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An Internally Stressed Body in a Gravitational Field

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An isolated light body whose world tube is nongeodesic because of the body's internal tensions is considered within the framework of general relativity theory.

AN internally stressed body located in a nonuniformly curved space will tend to move in such a manner that the deformation accompanying its motion will lead to a reduction of the body's energy. Consequently, the world line of an internally stressed body can be strongly nongeodesic.

When the world tube of a specific body is calculated it is, of course, sufficient to require that the covariant divergence of the energy-momentum tensor should vanish:

$$\nabla_i T^{ij} = 0 \tag{1}$$

and that no flux should pass through the boundary. The calculation is greatly simplified when the body is a thin shell.[1)] Thinness is here understood to mean that the body is subjected to no bending stresses but only to a tensile or compressional stress that is uniform throughout its thickness; we can therefore disregard the thickness. The shell can then be regarded as a two-dimensional surface having a world tube that is represented by the equations

$$y^i = y^i(x^\alpha);\ i = 0, 1, 2, 3,\ \alpha = 0, 2, 3. \tag{2}$$

In this case the derivative in (1) should only be taken with respect to directions within the tube:

$$\nabla_\alpha T^{\alpha j} \equiv \partial_\alpha T^{\alpha j} + \Gamma_{\alpha\beta}{}^\alpha T^{\beta j} + \Gamma_{hi}{}^j T^{\alpha h}\partial_\alpha y^i = 0; \tag{3}$$

here $T^{\alpha j}(x) = T^{\alpha\beta}\partial_\beta y^j$ is the energy-momentum tensor of the shell with one index transformed to the coordinate system of the enveloping space time; $\Gamma^d_{\gamma\beta}$ is the Christoffel symbol of the coordinate system x^α introduced for the shell.

As a concrete example we attempt to satisfy (3) for a surface of rotation in the form of a saucer:

$$r = r(\theta),\ 0 < \theta < \theta_m, \tag{4}$$

which is suspended motionless in Schwarzschild space; we shall use the notation in Sec. 97 of [1], but with $c = 1$, $r_g = 1$. We make the important assumption that the density and total mass of the body are small, and that we can neglect distortion of the Schwarzschild field.

The nonvanishing components of the tensor $T^\alpha_\beta = T^{\alpha\gamma} g_{\gamma\beta}$ are denoted by $T^0_0 = \epsilon$, $T^2_2 = -\sigma_2$, $T^3_3 = -\sigma_3$. Substituting these into (3), we obtain two equations for the four undetermined functions of θ: ϵ, σ_2, σ_3, r. The other two equations of the system (3) are satisfied identically because of the static conditions and the symmetry with respect to φ. The elimination of σ_3 leads to

$$\varepsilon = \frac{2r(r-1)(\sigma_2\sin\theta)'}{r'^2 + r^2 - r}\left(\frac{r-1}{\cos\theta} - \frac{r'}{\sin\theta}\right) + \sigma_2 f(r, r', r'', \theta), \tag{5}$$

where f is a complicated known function and the primes denote differentiation with respect to θ.

If the shape of the surface $r(\theta)$ is given by (6)

$$r' = (r-1)\,\mathrm{tg}\,\theta,\ \text{i. e. } r = 1 + (r_0 - 1)/\cos\theta \tag{6}$$

(where r_0 is the Schwarzschild coordinate of the center of the saucer) then in (5) the first term, whose sign changes necessarily [because $\sigma_2(\theta)_m = 0$], vanishes, and for ϵ there remains the simple expression

$$\varepsilon = -\sigma_2 3(r_0 - 1)\sin^2\theta / (\cos^3\theta + r_0 - 1). \tag{7}$$

Thus with $\sigma_2 < 0$ (a radially stretched saucer) we have $\epsilon > 0$. For the same shape of the surface, σ_3 is given by

$$\sigma_3 = \sigma_2'(\cos\theta + r_0 - 1)\sin\theta\cos\theta / (\cos^3\theta + r_0 - 1) + \sigma_2 a(\theta), \tag{7a}$$

where $a(\theta)$ designates a complicated function that equals unity approximately. Therefore the scalar $T^\alpha_\alpha \equiv \epsilon - \sigma_2 - \sigma_3$ unfortunately becomes negative in a small region on the edge of the described saucer. For exam-

[1)] The terminology is taken from the theory of the strength of materials.

ple, in the case of small θ ($\theta_m \ll 1$) and large $r_0 \gg 1$ we have

$$T_\alpha^\alpha \approx -3\theta^2\sigma_2 - 2\sigma_2 - \theta\sigma_2' = -3\theta^2\sigma_2 - \theta^{-1}(\theta^2\sigma_2)', \quad (7b)$$

so that the term which necessarily undergoes a change of sign now appears in explicit form. However, for the integral over the saucer we have

$$\int T_\alpha^\alpha dV = r_0^2 \int_0^{\theta_m} d\theta \int_0^{2\pi} \theta\, d\varphi \left[-3\theta^2\sigma_2 - \frac{1}{\theta}(\theta^2\sigma_2)' \right] > 0 \quad (8)$$

because $\epsilon > 0$. Moreover, as the size of the saucer is increased the region where $T_\alpha^\alpha < 0$ becomes relatively smaller. If the saucer is extended to infinity ($\theta_m = \pi/2$) while both ϵ and σ_2 decrease like r^{-5} or more slowly, we shall have $T_\alpha^\alpha \geq 0$ everywhere.

The total energy of the matter and gravitational field of the saucer, given by the integral (in Sec. 101 of [1])

$$\int (T_0^0 - T_2^2 - T_3^3)\, dV,$$

is, like (8), positive for $\theta_m \ll 1$.

The condition of "energy dominance" (i.e., the requirement that an observer moving in an arbitrary manner find positive energy density) is reduced to the inequalities $\epsilon \geq |\sigma_2|$, $\epsilon \geq |\sigma_3|$ and is violated for the material of the saucer because too large tensions are required.

The condition (6) has the interesting geometric meaning that the angular component of the second quadratic form[2] of the surface (2) vanishes: $\Omega_{33} = 0$. Only a thin shell of this form remains undeformed under internal stresses.

It should be noted that the foregoing results essentially provide a static solution of Einstein's equation for two bodies or, at least, prove that such a solution exists.

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[1]L. D. Landau and E. M. Lifshitz, Teoriya polya, Nauka, 1967 [The Classical Theory of Fields, 2nd ed., Addison-Wesley, Reading, Mass., 1962].

[2]L. P. Eisenhart, Riemannian Geometry, Princeton Univ. Press, 1926 (Russ. transl., 1948).

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