Cylindrical Nonlinear Waveguides

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Scalar TE- and vector TM-types of waveguide localized electromagnetic-field distributions in a nonlinear medium are investigated for the case of cylindrical geometry. It is shown that, in contrast to the case of plane geometry, for both the TE- and TM-types of self-trapped cylindrical waveguides, there is a complete set of various modes (fundamental and higher). Besides a qualitative phase analysis of cylindrical waveguide distributions, numerical calculations are carried out that yield the proper values of the parameters of the TE- and TM-waveguides.

STUDIES of the equations of nonlinear electrodynamics [1-3] have shown that in the case of a plane geometry, in addition to the well known [4,5] scalar localized TE-waveguide distributions of the field in a nonlinear medium, vector localized TM-distributions of the field of the plane waveguide type are possible. From the point of view of the real experimental study of self-focused waveguide distributions, there is interest in the study of the consequences of the equations of nonlinear electrodynamics in the case of cylindrical geometry. In the present paper, the results are given of a similar study that applies both to TE- and TM-waveguide distributions of the electromagnetic field in a nonlinear medium. In contrast with the problem of plane geometry, where most of the results can be represented in analytical form, the theory of cylindrical self-focused waveguides is primarily based on the qualitative analysis of phase trajectories corresponding to solutions of nonlinear electrodynamics. Such an analysis is supported by the results of numerical calculations, which also permit us to determine the values of the proper parameters of the lower TE- and TM-field distributions in nonlinear waveguides. It is shown that for TE field distributions in a nonlinear medium, along with the fundamental localized waveguide mode previously known, there also exist higher localized waveguide modes, which possess nodes at finite distances from the axis of the waveguide. For the lowest nonaxial TE-mode, a distribution of the azimuthal electric field in the self-focused waveguide is obtained as the result of numerical integration. It is shown that field distributions exist in a transparent nonlinear medium of the type of localized TM-modes (fundamental and higher), corresponding to focused waveguide filaments, in which both radial and longitudinal electric field is excited.

1. For the electric field

$$\mathbf{E}(\mathbf{r},t)=\mathbf{E}^{+}(\mathbf{r})\cos\omega t+\mathbf{E}^{-}(\mathbf{r})\sin\omega t \tag{1.1}$$

the investigated equations of nonlinear electrodynamics have the form

$$\operatorname{rot}\operatorname{rot}\vec{\mathscr{E}}=k^2\varepsilon(\omega,|\vec{\mathscr{E}}|^2)\vec{\mathscr{E}}. \tag{1.2}$$

Here

$$\vec{\mathscr{E}}(\mathbf{r})=\mathbf{E}^{+}(\mathbf{r})+i\mathbf{E}^{-}(\mathbf{r}),\qquad k^2=(\omega/c)^2,$$

and ϵ is a real nonlinear dielectric constant of the medium. In the case of a plane geometry, (1.2) has solutions of the form

$$\mathscr{E}_j(\mathbf{r})\to\mathscr{E}_j(x)\exp\{ik_zz+ik_yy+i\delta_j\},\qquad j=x,y,z \tag{1.3}$$

and leads to the following set of equations:

$$\begin{aligned}&-\mathscr{E}_z''+k_z\mathscr{E}_x'=k^2\varepsilon(\omega,\mathscr{E}_x^2+\mathscr{E}_y^2+\mathscr{E}_z^2)\mathscr{E}_z,\\&k_z^2\mathscr{E}_x-k_z\mathscr{E}_z'=k^2\varepsilon(\omega,\mathscr{E}_x^2+\mathscr{E}_y^2+\mathscr{E}_z^2)\mathscr{E}_x,\\&-\mathscr{E}_y''+k_z^2\mathscr{E}_y=k^2\varepsilon(\omega,\mathscr{E}_x^2+\mathscr{E}_y^2+\mathscr{E}_z^2)\mathscr{E}_y.\end{aligned} \tag{1.4}$$

In the derivation of (1.4), we have assumed $k_y = 0$, $\delta_z - \delta_x = \pi/2$ for an arbitrary phase δ_y and real functions $\mathscr{E}_j$, which corresponds to the absence of energy flux along the x axis. Among the solutions of the set (1.4), we can isolate two types of exact solutions, similar to the TE- and TM-field distributions in the linear case. For the TE-distribution, (1.4) is degenerate in the equation

$$\mathscr{E}_y''+[k^2\varepsilon(\omega,\mathscr{E}_y^2)-k_z^2]\mathscr{E}_y=0, \tag{1.5}$$

which leads, in particular, to a localized plane TE-layer.[3,5] For the TM distribution, (1.4) degenerates into the system

$$\begin{aligned}&-\mathscr{E}_z''+k_z\mathscr{E}_x'=k^2\varepsilon(\omega,\mathscr{E}_x^2+\mathscr{E}_z^2)\mathscr{E}_z,\\&-k_z\mathscr{E}_z'=[k^2\varepsilon(\omega,\mathscr{E}_x^2+\mathscr{E}_z^2)-k_z^2]\mathscr{E}_x,\end{aligned} \tag{1.6}$$

the study of which, carried out in [1-3], indicates the existence of a localized plane waveguide TM-layer.

2. There is special interest in the investigation of localized field distributions for cylindrical geometry. Here, the Eqs. (1.2) have a solution of the type:

$$\mathscr{E}_j(r)\to\mathscr{E}_j(r)\exp\{ik_zz+im\varphi+i\delta_j\},\quad j=r,\varphi,z \tag{2.1}$$

and for $\delta_r - \delta_z = \pi/2$, $\delta_\varphi = \delta_z$, which corresponds to the absence of energy flux perpendicular to the axis of the cylindrical coordinate system, lead to the system

$$\begin{aligned}&-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathscr{E}_z}{dr}\right)+\frac{m^2}{r}\mathscr{E}_z-\frac{k_z}{r}\frac{d}{dr}(r\mathscr{E}_r)-\frac{mk_z}{r}\mathscr{E}_\varphi=k^2\varepsilon\mathscr{E}_z,\\&\left(k_z^2+\frac{m^2}{r^2}\right)\mathscr{E}_r+k_z\frac{d\mathscr{E}_z}{dr}+\frac{m}{r^2}\frac{d}{dr}(r\mathscr{E}_\varphi)=k^2\varepsilon\mathscr{E}_r,\\&k_z^2\mathscr{E}_\varphi-\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\mathscr{E}_\varphi)\right]-\frac{mk_z}{r}\mathscr{E}_z-m\frac{d}{dr}\left(\frac{\mathscr{E}_r}{r}\right)=k^2\varepsilon\mathscr{E}_\varphi.\end{aligned} \tag{2.2}$$

For azimuthal one-dimensional case m = 0 and among solutions to the set (2.2), we can find two types of distributions. To be precise, for the TE distributions, (2.2) leads to the equation

$$-\mathscr{E}_\varphi'' - \frac{1}{r}\mathscr{E}_\varphi' + \left(k_z^2 + \frac{1}{r^2}\right)\mathscr{E}_\varphi = k^2\varepsilon(\omega, \mathscr{E}_\varphi^2)\mathscr{E}_\varphi. \tag{2.3}$$

For TM-distributions, (2.2) leads to the system

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d\mathscr{E}_z}{dr}\right) - \frac{k_z}{r}\frac{d}{dr}(r\mathscr{E}_r) = k^2\varepsilon(\omega, \mathscr{E}_z^2 + \mathscr{E}_r^2)\mathscr{E}_z, \tag{2.4}$$

$$k_z\frac{d\mathscr{E}_z}{dr} = [k^2\varepsilon(\omega, \mathscr{E}_z^2 + \mathscr{E}_r^2) - k_z^2]\mathscr{E}_r.$$

3. The system (2.4) for the dielectric permittivity

$$\varepsilon = \varepsilon_0(\omega) + \Delta(\omega)h, \quad h \equiv \mathscr{E}_z^2 + \mathscr{E}_r^2 \tag{3.1}$$

at $\epsilon_0(\omega) > 0$ and $\Delta(\omega) > 0$, which corresponds to a transparent medium, can be written in the form

$$\ddot{e}_z + \frac{1}{\xi}\dot{e}_z + \sqrt{\alpha}\,\dot{e}_r + \frac{\sqrt{\alpha}}{\xi}e_r + (1 + e_z^2 + e_r^2)e_z = 0,$$

$$\sqrt{\alpha}\dot{e}_z = (1 + e_z^2 + e_r^2)e_r - \alpha e_r. \tag{3.2}$$

Here we have used the following notation:

$$e_j = \sqrt{\frac{\Delta(\omega)}{\varepsilon_0(\omega)}}\mathscr{E}_j; \quad \xi = \sqrt{k^2\varepsilon_0(\omega)}\,r, \quad \alpha = k_z^2/k^2\varepsilon_0(\omega).$$

As $r \to \infty$, (3.2) degenerates into a set of equations that have the first integral[3]

$$\mathscr{H} = (\mathscr{E}_z')^2 - k_z^2\mathscr{E}_r^2 + k^2\int_0^{\mathscr{E}_z^2+\mathscr{E}_r^2} dq\,\varepsilon(\omega, q). \tag{3.3}$$

For the dielectric permittivity (3.1), the integral curves on the (e_z, e_r) plane have the following representation:

$$e_r^2 = \frac{\alpha(2\beta - 2s - s^2)}{2(1+s)(1+s-2\alpha)}, \quad e_r^2 + e_z^2 = s. \tag{3.4}$$

Here

$$\beta = \Delta(\omega)\mathscr{H} / k^2\varepsilon_0^2(\omega).$$

In the case of cylindrical geometry, the change in the quantity $\mathscr{H}$, which is conserved in the plane geometry, is determined by the relation

$$\frac{d\mathscr{H}}{dr} = -\frac{2\mathscr{E}_r^2}{k_z^2 r}k^2\varepsilon(k^2\varepsilon - k_z^2). \tag{3.5}$$

Using (3.1), we find that

$$d\beta / d\xi = -2e_r^2(1+s)(1+s-\alpha) / \alpha\xi. \tag{3.6}$$

The behavior of the integral curves (3.4) for $\alpha > 1$, i.e., $k_z^2 > k^2\epsilon_0(\omega)$, is shown in Fig. 1. The system (3.2) for $\alpha > 1$, which corresponds to transverse nontransparency in linear electrodynamics, possesses three equilibrium positions. These are the saddle point $e_z = e_r = 0$ and the points of the type of the center, $e_r = \pm\sqrt{\alpha - 1}$, $e_z = 0$. The integral curves with the parameter $\beta = 0$ corresponds to a localized plane TM-layer and intersects the e_r axis at the points

$$e_r = \pm\sqrt{s_+}, \quad s_+ = {}^3/_4\alpha - 1 + [({}^3/_4\alpha - 1)^2 + \alpha - 1]^{1/2} > \alpha - 1.$$

Figure 1 also shows a circle of radius $\sqrt{\alpha - 1}$, which separates in the (e_r, e_z) plane the region $e_z^2 + e_r^2 < \alpha - 1$ in which $d\beta/d\xi > 0$, and the region $e_z^2 + e_r^2 > \alpha - 1$, in which $d\beta/d\xi < 0$. In the first of these regions, the true integral curves, which correspond to cylindrical geometry, intersect the integral curves of the plane problem, being directed toward an increase in the parameter β, since the real integral curves in the second region of the phase plane are directed, with increasing ξ, toward smaller β. For localized distributions of the field, the boundary conditions take the form

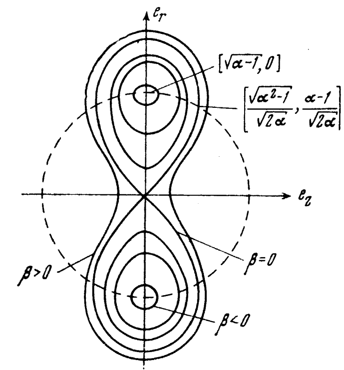

FIG. 1. Phase trajectories for TM field distributions for $K_z^2 > k^2\epsilon_0(\omega)$.

$$\lim_{\xi\to 0} e_r = 0, \quad \lim_{\xi\to 0} e_z = e_z(0), \quad \lim_{\xi\to\infty} e_z = \lim_{\xi\to\infty} e_r = 0. \tag{3.7}$$

Here $e_z(0)$ is the desired proper parameter.

We now investigate the behavior of the integral curves near the special points of the type of the center $(\pm\sqrt{\alpha - 1}, 0)$. Assuming

$$e_r = \pm\sqrt{\alpha - 1} + \tilde{e}_r, \quad e_z = 0 + \tilde{e}_z, \tag{3.8}$$

and after linearization of the system (3.2), we get the equation

$$\ddot{\tilde{e}}_z + \frac{1}{\xi}\dot{\tilde{e}}_z + \frac{\alpha(\alpha-1)}{3\alpha/2 - 1}\tilde{e}_z = \mp\frac{\sqrt{\alpha}(\alpha-1)^{3/2}}{(3\alpha/2-1)\xi}, \tag{3.9}$$

the general solution of which has the form

$$\tilde{e}_z = c_+J_0(k\xi) + c_-N_0(k\xi) \mp \frac{\pi}{2}\sqrt{\frac{\alpha-1}{\alpha}}H_0(k\xi). \tag{3.10}$$

Here $k^2 = \alpha(\alpha - 1)/({}^3/_2\alpha - 1)$, J_0 and N_0 are the Bessel functions of zeroth order, H_0 is the Struve function,[7] $c_\pm$ are arbitrary constants, and the choice of the sign corresponds to choice of the sign in (3.8). For we find that

$$\tilde{e}_z \sim \frac{A}{\sqrt{k\xi}}\cos(k\xi + \Phi) \mp \frac{1}{\xi}, \tag{3.11}$$

where A and Φ are arbitrary constants. It is obvious that such a behavior of the solutions corresponds to a winding of the integral curves about the position of equilibrium $(\pm\sqrt{\alpha - 1}, 0)$. The results of numerical integration of the system (3.2) by a high-speed computer for a number of values of $e_z(0)$ and a value of the parameter $\alpha = 2$ are shown in Figs. 2 and 3. For all values of $e_z(0) < 2$, the integral curves wind about the equilibrium point $e_z = 0$, $e_r = -1$ as $\xi \to \infty$. However, even for $e_z(0) > 3$, the integral curves go past the equilibrium position $e_z = 0$, $e_r = -1$ and wind about another equilibrium position $e_z = 0$, $e_r = +1$. With further increase in $e_z(0)$ from 3 to 4, and also from 4 to 5, another such shift occurs in the equlibrium position, about which the integral curves are wound (Fig. 3). Such a behavior of the integral curves indicates the existence of eigenvalues $e_z(0)$ which lead to a succession of localized TM modes (fundamental and higher). Detailed in-

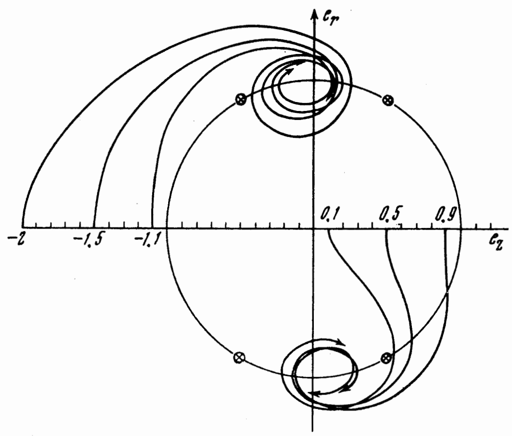

FIG. 2. Integral curves for TM field distributions for $e_z(0) \geqslant 2$. The points ⊗ denote intersections of the phase trajectories $\beta = 0$ with the circle on which $d\beta/d\xi = 0$.

vestigation permits us to establish the fact that the proper value $e_z(0)$—the projection of the electric field on the axis of the cylindrical waveguide—for the fundamental localized TM-mode is included within the limits

$$2.61 < e_z(0) < 2.62.$$

The next localized TM-mode is characterized by the condition

$$3.88 < e_z(0) < 3.89.$$

4. It has been shown previously[2,3] that, for a plane geometry and a medium that is nontransparent in a weak field, $\epsilon_0(\omega) < 0$, the equations of nonlinear electrodynamics yield solutions of the type of a plane TM-waveguide layer. The set of equations (1.6), and also (2.4) as $r \to \infty$, have the following positions of equilibrium:

$$e_z = e_r = 0; \quad e_z = 0, \quad e_r = \pm\sqrt{1+\alpha}; \quad e_r = 0, \quad e_z = \pm 1. \tag{4.1}$$

Here the values $(\pm\sqrt{1+\alpha}, 0)$ correspond to points of nonlinear transverse transparency

$$k^2\varepsilon(\omega, \mathscr{E}_r^2) = k_z^2, \quad \mathscr{E}_z = 0, \tag{4.2}$$

while $(0, \pm 1)$ are points of nonlinear longitudinal transparency

$$\varepsilon(\omega, \mathscr{E}_z^2) = 0, \quad \mathscr{E}_r = 0. \tag{4.3}$$

For a plane geometry, the phase trajectories are such that

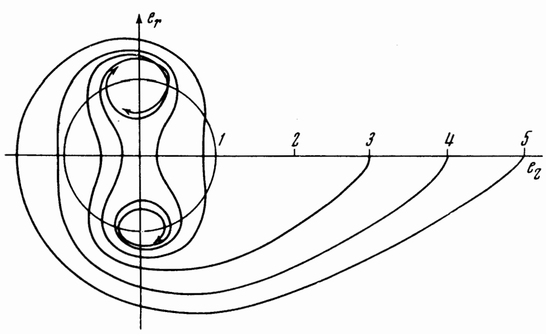

FIG. 3. Integral curves for the TM field distributions for $e_z(0) \geqslant 3$.

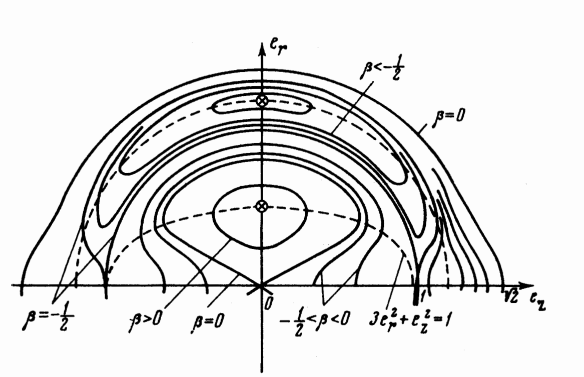

FIG. 4. Phase trajcetories for TM field distributions for $\epsilon_0(\omega) < 0$. The behavior of the trajectories for $e_r < 0$ is symmetric relative to the e_z axis.

$$\mathscr{E}_r^2 = k_z^2 \frac{\mathscr{H} - k^2 \int_0^h dq\,\varepsilon(\omega, q)}{k^2\varepsilon(\omega, h)\,[k^2\varepsilon(\omega, h) - k_z^2]}, \quad \mathscr{E}_z^2 + \mathscr{E}_r^2 = h. \tag{4.4}$$

For the case of the dielectric permittivity (3.1) and $\epsilon_0(\omega) < 0$, the relations (4.4) lead to the equations

$$e_r^2 = \alpha \frac{2\beta + 2s - s^2}{2(1-s)(1+2\alpha-s)}, \quad e_r^2 + e_z^2 = s. \tag{4.5}$$

Here we have used the following notation

$$\mathscr{E}_j = \sqrt{-\varepsilon_0(\omega)/\Delta(\omega)}\,e_j, \quad \xi = \sqrt{-k^2\varepsilon_0(\omega)}\,r,$$
$$\alpha = -k_z^2/k^2\varepsilon_0(\omega), \quad \beta = \Delta(\omega)\mathscr{H}/k^2\varepsilon_0^2(\omega). \tag{4.6}$$

The topology of the integral curves (4.5) for the values of the parameter $\alpha < 1$ is shown in Fig. 4. The phase trajectories with $\beta = 0$ correspond to the localized and periodic TM field distributions studied previously.[2,3] Closed trajectories with $\beta \neq 0$ correspond to periodic TM field distributions of a different nature in the nonlinear medium with $\epsilon_0(\omega) < 0$. The points of the phase plane $e_z = 0$, $e_r = \pm 1/\sqrt{3}$ are singular points in correspondence with the topologies of the phase trajectories (Fig. 5). It is important that the given points correspond to the maximum value of the parameter

$$\beta = 4/27\alpha + 1/6. \tag{4.7}$$

in the region of the phase plane $e_z^2 + e_r^2 < 1$.

For a cylindrical geometry, the quantity $\mathscr{H}$, which is conserved in the case of a plane geometry, changes

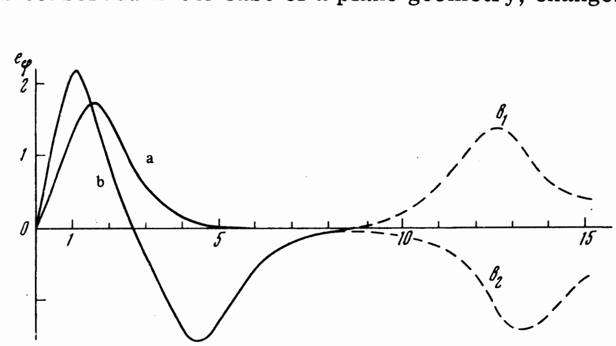

FIG. 5. Distributions of the azimuthal field in the TE waveguide: a—fundamental mode, b—lowest nonfundamental mode. The dashed curves b_1 and b_2 correspond to field distributions for values of the parameter c that are smaller and larger, respectively, than the proper value.

according to the law (3.5). In the case under study, (3.5) takes the form

$$d\beta / d\xi = -2e_r^2(1-s)(1+\alpha-s) / \alpha\xi. \tag{4.8}$$

Consequently, the derivative $d\beta/d\xi < 0$ on the entire phase plane with the exception of the annular region

$$1 < e_z^2 + e_r^2 < 1+\alpha, \tag{4.9}$$

while in the ring (4.9) we have $d\beta/d\xi > 0$.

We shall show that for cylindrical geometry and $\epsilon_0(\omega) < 0$, the equations of nonlinear electrodynamics do not yield solutions of the type of a localized TM waveguide. Actually, let us consider the integral curve which starts out from the point $e_z = e_r = 0$ for $\xi = \infty$, corresponding to the vanishing of the field at infinity. Upon decrease in ξ, the point emerges from the saddle point $e_z = e_r = 0$ along one of the separatrices and moves inside the region of the phase plane bounded by the trajectories with $\beta = 0$, intersecting the trajectory with $\beta > 0$. The motion of the representative point, upon decrease in the spatial variable, is accompanied by an increase in the value of β. In order that the axis of the cylindrical set of coordinates be reached in the motion, it is necessary that $\beta \to +\infty$. However, in the region of the phase plane $e_z^2 + e_r^2 < 1$, the largest value of the parameter β is bounded and is given by the expression (4.7). Consequently, the motion associated with the integral curve, which starts out from the saddle point at $\xi = \infty$, cannot be continued to the axis of the cylindrical system. It is evident that this statement is connected with the fact that the trajectory of the plane problem with $\beta = 0$, which corresponds to a localized TM distribution, cannot have intersections with the curves $d\beta/d\xi = 0$ which separate the phase plane into regions in which the sign of the derivative $d\beta/d\xi$ is the same. A similar situation is preserved also in the case $\alpha > 1$. We recall that in the case of a transparent medium investigated above, there was an intersection of the curves $\beta = 0$ and $d\beta/d\xi = 0$.

This, for $\epsilon_0(\omega) < 0$, there are localized TM field distributions with cylindrical geometries, satisfying the equations (2.4).

5. We now consider the problem of localized TE field distributions in the nonlinear medium. Equation (2.3) for the dielectric permittivity (3.1) and $\epsilon_0(\omega) < 0$ or $k_z^2 > k^2\epsilon_0(\omega) > 0$ takes the form

$$\ddot{e}_\varphi + \frac{1}{\xi}\dot{e}_\varphi - \frac{1}{\xi^2}e_\varphi - e_\varphi + e_\varphi^3 = 0. \tag{5.1}$$

Here we have used the notation

$$e_\varphi = \frac{k\sqrt{\Delta(\omega)}}{\sqrt{k_z^2 - k^2\varepsilon_0(\omega)}}\mathscr{E}_\varphi, \quad \xi = \sqrt{k_z^2 - k^2\varepsilon_0(\omega)}\, r.$$

The problem of localized TE distributions of the azimuthal field corresponds to the boundary conditions

$$\lim_{\xi\to 0} e_\varphi = 0, \quad \lim_{\xi\to\infty} e_\varphi = 0. \tag{5.2}$$

By investigating the behavior of solutions on the axis, we find that $e_\varphi \sim c\xi$ as $\xi \to 0$. In the numerical integration of Eq. (5.1), the value $c = de_\varphi/d\xi|_{\xi=0}$ was taken for the desired proper parameter of the problem. Calculations showed that for the fundamental localized TE mode, the characteristic parameter $c_0 \approx 1.252$. We note that the distribution of the azimuthal field over the radius for the fundamental mode is identical with the distribution found previously.[6] Computer calculations have shown that, together with the fundamental mode, there also exist higher localized TE modes. For example, the first mode shown in Fig. 5 is characterized by the value of the parameter $c_1 \approx 2.415$ and by the node at $\xi \approx 2.7$.

Thus, in contrast with the results of [5,8], we have shown here that the equations of nonlinear electrodynamics achieve exact solutions of the form of self-trapped cylindrical waveguides for both the TE type[6] and the TM type. In both cases here there is a whole set of different modes, which differ qualitatively from the solutions of plane waveguides considered earlier.

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12