smallness of the deviation of m from n, we will write the solution m in the form

$$\mathbf{m} \approx \mathbf{n} + \mathrm{Re}\, C\mathbf{e}. \tag{3.5}$$

Substituting (3.5) into (2.5), with inclusion of (2.6) we obtain in the first approximation

$$C = ie^{-i\nu\theta} \int_{-\infty}^{\theta} (\mathbf{w}\mathbf{e}^*)_{\theta'} e^{i\nu\theta'} d\theta' \equiv \hat{L}\mathbf{w}\mathbf{e}^*. \tag{3.6}$$

(The integration is carried out with a negative imaginary addition to ν.) From (3.3) and (3.6) we obtain a formula for τ_d^{-1}:

$$\tau_d^{-1} = {}^1/_2\langle |\delta C|^2 \rangle = {}^1/_2\langle |\delta \hat{L}\mathbf{w}\mathbf{e}^*|^2 \rangle, \tag{3.7}$$

where $\mathbf{w}\cdot\mathbf{e}^*$ can be represented by a Fourier series:

$$\mathbf{w}\mathbf{e}^* = \sum_k (\mathbf{w}\mathbf{e}^*)_k e^{-i\varphi_k} \equiv \sum_k \omega_k. \tag{3.8}$$

Then

$$\tau_d^{-1} = \frac{1}{2} \sum_{k,k'} \frac{\langle \delta\omega_k^* \, \delta\omega_{k'} \rangle}{(\nu - \nu_k)(\nu - \nu_{k'})}. \tag{3.9}$$

4. DEPOLARIZATION EFFECTS IN THE LINEAR APPROXIMATION

Let us consider the main effects arising in an approximation linear in the deviation of the particle trajectory from the equilibrium orbit. The following corrections can be important only when ν is in the immediate vicinity of the frequencies of the higher order terms. In the linear approximation we have

$$\mathbf{w}\mathbf{e}^* \equiv w_\perp \approx \frac{\partial w_\perp}{\partial \gamma}(\gamma - \gamma_s) + \frac{\partial w_\perp}{\partial q_\alpha} q_\alpha, \tag{4.1}$$

where q_α (α = 1, 2, 3, 4) are the transverse deviations of the coordinates and momenta of the particles from the equilibrium trajectory.

The solution for q_α has the form[11]

$$q_\alpha = (\gamma - \gamma_s)\rho_\alpha(\theta) + U_{\alpha i}(\theta) A_i, \quad A_i = \mathrm{const}, \tag{4.2}$$

where $(\gamma - \gamma_s)\rho_\alpha(\theta)$ determines a closed trajectory as a function of energy, and $U_{\alpha i}A_i$ describes the betatron oscillations. The quantity $U_{\alpha i}$ is a 4×4 matrix of the normal solutions (the Floquet solutions):

$$U_{\alpha i} = (Q_1, Q_1^*, Q_2, Q_2^*),$$

$$Q_i(\theta + 2\pi) = \exp(-2\pi i \nu_i) Q_i(\theta); \quad \nu_2 = -\nu_1, \quad \nu_4 = -\nu_3$$

(ν_1 and ν_3 are the frequencies of betatron oscillations).

In view of the low value of the synchrotron frequency, far from spin resonances

$$\nu \approx k, \quad k \pm \nu_{1,3} \tag{4.3}$$

the energy oscillations can be taken into account adiabatically, setting γ = const in determination of m. The effects of synchrotron modulation near resonances with high frequencies (4.3) have been discussed in Sec. 6.

The solution m can be written in the form

$$\mathbf{m} \approx \mathbf{n}(\gamma, \theta) + \Delta\mathbf{m}_b, \tag{4.4}$$

where $\mathbf{n}(\gamma, \theta) = \mathbf{n}(\gamma, \theta + 2\pi)$ is the solution m for a closed trajectory ($\mathbf{n}(\gamma_s, \theta) \equiv \mathbf{n}(\theta)$), and $\Delta \mathbf{m}_b$ is due to betatron oscillations.

By means of Eq. (3.6) we obtain

$$\mathbf{n}(\gamma, \theta) = \mathbf{n}(\theta) + \mathrm{Re}\, C_\gamma \mathbf{e},$$

$$C_\gamma = \frac{i(\gamma - \gamma_s)}{\exp(2\pi i\nu) - 1} \int_0^{2\pi} \left(\frac{\partial w_\perp}{\partial \gamma} + \frac{\partial w_\perp}{\partial q_\alpha} \rho_\alpha \right)_{\theta+\tau} e^{i\nu\tau} d\tau; \tag{4.5}$$

$$\Delta\mathbf{m}_b = \mathrm{Re}\, C_b \mathbf{e},$$

$$C_b = i \sum_{i,\alpha} \frac{A_i}{\exp[2\pi i(\nu - \nu_i)] - 1} \int_0^{2\pi} \left(\frac{\partial w_\perp}{\partial q_\alpha} U_{\alpha i} \right)_{\theta+\tau} e^{i\nu\tau} d\tau. \quad (4.6)$$

For functions $F(\theta + 2\pi) = \exp(-2\pi i\mu) F(\theta)$ the action of the operator $\hat{L}$ is:

$$\hat{L}F = i(e^{2\pi i(\nu-\mu)} - 1)^{-1} \int_0^{2\pi} F(\theta + \tau) e^{i\nu\tau} d\tau.$$

Thus,

$$\tau_d^{-1} = \frac{1}{2} \langle |\delta \mathbf{n}(\gamma, \theta) + \delta \mathbf{m}_b|^2 \rangle = \frac{1}{2} \left\langle \left| \frac{\partial C_\nu}{\partial \gamma} \delta\gamma + \frac{\partial C_b}{\partial A_i} \delta A_i \right|^2 \right\rangle. \quad (4.7)$$

The expression for the jump in betatron oscillation amplitudes we obtain from (4.2):

$$\delta A_i = U_{i\alpha}^{-1} (\delta q_\alpha - \rho_\alpha \delta\gamma) \quad (4.8)$$

($\delta q_\alpha \neq 0$, of course, only for the momentum components).

As can be seen, energy fluctuations lead to jumps not only in $\mathbf{n}(\gamma, \theta)$, but also in the betatron part $\Delta \mathbf{m}_b$, as the result of coupling of the transverse and longitudinal motions of the particle about the equilibrium trajectory. Equation (4.7) takes into account also the correlation of these effects.

Let us apply Eq. (4.7) to the common situation in which both the coupling of the vertical motion with the radial and longitudinal motions and the departure of $\mathbf{n}$ from the vertical direction are small.

1) The case of ideal geometry (complete separation of vertical and radial oscillations about a plane closed orbit). As usual,[11] we will represent the radius vector $\mathbf{r}$ of the particle in the form

$$\mathbf{r} = \mathbf{r}_s(\theta) + x\mathbf{e}_x(\theta) + z\mathbf{e}_z, \quad (4.9)$$

where x and z are the radial and vertical deviations from the plane equilibrium orbit $\mathbf{r}_S$. Here

$$\mathbf{e}_y = \mathbf{r}_s', \quad \mathbf{e}_x' = K\mathbf{e}_y, \quad \mathbf{e}_y' = -K\mathbf{e}_x, \quad \mathbf{n}(\theta) = \mathbf{e}_z = \text{const},$$
$$\nu = \gamma_s \frac{q'}{q_0}, \quad \mathbf{W}_s = (1 + \nu) K \mathbf{e}_z \omega_s; \quad (4.10)$$
$$\mathbf{e} = (\mathbf{e}_x + i\mathbf{e}_y) \exp\left[-i\nu \int (K-1) d\theta\right] \equiv \mathbf{e}\,\text{id}.$$

$\mathbf{r}$, $\mathbf{r}_S$, x, and z are measured in units of the reciprocal of the average radius of curvature of the equilibrium orbit

$$\left(\langle K \rangle = \frac{1}{2\pi} \int_0^{2\pi} K d\theta = 1 \right).$$

In the linear approximation we obtain from (1.1) and (2.5)

$$w_\perp = [(1+\nu) z'' + i\nu K z' - iK'z] \exp\left[i\nu \int^\theta (K-1) d\theta\right]. \quad (4.11)$$

(Since at the present time the systems with damping are storage rings for light particles, we will make use of the fact that q'/q_0 is small.) As can be seen, the spin diffusion in this case is due only to scattering in the z direction. The formula for τ_d^{-1} has the form

$$\tau_d^{-1} = \frac{1}{2} \left\langle \delta z'^2 \left| 1 + \frac{1}{2} \int_\theta^{\theta+2\pi} (K' - i\nu g_z) \left[\frac{f_z^*(\theta) f_z(\theta')}{1 - \exp 2\pi i(\nu + \nu_z)} \right.\right.\right.$$
$$\left.\left.\left. - \frac{f_z(\theta) f_z^*(\theta')}{1 - \exp 2\pi i(\nu - \nu_z)} \right] \exp\left(i\nu \int_\theta^{\theta'} K d\theta \right) d\theta' \right|^2 \right\rangle, \quad (4.12)$$

where $f_z(\theta + 2\pi) = e^{2\pi i\nu_z} f_z(\theta)$ is the normal solution of the Floquet equation for z oscillations

$$z'' + g_z z = 0, \quad \operatorname{Im} f_z^* f_z' = 1.$$

In the azimuthally uniform case

$$K = 1, \quad g_z = \nu_z^2, \quad f_z(\theta) = \nu_z^{-1/2} e^{i\nu_z\theta},$$
$$\tau_d^{-1} = \frac{1}{2} \left(1 - \frac{\nu\nu_z^2}{\nu^2 - \nu_z^2} \right)^2 \langle \delta z'^2 \rangle. \quad (4.13)$$

For $\nu \ll \nu_z$ the effect is due to the radial field which appears in the z oscillations. On the other hand, for $\nu \gg \nu_z$ the decisive contribution is from the longitudinal magnetic field $\mathbf{H} \cdot \mathbf{v} \approx H_z z'$. The latter effect is not associated with nonuniformity of the magnetic field. At the point $\nu_z^2 = \nu^2/(1 + \nu)$ the interference of these two effects leads to independence of the direction of $\mathbf{m}$ on z'. This explains the disappearance of diffusion in this case ($\tau_d^{-1} = 0$).

In the general case the result (4.12) depends to a significant degree on the structure of the magnetic system. Here an increase of τ_d^{-1} occurs near the resonances possible in the linear approximation,

$$\nu \approx \nu_k = \pm\nu_z + kN, \quad k = 0, 1 \ldots, \quad (4.14)$$

in proportion to $(\nu - \nu_k)^{-2}$, where N is the number of elements of periodicity in the orbit.

Let us estimate the depolarization time (for electrons, positrons) due to quantum fluctuations of synchrotron radiation, which give

$$\langle \delta z'^2 \rangle \sim \frac{\hbar}{mR} \frac{\delta_{\text{rad}}}{2}$$

($\frac{1}{2}\delta_{\text{rad}}$ is the radiative damping decrement). Comparing τ_d^{-1} near resonances (4.14)

$$\tau_d^{-1} \sim \frac{\nu^2 \langle g_z^2 \rangle}{\nu_z^2 (\nu - \nu_k)^2} \frac{\lambda\!\!\!^{-}}{R} \delta_{\text{rad}} \quad (4.15)$$

with (1.2), we see that, in the approximation of ideal geometry, quantum fluctuations of the radiation lead to depolarization only rather close to resonance:

$$|\nu - \nu_k| < \frac{\nu \langle |g_z| \rangle}{\gamma \nu_z} = \frac{q'}{q_0} \frac{\langle |g_z| \rangle}{\nu_z} \sim 10^{-3} \frac{\langle |g_z| \rangle}{\nu_z}. \quad (4.16)$$

2) Effects of energy fluctuation for small departures from idealness. In the linear approximation the effect on the polarization of a scatter in energy appears in taking into account deviations from idealness and can become determining if the momentum spread in the transverse direction is sufficiently small.

Inclusion of the nonideal part of the field leads to a dependence of the periodic part of $\mathbf{m}$ on γ ($\partial \mathbf{n}(\gamma, \theta)/\partial\gamma \neq 0$) and of the betatron part $\Delta \mathbf{m}_b$ on the radial betatron oscillations. Here the amplitudes of betatron oscillations, as a result of coupling of transverse and longitudinal motions, are in turn functions of energy. Let us consider typical examples.

a) Effect of radial field gradient. Assume that in the equilibrium orbit

$$H_x - E_z = H_y = 0, \quad g \equiv \langle H_z \rangle^{-1} \frac{\partial}{\partial x} (H_x - E_z) \neq 0. \quad (4.17)$$

(The dependence of τ_d^{-1} on $E_x(\mathbf{r}_S)$ and $E_y(\mathbf{r}_S)$ is not important in the linear approximation.) The equations of the z and x motion have the form

$$z'' + g_z z = gx, \quad x'' + g_x x = \frac{\Delta\gamma}{\gamma} K, \quad \Delta\gamma = \gamma - \gamma_s. \quad (4.18)$$

In this case the equilibrium motion of the particle and the spin are not distorted: $\mathbf{n}(\theta) = \mathbf{e}_z$. Therefore $w_\perp$ as before is due to vertical departures from the equilibrium orbit and has the form (4.11). The dependence of $\mathbf{m}$ on energy is due to coupling of the vertical and radial motions for motion of the particle about the equilibrium orbit.

Having determined ρ and U in (4.2) from (4.18), we can reduce (4.7) to the form

$$\tau_d^{-1} = \frac{1}{32}\left\langle \frac{\delta\gamma^2}{\gamma^2} \left| \hat{L}Kf_x^* \hat{L}gf_x(f_z^*\hat{L}\alpha f_z - f_z\hat{L}\alpha f_z^*) - \hat{L}f_xK\hat{L}gf_x^*(f_z^*\hat{L}\alpha f_z - f_z\hat{L}\alpha f_z^*)\right|^2 \right\rangle, \tag{4.19}$$

$$\alpha = [i(\nu^2-1)K' - \nu^3K^2]\exp\left[i\nu\int^\theta (K-1)\,d\theta\right],$$

where f_x and f_z are ideal Floquet solutions. It is evident from (4.19) that resonances are possible at $\nu \approx \nu_k = k$, $\pm\nu_z + k$, $\pm\nu_x + k$ with a $(\nu - \nu_k)^{-2}$ dependence near resonance.

The resulting τ_d^{-1} is the sum of (4.19) and (4.12). In particular, in the azimuthally uniform case $K' = g_x' = g_z' = 0$ we have

$$\tau_d^{-1} = \frac{1}{2}\left\langle\frac{\delta\gamma^2}{\gamma^2}\right\rangle \frac{\nu^6}{(\nu^2-\nu_z^2)^2}\sum_{k=-\infty}^{\infty}\frac{|g_k|^2}{(\nu-k)^2[(\nu-k)^2-\nu_x^2]^2} + \frac{1}{2}\langle\delta z'^2\rangle\left(1-\frac{\nu\nu_z^2}{\nu^2-\nu_z^2}\right)^2. \tag{4.20}$$

Note that near a resonance $\nu \approx \nu_z$ in the azimuthally uniform case the spin diffusion is determined by the mean-square amplitude A_z^2 of free z oscillations. Actually

$$A_z^2 = \frac{\tau_z}{2}\langle\delta A_z^2\rangle = \frac{\tau_z}{2\nu_z^2}\left\{\langle\delta z'^2\rangle + \left\langle\frac{\delta\gamma^2}{\gamma^2}\right\rangle\sum_k{}' \frac{|g_k|^2}{(\nu_z-k)^2[(\nu_z-k)^2-\nu_x^2]^2}\right\}, \tag{4.21}$$

where τ_z^{-1} is the total damping decrement of z oscillations. For $\nu \approx \nu_z$ we obtain

$$\tau_d^{-1} \approx \frac{\nu_z^6 A_z^2}{4(\nu-\nu_z)^2}\tau_z^{-1}. \tag{4.22}$$

b) Equilibrium motion distortion effects. Appearance of H_x, H_y, and E_z, in addition to the obvious change in the perturbation w, leads to a direct dependence of $w_\perp$ on w_z, associated with the inclination of $\mathbf{n}$ to the plane of the ideal equilibrium trajectory $(\mathbf{n}(\theta) \neq \mathbf{e}_z)$. The formula for τ_d in the general case is unwieldy and we will not present it completely.

Let us consider the $\mathbf{n}$ distortion effect in an azimuthally uniform storage ring for $\nu \sim k \gg 1$, where k is the azimuthal harmonic of the perturbation H_x. (This case has been discussed in ref. 7). Here

$$w_\perp \approx w_z\mathbf{e}_z\mathbf{e}^* \approx -w_z\mathbf{n}\mathbf{e}_{\text{id}}^* \approx -\nu g_x x\mathbf{n}\mathbf{e}_{\text{id}}^*. \tag{4.23}$$

The solution for $\mathbf{n}$ can be found by means of (4.11), substituting $z \to z_s$ (z_s are the vertical distortions of the equilibrium orbit) and retaining the leading term in ν:

$$\mathbf{n} \approx \mathbf{e}_z + \nu\,\mathrm{Re}\,\mathbf{e}_{\text{id}}\hat{L}z_s''. \tag{4.24}$$

From (4.23) and (4.24) we find

$$\tau_d^{-1} = \frac{(\nu_x\nu)^4}{2}\left\langle\delta\gamma^2\left|\frac{d}{d\gamma}\hat{L}x\,\hat{L}z_s''\right|^2\right\rangle = \frac{1}{2}\left\langle\frac{\delta\gamma^2}{\gamma^2}\right\rangle\frac{(\nu_x\nu k)^4|z_s^k|^2}{(\nu-k)^4[(\nu-k)^2-\nu_x^2]^2}, \tag{4.25}$$

where $z_s^k \approx -k^{-2}H_x^k/\langle H_z\rangle$ is the harmonic of number k of the vertical distortion of the equilibrium orbit.

In ref. 7 where the depolarization effect due to energy jumps was first discussed, Baĭer et al., derived the equation

$$\tau_d^{-1} \approx \frac{1}{2}\left\langle\frac{\delta\gamma^2}{\gamma^2}\right\rangle\frac{\nu^4k^4}{(\nu-k)^4}|z_s^k|^2. \tag{4.26}$$

The difference between (4.26) and (4.25) is explained by the fact that amplitude jumps in the betatron x oscillations were not taken into account in ref. 7. This is justified only for $|\nu - k| \ll \nu_x$.

Let us compare the relative role of the field H_x and its gradient $\partial H_x/\partial x$ (Eqs. (4.25) and (4.20)). In practice, if special measures are not taken,

$$\left|\frac{\partial H_x}{\partial x}\Big/H_x\right| \sim \frac{R}{\Delta R} \gg 1,$$

where R is the orbit radius, ΔR is the effective length for variation of H_x with radius, which is usually of the order of the storage-ring chamber dimension. Here the term with the gradient will be the main term, except very close to a resonance $\nu \approx k$:

$$|\nu - k| < \nu\Delta R/R. \tag{4.27}$$

5. DIFFUSION NEAR RESONANCES

For definiteness we will consider spin diffusion near an isolated resonance of first order in the presence of damping in the orbital motion of the particle. Examples with more complicated dynamical motion (the case of overlapping resonances) are discussed in Sec. 6. The correct solution for m, which is applicable also for $|\nu - \nu_k| \lesssim |\omega_k| \equiv u$, has the form[6]

$$\mathbf{m} = \mathbf{h}/h, \quad \mathbf{h} = \varepsilon\mathbf{n} + \mathrm{Re}\,ue^{i\Psi_k}\mathbf{e}^*, \qquad \varepsilon = \nu - \nu_k + \langle\mathbf{w}\mathbf{n}\rangle, \quad \Psi_k' = \nu_k. \tag{5.1}$$

In a resonant system (rotating with respect to (2.2) with a velocity $\nu_k\mathbf{n}$) the spin precesses around $\mathbf{m}$ with a constant angular velocity h. In the case $h > \lambda$,[1)] where λ is the damping decrement of h resulting from friction ($\tau_m \sim \lambda^{-1}$), the depolarization time can be found by means of (2.14). In order of magnitude

$$\langle|\delta ue^{i\Psi_k}|^2\rangle \sim u^2\lambda, \quad \overline{\delta\varepsilon^2} \sim \Delta^2\lambda, \tag{5.2}$$

where u^2 and Δ^2 are the mean-square spread in the equilibrium state of the beam.

For $\Delta^2 < u^2 + \langle\epsilon\rangle^2$ the jumps in ϵ can be neglected, and the estimate

$$\tau_d^{-1} \sim \langle\delta\mathbf{m}^2\rangle \sim u^2\lambda/h^2 \tag{5.3}$$

is applicable also to the resonance region, where the spin diffusion rate reaches a maximum value $\tau_d^{-1} \sim \lambda$.[2)]

In the opposite case $\Delta^2 > u^2 + \langle\epsilon\rangle^2$ the depolarization is due to diffusion in ϵ. Here τ_d is determined by the region of small $|\epsilon|$, which all particles visit in a

---

[1)]We limit ourselves to the case $\lambda_1 \sim \lambda_2 \sim \lambda_n \sim \lambda$.

[2)]In the resonance region, when the spread in $\mathbf{m}$ is large, there first occurs a dynamic mixing of the particle spins (dynamic depolarization) which, as a rule, does not lead to complete disappearance of $\langle\zeta\rangle$ and an isotropic distribution of spins. The value of τ_d in this case characterizes the damping time of the "residual" polarization.

time $\sim\lambda^{-1}$ as the result of diffusion processes. If $\mu^3 > \overline{\delta\epsilon^2}$, then condition (2.13b) (in our case $\Phi' = h$) is satisfied for all ϵ. Here in order of magnitude τ_d is equal to the time to reach the resonance region $|\epsilon| \sim u$, where inversion of m occurs. Thus, for an equilibrium distribution of particles in the beam, $\tau_d \sim \lambda^{-1}$. And if

$$u^3 < \overline{\delta\varepsilon^2}, \tag{5.4}$$

then (2.14) is applicable only up to $|\epsilon| \lesssim \epsilon_b = (\overline{\delta\epsilon^2})^{1/3}$. An estimate of τ_d can be obtained by considering the depolarization process as uncorrelated fast traversals with a frequency $\sim\lambda$ of an effective region $|\epsilon| \lesssim \epsilon_b$. For one traversal the change in ζ_n to first order is:

$$\delta\zeta_n \sim \zeta_\perp u\tau, \quad \zeta_\perp = \sqrt{1-\zeta_n^2}, \tag{5.5}$$

where $\tau \sim \epsilon_b^{-1}$ is the time of rotation of the spin around n by an angle of order unity in traversal of the region $|\epsilon| < \epsilon_b$. The mean-square rate of change of

$$\dot{\zeta}_n \sim \frac{1}{2}\frac{\partial}{\partial\zeta_n}(\delta\zeta_n)^2\lambda \sim -\frac{u^2}{\varepsilon_b^2}\lambda\zeta_n.$$

Consequently, for the condition (5.4) we have

$$\tau_d^{-1} \sim \frac{u^2}{\Delta}\left(\frac{\lambda}{\Delta}\right)^{1/3}. \tag{5.6}$$

Evaluation of τ_d by means of (2.14), of course, gives the same order of magnitude:

$$\tau_d^{-1} \sim \lambda\int_{\varepsilon_b^2}^{\Delta^2}\langle\delta m^2\rangle\frac{d\varepsilon^2}{\overline{\delta\varepsilon^2}} \sim \frac{u^2}{\Delta}\left(\frac{\lambda}{\Delta}\right)^{1/3}.$$

It remains to consider the case $h < \lambda$. Here (2.14) is inapplicable. Let us estimate the rate of damping of $\langle\zeta_n\rangle$. In times $\sim\lambda^{-1}$ the field h can be considered as a small perturbation. In a time λ^{-1} (the flipping time is $h - \epsilon n$) the change is $\delta\zeta_n \sim \zeta_\perp u\lambda^{-1}$. The mean-square rate of change is $\dot{\zeta}_n \sim -u^2\zeta_n/\lambda$. Hence

$$\tau_d^{-1} \sim u^2/\lambda. \tag{5.7}$$

The maximum diffusion rate $\tau_{d\,max}^{-1} \sim \lambda$ is reached near resonances with a width $u \gtrsim \lambda, \overline{\delta\epsilon^2}$.

As should be the case, the depolarization time is always greater than the times of the dynamical motion ($\tau_d^{-1} < h$).

For a single crossing of a resonance from outside, diffusion processes can be important only in the case of a slow crossing ($|\dot{\epsilon}_{ext}| \ll u^2$). For preservation of the degree of polarization the obvious necessary condition is[3)]

$$\tau_{d_{\varepsilon=0}}|\dot{\varepsilon}_{id}| \gg \max(\Delta, u, \lambda). \tag{5.8}$$

6. SPIN DIFFUSION IN SYNCHROTRON CROSSINGS OF A RESONANCE

Let us consider an example of spin diffusion for a more complex dynamical motion, when many resonances overlap. Such a situation, for example, can occur near a resonance (4.3) at high frequencies (in the linear approximation) during slow synchrotron oscillations of the energy. In a resonant system the spin moves in average field modulated by energy oscillations,

$$\mathbf{h} = \mathbf{h}_s + \Delta(\gamma), \quad \Delta(\gamma) = \Delta_0\sin\Psi_\gamma\mathbf{e}_3, \quad \Psi_\gamma' = \nu_\gamma; \tag{6.1}$$

Ψ_γ and ν_γ are the phase and frequency of synchrotron oscillations. For $\Delta \ll \nu_\gamma$ or $\Delta \ll h_s$ the diffusion rate can easily be found on the basis of the discussion above. Overlapping of resonances (in terms of the theory of isolated resonances) occurs for $\Delta \gg \nu_\gamma, h_s$. In this case we are dealing in essence with periodic traversals of a resonance (4.3). We will use the designations

$$\mathbf{h} = \varepsilon\mathbf{e}_3 + u\mathbf{e}_1, \quad \varepsilon = \mathbf{h}\mathbf{e}_3 = \varepsilon_s + \Delta(\gamma),$$
$$u\mathbf{e}_1 = \mathbf{h}_s - (\mathbf{h}_s\mathbf{e}_3)\mathbf{e}_3 = \text{const}. \tag{6.2}$$

The dynamics of the spin motion in a field of the form of (6.2) have been studied in ref. 6, where the periodic solution for m and the frequency μ of spin precession around m were found. Here the main parameters determining m and μ are

$$\delta \approx \frac{\pi}{4}\frac{u^2}{\nu_\gamma\Delta_0}, \quad X = \frac{1}{2\nu_\gamma}\int_0^{2\pi} h\,d\Psi_\gamma, \quad Y = \frac{1}{2\nu_\gamma}\int_0^{2\pi} h\,\text{sign}\,\varepsilon\,d\Psi_\gamma. \tag{6.3}$$

Here we will limit ourselves to discussion of the limiting cases of fast or slow traversals.

1) Fast traversal ($\delta \ll 1$). For a single fast traversal ζ_3 changes by an amount $\Delta\zeta_3 \sim \sqrt{\delta}$. Periodic crossing leads to oscillations of ζ_3 whose amplitude and frequency are determined by the closeness of $Y/\pi \approx \epsilon_s/\nu_\gamma$ to an integer: $\Delta\zeta_{3max} \sim \sqrt{\delta}/\sin Y$, and the frequency of the oscillations is $\mu \sim \nu_\gamma \sin Y$. A diffusion Δ_0 leads to development of chaotic phases of the traversals of the resonance. The reciprocal correlation time of the traversal phase is

$$\tau_{corr}^{-1} \sim \min\left(\nu_\gamma;\ \overline{\delta X^2} \sim \frac{\overline{\delta\Delta^2}}{\nu_\gamma^2} \sim \frac{\Delta^2}{\nu_\gamma^2}\lambda\right).$$

The depolarization rate is determined by the dynamic change $\Delta\zeta_{3corr}$ in a time t_{corr}:

$$\tau_d^{-1} \sim (\Delta\zeta_{3corr})^2\tau_{corr}^{-1}. \tag{6.4}$$

The following cases can be distinguished:

a) $\nu_\gamma|\sin Y| > \overline{\delta X^2}$; here $\Delta\zeta_{3corr} \sim \Delta\zeta_{3max}$ and

$$\tau_d^{-1} \sim \frac{\delta_0}{\sin^2 Y + \delta_0}\overline{\delta X^2} \tag{6.5}$$

($\sqrt{\delta_0}$ is the width of a resonance $Y = k\pi$);

b) $\nu_\gamma|\sin Y| < \overline{\delta X^2} < \nu_\gamma$; $\Delta\zeta_{3corr} \sim \tau_{corr}\delta_0$,

$$\tau_d^{-1} \sim \nu_\gamma^2\delta_0/\overline{\delta X^2}; \tag{6.6}$$

c) $\nu_\gamma < \overline{\delta X^2}$, $\sqrt{\Delta\nu_\gamma} > (\overline{\delta\epsilon^2})^{1/3}$, successive traversals are completely uncorrelated:

$$\tau_d^{-1} \sim \nu_\gamma\delta_0; \tag{6.7}$$

d) $\sqrt{\nu_\gamma\Delta} < (\overline{\delta\epsilon^2})^{1/3}$; $\nu_\gamma > \lambda$; the change $\Delta\zeta_3$ for a single traversal is limited by the diffusion of ϵ (see Section 5):

$$\Delta\zeta_{3\,corr} \sim u(\overline{\delta\varepsilon^2})^{-1/3}, \quad \tau_d^{-1} \sim u^2\nu_\gamma(\overline{\delta\varepsilon^2})^{-2/3}. \tag{6.8}$$

2) Slow traversal ($\delta \gg 1$). Here the departure of m from h/h is almost always exponentially small. In the adiabatic approximation, if $u > (\overline{\delta\epsilon^2})^{1/3}$, the resonance occurs dynamically with a rate $\dot{\epsilon} = \nu_\gamma\Delta_0$. Here

$$\tau_d^{-1} \sim \min(\nu_\gamma;\ \langle\delta m^2\rangle),\ \langle\delta m^2\rangle \sim \Delta\lambda/u. \tag{6.9}$$

[3)]For $u > \lambda, \overline{\delta\epsilon^2}$, a dynamic depolarization occurs in the course of an adiabatic traversal. The condition (5.8) assures the restoration of the initial degree of polarization after traversal of the resonance.

The case $u < (\overline{\delta\epsilon^2})^{1/3}$ does not differ from case d).

The adiabatic nature is destroyed near resonances $X \approx k\pi$ ($X \sim \Delta/\nu_\gamma$) with width $e^{-\delta}$ which occur as the result of diffusion in X. The change in traversal of one resonance is

$$\Delta\zeta_3 \sim e^{-\delta_0}\left[e^{-2\delta_0} + \left(\frac{\overline{\delta X^2}}{\nu_\gamma}\right)^{2/3}\right]^{-1/2}. \qquad (6.10)$$

Since $\overline{\delta X^2}$ resonances are traversed per unit time, we have

$$\tau_{d\,\text{non-ideal}}^{-1} \sim \overline{\delta X^2}\left[1 + e^{2\delta_0}\left(\frac{\overline{\delta X^2}}{\nu_\gamma}\right)^{2/3}\right]^{-1}. \qquad (6.11)$$

The resulting τ_d^{-1} is determined by the sum of (6.9) and (6.11). As can be seen, nonadiabatic effects can be neglected for

$$\delta_0 e^{-4\delta_0} \ll \left(\frac{\lambda}{\Delta}\right)^{1/2} \frac{\lambda}{\nu_\gamma}. \qquad (6.12)$$

The maximum rate (for a given $\overline{\delta X^2}$) $\tau_d^{-1} \sim \overline{\delta X^2}$ is reached for $e^{-\delta_0} > (\overline{\delta X^2}/\nu_\gamma)^{1/3}$.

We take this occasion to thank V. N. Baĭer and S. T. Belyaev for their interest in this work and for discussions, and Yu. M. Shatunov for useful advice on experimental questions. We are particularly indebted to A. N. Skrinskiĭ for stimulating discussions during the course of this work.

[1]V. Bargmann, L. Michel, and V. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).

[2]V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, Relyativistskaya kvantovaya teoriya (Relativistic Quantum Theory), Part 1, Fizmatgiz, 1968.

[3]V. N. Baĭer, V. M. Katkov, and V. M. Strakhovenko, Phys. Lett. A **31**, 198 (1970).

[4]Ya. S. Derbenev, A. M. Kondratenko, and A. N. Skrinskiĭ, Dokl. Akad. Nauk SSSR **192**, 1255 (1970) [Sov. Phys. Dokl. **15**, 583 (1970)].

[5]Kh. A. Simonyan and Yu. F. Orlov, Zh. Eksp. Teor. Fiz. **45**, 173 (1963) [Sov. Phys. JETP **18**, 123 (1964)].

[6]Ya. S. Derbenev, A. M. Kondratenko, and A. N. Skrinskiĭ, Zh. Eksp. Teor. Fiz. **60**, 1216 (1971) [Sov. Phys. JETP **33**, 658 (1971)]; Preprint, Institute of Nuclear Physics, Siberian Division, Academy of Sciences, USSR, No. 44–70.

[7]V. N. Baĭer and Yu. F. Orlov, V International Conference on High Energy Accelerators, Frascati, 1965, p. 569; Dokl. Akad. Nauk SSSR **165**, 783 (1965) [Sov. Phys. Dokl. **10**, 1145 (1966)].

[8]A. A. Sokolov and I. M. Ternov, Dokl. Akad. Nauk SSSR **153**, 1052 (1963) [Sov. Phys. Dokl. **8**, 1203 (1964)].

[9]V. N. Baĭer and V. M. Katkov Phys. Lett. A **24**, 327 (1967).

[10]V. N. Baĭer and V. M. Katkov, Zh. Eksp. Teor. Fiz. **52**, 1422 (1967) [Sov. Phys. JETP **25**, 944 (1967)].

[11]A. A. Kolomenskiĭ and A. N. Lebedev, Teoriya tsiklicheskikh uskoriteleĭ (Theory of Cyclic Accelerators), Fizmatgiz, 1962.

Translated by C. S. Robinson
55