ON THE QUANTUM THEORY OF LASERS

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Submitted May 13, 1971

Zh. Eksp. Teor. Fiz. 61, 1790–1800 (November, 1971)

Fluctuations of single-mode laser radiation are considered for the case of high density of excited atoms. In this case it is necessary to take into account effects of the self-consistent field in discussing spontaneous emission by atoms. Of greatest interest in this case is the study of the domain of instability of stationary generation, since near the boundary of this region fluctuations become large. It is shown that on the boundary of the instability region the correlator of the fluctuations of the number of photons oscillates with time while the damping tends to zero. The effect of inhomogeneous broadening on collective processes is also considered. It turns out that for low energies of the radiation the thermal motion of the atoms converts the coherent interaction of long electromagnetic wave-trains produced in the course of spontaneous emission by atoms into incoherent interaction. For high energies the interaction again becomes coherent. A simple physical interpretation of the effects of the self-consistent field is given.

1. INTRODUCTION

QUANTUM fluctuations of laser radiation have been studied in greatest detail both experimentally[1] and theoretically[2] in the case of a gas laser near the generation threshold. In this domain the fluctuations in the number of photons are much greater than in the Poisson distribution: $\overline{\Delta n^2} \gg \bar{n}$. This circumstance makes it significantly easier to observe them. Theoretically the threshold domain is convenient because here the fluctuations can be described by means of random forces: below the threshold one can introduce an effective temperature and determine the correlators of the random forces from thermodynamic considerations. In going over into the domain situated somewhat above the threshold these correlators are not changed. We emphasize that the density of excited atoms in the gas laser, as a rule, is small, so that the spontaneous radiation of individual atoms can be regarded as statistically independent.[3]

In the present paper we wish to draw attention to the fact that for the observation of fluctuations the study of a laser near the domain of instability of stationary generation is of definite interest since here the fluctuations can become large. In this case we have in mind the generation in the single-mode regime. In the model of a laser with fixed atoms and a single relaxation time τ instability arises for $\nu\tau > 2$ and for a sufficiently high energy of the radiation,[4] ν is the inverse lifetime for a photon in the resonator. The parameter $\nu\tau$ is large in the case of molecular generators. In gas lasers in some cases[1)] it can be of the order of magnitude unity; it is quite possible that by varying the parameters of the laser one can enter the domain of instability.

The condition $\nu\tau \gtrsim 1$ (the photon mean free path is smaller than the spontaneous electromagnetic wave train $c\tau$) corresponds to a high density of excited atoms. The spontaneous radiation of individual atoms can now no longer be regarded as statistically independent and one must take into account the effects of the self-consistent field.

In the domain of generation distant from the threshold thermodynamic considerations are not applicable for describing fluctuations and it is necessary to solve the corresponding quantum problem. The solution of this problem for arbitrary $\nu\tau$ and for arbitrary energy of radiation has been obtained previously in [7]. In the present paper this solution is utilized for the study of the following two questions: the time correlation of photons (Sec. 2) and the interaction of electromagnetic spontaneous wave trains under the conditions of strong inhomogeneous broadening (Sec. 3).

In Sec. 2 attention is principally devoted to the region close to the instability boundary. The statistical properties of fluctuations in this region are essentially different from those near the generation threshold. It turns out that the correlator of the fluctuations at the boundary of the instability region oscillates with time and damping tends to zero, while in the threshold region the correlator of the fluctuations decays exponentially with time.

In the case of a single atomic relaxation time τ and $\nu\tau > 1$ one can speak of coherent interaction of spontaneous electromagnetic wave trains since phase correlation is preserved within the limits of the whole wave train $c\tau$.

In some cases (solid state lasers and also CO_2 lasers) the transverse relaxation time $1/\gamma$ is small; $\gamma \gg 1/\tau$, $\gamma \gg \nu$ (τ is the longitudinal relaxation time). Now for $\nu\tau > 1$ incoherent interaction of wave trains occurs, since phase correlation is preserved only within the limits c/γ.

In Sec. 3 it will be shown that under the conditions of strong inhomogeneous broadening the coherent interaction of wave trains is converted into an incoherent interaction. But this occurs only for low energies of the ra-

[1)]For example, in a mercury laser in the case of the 1.52μ transition [5]. The parameter $\nu\tau$ depends in an essential manner on the quality factor of the resonator and on the gas pressure. We note that in a helium-neon laser the natural widths of the 0.63 and 3.39μ transitions are quite small [6].

diation. In the case of high energies the coherence effects again become essential, and this is manifested by the fact that the generation can become unstable.

In conclusion a simple physical interpretation of the effects of the self-consistent field is given.

2. CORRELATION PROPERTIES OF THE RADIATION

The quantum model of a single-mode laser consists of a quantum oscillator of frequency Ω interacting with a system of N two-level atoms with a transition frequency ω_0. In the resonance approximation this model corresponds to the spin Hamiltonian ($\hbar = 1$)

$$H = \Omega a^+ a + {}^1/_2 \sum_i \omega_0 \sigma_3^i + \sum_i (g_i \sigma_+^i a + \bar{g}_i \sigma_-^i a^+). \tag{1}$$

Here a^+ and a are the Bose creation and annihilation operators for a photon, σ_3 is the diagonal Pauli matrix, σ_+ and σ_- are the matrices for rotating the spin up or down. The coupling constants g_i depend on the spatial structure of the mode of the field, the bar denotes complex conjugation. In the simplest case of a progressive wave and stationary atoms one can assume that $g_i = g = d\sqrt{2\omega_0/V}$, where d is the dipole moment for the transition, V is the volume of the system. For the sake of simplicity we assume the case of exact resonance $\Omega = \omega_0$.

In future it will be convenient to work in the representation of coherent states in which the annihilation operator is diagonal:

$$a|z\rangle = z|z\rangle, \quad z = x + iy, \tag{2}$$

while for the density matrices it is convenient to utilize the diagonal form.[8] Thus, the density matrix for a quantum oscillator $\hat{\rho}(t)$ has the form

$$\hat{\rho}(t) = \int d^2z|z\rangle\langle z|\rho(tz), \quad \int d^2z\rho(tz) = 1. \tag{3}$$

Going over to the interaction representation and utilizing the z-representation we obtain

$$H \to \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \quad \mathcal{H}_0 = \sum_i g_i \sigma_+^i z + \text{h.c.}, \tag{4}$$

$$\mathcal{H}_1 = -\sum_i \bar{g}_i \sigma_-^i \frac{\partial}{\partial z}.$$

The Hamiltonian $\mathcal{H}_0$ describes the behavior of spins in a classical field z; the non-Hermitean part of the Hamiltonian $\mathcal{H}_1$ which describes the absorption and emission of photons contains within itself effects of the self-consistent field and quantum effects associated with the finite width of the distribution function for the field ρ(tz). After the effects of the self-consistent field have been separated out quantum corrections of order $1/\sqrt{\bar{n}}$ can be taken into account in accordance with perturbation theory and one can obtain a closed system of equations for the following first distribution functions: ρ(tz), the density matrix for the i-th spin r_i(tz) and the correlation density matrix for the i-th and j-th spins δr_{ij}(tz).[7]

Counting Rate for Pair Coincidences

The correlation properties of the fluctuations of photon numbers in laser radiation have been investigated in a number of articles,[2, 11] but in these cases either the condition $\nu\tau \ll 1$, or the condition $\gamma \gg 1/\tau$, $\gamma \gg \nu$ was essential. Also the radiation fluxes could be regarded as correlated to a δ-function in time. Here we shall examine on the example of the simplest correlator (5) an approach which is applicable for any ratios of the relaxation times and for arbitrary energies of the radiation. Principally we shall be interested in the region close to the instability boundary.

In experiments of the Brown and Twiss type the following quantity[2)] is measured

$$G(t) = \text{Sp}(a^+(0)a^+(t)a(t)a(0)\hat{R}_0), \tag{5}$$

which determines the counting rate of pair coincidences.[8] Here $a^+(t)$ and a(t) are the Heisenberg creation and annihilation operators, $\hat{R}_0$ is the density matrix for the system "spins + radiation" for t = 0.

The quantity $G(t)(\Delta t)^2$ determines the probability of simultaneous recording of one photon in the small time interval (0, Δt) and of another photon in the interval (t, t + Δt). It is more convenient to write G(t) in the form

$$G(t) = \text{Sp}(a^+(0)a(0)\hat{R}(t)), \tag{6}$$

where $\hat{R}(t) = \exp(-iHt)\, a(0)\hat{R}_0 a^+(0) \exp(iHt)$ and H is the Hamiltonian of the system (1).

If we now utilize for the density matrices $\hat{R}_0$ and $\hat{R}$ the diagonal z-representation

$$\hat{R}(t) = \int d^2z|z\rangle\langle z|R(t,z), \quad \hat{R}_0 = \int d^2z|z\rangle\langle z|R_0, \tag{7}$$

then it is clear that the initial condition

$$R(0, z) = \xi R_0(z), \quad \xi = |z|^2.$$

holds. Since R(tz) obeys the same equation of motion as does the density matrix for the system $\hat{R}_0$(tz) in the nonstationary case, then for the quantities ρ(tz), r_i(tz) and δr_{ij}(tz) obtained from the density matrix R(tz) the equations of motion obtained in [7] hold.

In the new representation G(t) has the form

$$G(t) = \int d^2z\, \xi\rho(tz). \tag{8}$$

Coherent Interaction of Wave Trains

Further development depends on the properties of the specific laser model. We first consider a model with one relaxation time ($\gamma = 1/\tau$). Moreover, we shall treat the wave as a travelling wave, and the atoms as fixed. In such a model all the atoms experience the same conditions.

We introduce the macroscopic quantities (we omit the spin indices i and j and we replace summation by multiplication by N-the number of spins):

$$P_\alpha = N\,\text{Sp}(\sigma_\alpha r), \quad P_{\alpha-} = N\,\text{Sp}(\sigma_\alpha \tilde{\sigma}_- r), \; \tilde{\sigma}_\pm = \sigma_\pm - P_\pm/N, \tag{9}$$

$$P_{+\alpha} = N\,\text{Sp}(\tilde{\sigma}_+ \sigma_\alpha r), \quad \delta P_{\alpha\beta} = N^2\,\text{Sp}_{(1,2)}(\sigma_\alpha^{(1)} \sigma_\beta^{(2)} \delta r_{(1,2)}(tz)). \tag{10}$$

In accordance with [7] we have the following system of equations of motion:

[2)]In what follows the generation is assumed to be stationary.

$$i\frac{\partial\rho}{\partial t}=\frac{\partial}{\partial\bar z}(v_+\rho)-\frac{\partial}{\partial z}(v_-\rho),\quad v_+=gP_++i\nu\bar z,\quad v_-=\bar v_+,\tag{11}$$

$$i\frac{\partial P_\alpha}{\partial t}+\frac{i}{\tau}(P_\alpha-P_\alpha^{(0)})=\Big(v_+\frac{\partial}{\partial\bar z}-v_-\frac{\partial}{\partial z}\Big)P_\alpha+\sum_\beta\mathscr{H}_{0\alpha\beta}P_\beta$$
$$+g\frac{\partial\ln\rho}{\partial\xi}[(P_{+\alpha}+\delta P_{+\alpha})\bar z-(P_{\alpha-}+\delta P_{\alpha-})z],\tag{12}$$

where $P_3^{(0)}=N$ is the magnitude of population excess in the absence of radiation, $P_\pm^{(0)}=0$, $\mathscr{H}_{0\alpha\beta}$ are the matrix elements of the Hamiltonian $\mathscr{H}_0$ from (1). Since the stationary distribution function $\rho(\xi)$ depends only on $\xi=|z|^2$ (we recall that the angular dependence disappears due to the fluctuations of the phase of the field z), then $\rho(t,\xi)$ also has the same property, and this is utilized in Eq. (12).

The initial conditions for nonstationary quantities (we shall write the stationary ones without an index) have the form

$$\rho(0,\xi)=\xi\rho(\xi),\quad P_\alpha(0,z)=P_\alpha(z),\quad \delta P_{\alpha\beta}(0,z)=\delta P_{\alpha\beta}(z).\tag{13}$$

Restricting ourselves to generation above the threshold we can set $\xi=\bar n+\delta\xi$, where $\bar n$ is the mean number of photons in stationary generation, while $\delta\xi$ is the fluctuation in the number of photons. Since everywhere apart from the region of instability $\delta\xi\ll\bar n$, then the nonstationary quantities will differ little from the stationary ones.

Breaking up all the functions into large stationary parts and small nonstationary ones we set

$$\rho(t\xi)=\bar n\rho(\xi)(1+\chi(t,\xi)),\quad \chi\ll 1,\tag{14}$$
$$P_\alpha(tz)=P_\alpha(z)+\Delta P_\alpha(tz),\tag{15}$$
$$\delta P_{\alpha\beta}(tz)=\delta P_{\alpha\beta}(tz)+\Delta\delta P_{\alpha\beta}(tz).\tag{16}$$

At the initial time instant $\chi=\delta\xi/\bar n$, while ΔP_α and $\Delta\delta P_{\alpha\beta}$ are equal to zero.

Subtracting from G(t) the large constant "background" part we have for the fluctuation correlator

$$\Delta G(t)\equiv G(t)-\bar n^2=\pi\bar n\int_0^\infty d\xi\,\delta\xi\,\rho(\xi)\chi(t\xi).\tag{17}$$

In order to obtain equations of motion for $\chi(t\xi)$ and $\Delta P_\alpha(t\xi)$ it is sufficient to linearize the equations of motion (11) and (12). In this case one should keep in mind that we are not interested in small quantum corrections to the classical radiation flux, which do not contain the gradient of the perturbed part of the photon distribution function. For this reason in the additional quantum term in the equation for P_α it is sufficient to linearize only $\ln\rho(t\xi)\approx\ln(\bar n\rho(\xi))+\chi(t\xi)$.

The stationary distribution function for the photons has the Gaussian form:

$$\rho(\xi)=\frac{1}{\sqrt{2\pi\bar n d}}\exp\Big\{-\frac{(\delta\xi)^2}{2d\bar n}\Big\},\tag{18}$$

where d is the dispersion parameter associated with the fluctuations in the number of photons $\overline{\Delta n^2}$ by the relation

$$\overline{\Delta n^2}=(1+d)\bar n.\tag{19}$$

Under stationary conditions the probability flux is absent $(v_\pm(z)=0)$, so that $(v_\pm(tz)=g\Delta P_\pm(tz))$.

Replacing wherever possible ξ by $\bar n$ we seek the solution for $\chi(t\xi)$ in the form

$$\chi(t\xi)=\delta\xi\chi(t)/\bar n.\tag{20}$$

In this approximation ΔP_α does not depend on $\delta\xi$ and Eqs. (11) and (12) take on the form

$$i\frac{d\chi(t)}{dt}+\frac{1}{d}(gz\Delta P_+(t)-\text{c.c.})=0,\tag{21}$$

$$i\Big(\frac{d}{dt}+\frac{1}{\tau}+\nu\Big)\Delta P_+(t)=g\bar z\Delta P_3(t)-\frac{g\bar zNA}{\eta\bar n}\chi(t),\tag{22}$$

$$i\Big(\frac{d}{dt}+\frac{1}{\tau}\Big)\Delta P_3(t)=4gzk\Delta P_3(t)+\frac{iNB}{\tau\eta\bar n}\chi(t),\tag{23}$$

$$A=\frac{\eta}{N}\Big[P_{+-}+\delta P_{+-}-\frac{z}{\bar z}(P_{++}+\delta P_{++})\Big],\tag{24}$$
$$B=\frac{2gz\eta}{N}(P_{+3}+\delta P_{3+}),$$

$$k=1+\frac{1}{2}\frac{dP_3(\xi)}{d\xi},\quad \eta=\frac{g^2N\tau}{\nu}.\tag{25}$$

Here η is the generation parameter, $\eta=1$ corresponds to the threshold; in our case $\eta>1$. The coefficients A, B, k, and d were evaluated in [7].

Utilizing for the solution of Eqs. (21)–(23) the Laplace transformation $\chi_p=\int_0^\infty dt\exp(-pt)\times\chi(t)$ we obtain

$$\chi_p=\Big\{p+\frac{2\nu}{d\tau}\Big[A\Big(p+\frac{1}{\tau}\Big)-\frac{B}{\tau}\Big]\Big/\Big[\Big(p+\nu+\frac{1}{\tau}\Big)\Big(p+\frac{1}{\tau}\Big)+\frac{k(\eta-1)}{\tau^2}\Big]\Big\}^{-1}.\tag{26}$$

With the aid of this expression we consider the following two simple limiting cases. In a weak radiation field when $\eta-1\ll 1$, we have approximately

$$d=\frac{1}{(\eta-1)(1+\nu\tau)}\gg 1,\quad A=\frac{1}{1+\nu\tau},\quad B=0.\tag{27}$$

From here we find that the correlator (17) has the form

$$\Delta G(t)=\overline{\delta\xi^2(0)}e^{-p_0t},\quad p_0=2\nu(\eta-1)/[1+\nu\tau].\tag{28}$$

For small $\nu\tau$ the correlation time is determined by the photon lifetime in the resonator;[2] for $\nu\tau\gtrsim 1$ the relaxation is determined by the lifetime of the excited atom.

We now observe the behavior of the correlator $\Delta G(t)$ near the instability region. Near the curve $\eta=\eta_c(\nu\tau)$ which defines the boundary of the instability region, $A\to\infty$, $B\to\infty$, and $d\to\infty$, but their ratios are finite:[7]

$$\Big(\frac{A}{d}\Big)_c=\frac{1}{2}\frac{(\eta_c-1)}{1+\nu\tau},$$
$$\Big(\frac{B}{d}\Big)_c=-\frac{1}{2}\frac{(\eta_c-1)(1+\nu\tau)}{1+\nu\tau/2},\quad k_c=-\frac{1+\nu\tau}{\eta_c-1}.\tag{29}$$

Here the index c indicates that the values of the functions are taken on the curve $\eta=\eta_c(\nu\tau)$. Utilizing these limiting values of the coefficients we obtain

$$\Delta G(t)=\overline{\delta\xi^2(0)}\cos\omega t,\quad \omega^2=\frac{\nu}{\tau}\frac{(\eta_c-1)}{1+\nu\tau/2}.\tag{30}$$

Thus, the correlator of the fluctuations does not fall off with time, but oscillates with frequency ω. The system appears to "know" beforehand that within the instability region there must be set up an oscillatory regime of generation.[3)]

[3)]The oscillatory regime of generation in the instability region was discussed in [9].

The absence of damping is, of course, a result of the approximation (29). If we depart a little from the curve $\eta = \eta_c(\nu\tau)$ then on the right hand side of (30) there will appear the factor $\exp - \Gamma t$ where Γ is proportional to the deviation from the instability curve.

Incoherent Interaction of Wave Trains

The calculations given above can be easily carried over to the case $\gamma \gg 1/\tau$, $\gamma \gg \nu$, $\nu\tau > 1$. Therefore we give only the final result. We have[15]

$$\chi_p = \left\{ p + \frac{(\eta+1)\nu}{d} \frac{p + 1/\tau s}{p + s/\tau} \right\}^{-1}, \quad s = \eta(1 + 2\nu\tau(\eta - 1))^{-1}. \quad (31)$$

The solution of the stationary problem yields for the dispersion parameter d the following expression:

$$d = \frac{1}{2}\left(1 + \frac{1}{\eta}\right)\left(\frac{1}{\eta - 1} + 2\nu\tau\right). \quad (32)$$

Near the generation threshold, for $\eta - 1 \ll 1$ this expression coincides with that obtained earlier.[10-12] The poles χ_p are situated at $p_{1,2}$ equal to

$$p_{1,2} = -\frac{\eta}{2\tau}\left(1 \pm \sqrt{1 - 8\nu\tau(\eta - 1)/\eta^2}\right). \quad (33)$$

For large values of $\nu\tau$ the following cases are possible. Right near the threshold when $\eta - 1 \ll 1$ and $8\nu\tau(\eta - 1) < 1$, the fluctuation correlator falls off monotonically with time like $\exp(-t/2\tau)$.

For $8\nu\tau(\eta - 1) > \eta^2$ the correlator ceases to be monotonic and oscillations appear in it with frequency of the order of $\sqrt{2\nu(\eta - 1)/\eta^2}$ which is, generally speaking, large compared with the relaxation frequency $1/2\tau$. We note that the effect of the appearance of oscillations in the correlator at low energies of the radiation and large $\nu\tau$ was first noted in [11].

Finally, for large energies $\eta > 8\nu\tau$ the oscillations again disappear.

3. INTERACTION OF WAVE TRAINS UNDER CONDITIONS OF DOPPLER BROADENING

In this section we consider the question of the manner in which the thermal motion of the atoms affects the coherent interaction of wave trains.

The equations of motion for $\rho(tz)$, for the density matrix for the i-th spin $r_i(tz)$ and for the correlation matrix of the i-th and j-th spins $r_{ij}(tz)$ obtained in [7] can be easily generalized to the case of thermal motion of the spins by means of a transition to the trajectories of the moving particles.

It is natural to carry out the further arguments in phase space by introducing the single particle $r(\mathbf{r}\mathbf{v}tz)$ and the irreducible two particle $\delta r(\mathbf{r}_1\mathbf{v}_1\mathbf{r}_2\mathbf{v}_2tz)$ density matrices:

$$r(\mathbf{r}\mathbf{v}tz) = \left\langle \sum_i \delta(\mathbf{r} - \mathbf{r}_i(t))\,\delta(\mathbf{v} - \mathbf{v}_i)\,r_i(tz) \right\rangle, \quad (34)$$

$$\delta r(\mathbf{r}_1\mathbf{v}_1\mathbf{r}_2\mathbf{v}_2tz) = \left\langle \sum_{i \neq j} \delta(\mathbf{r}_1 - \mathbf{r}_i(t))\,\delta(\mathbf{v} - \mathbf{v}_i)\,\delta(\mathbf{r}_2 - \mathbf{r}_j(t)) \cdot \delta(\mathbf{v}_2 - \mathbf{v}_j)\,\delta r_{ij}(tz) \right\rangle. \quad (35)$$

Here the angular brackets denote averaging over the positions and the velocities of the particles. In future the trajectories of the particles are taken to be rectilinear. We utilize normalized density matrices: $\mathrm{Sp}(r_i) = 1$. Then

$$\mathrm{Sp}(r(\mathbf{r}\mathbf{v}tz)) = f(\mathbf{v}) \quad (36)$$

is the distribution function for atoms which we shall assume to be Maxwellian.

The total radiation flux now has the form

$$v_+ = \int d\mathbf{r}\,d\mathbf{v}\,\bar{g}(\mathbf{r})\,\mathrm{Sp}(\sigma_+ r) + i\nu\bar{z}. \quad (37)$$

The dependence of the coupling constant g on $\mathbf{r}$ reproduces the spatial structure of the field. The equation for the density matrix $r(\mathbf{r}\mathbf{v}tz)$ taking into account quantum fluctuations has the form

$$i\left(\frac{dr}{dt} + \mathbf{v}\frac{\partial r}{\partial \mathbf{r}} + \frac{r - r_0}{\tau}\right) = [\mathscr{H}_0(\mathbf{r}), r] + S(\mathbf{r}) + \delta S(\mathbf{r}), \quad (38)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{i}\left(v_- \frac{\partial}{\partial z} - v_+ \frac{\partial}{\partial \bar{z}}\right), \quad (39)$$

$$S(\mathbf{r}) = \frac{\partial \ln \rho}{\partial \bar{z}}\,\bar{g}(\mathbf{r})\,r\tilde{\sigma}_+ - \mathrm{h.c.};\ \tilde{\sigma}_\pm = \sigma_\pm - \frac{\mathrm{Sp}(\sigma_+ r)}{f}, \quad (40)$$

$$\delta S(\mathbf{r}) = \frac{\partial \ln \rho}{\partial \bar{z}} \int d\mathbf{r}_2\,d\mathbf{v}_2\,\bar{g}(\mathbf{r}_2)\,\mathrm{Sp}_{(2)}(\sigma_+^{(2)}\,\delta r(\mathbf{r}\mathbf{v}\mathbf{r}_2\mathbf{v}_2tz)) - \mathrm{h.c.} \quad (41)$$

The Hamiltonian $\mathscr{H}(\mathbf{r}) = g(\mathbf{r})\bar{z}\sigma + \mathrm{h.c.}$ describes the interaction of the spins with the classical field z. We note that $\mathrm{Sp}(S) = \mathrm{Sp}(\delta S) = 0$, $\mathrm{Sp}(r_0) = f$, and therefore the kinetic equation (38) agrees with condition (36).

Finally, we write the equation for the correlation density matrix $\delta r(1|2)$ using indices instead of arguments:

$$i\left(\frac{d}{dt} + \mathbf{v}_1\frac{\partial}{\partial \mathbf{r}_1} + \mathbf{v}_2\frac{\partial}{\partial \mathbf{r}_2} + \frac{2}{\tau}\right)\delta r(1|2) = [\mathscr{H}(1) + \mathscr{H}(2), \delta r(1|2)]$$
$$+ \int d\mathbf{r}_3\,d\mathbf{v}_3\,\mathrm{Sp}_{(3)}\left[\frac{\partial r(1)}{\partial z} g(3)\sigma_-^{(3)}\delta r(2|3) + 1 \rightleftarrows 2 - \mathrm{h.c.}\right]$$
$$- \left(g(1)\tilde{\sigma}_-(1)r(1)\frac{\partial r(2)}{\partial z} + 1 \rightleftarrows 2 - \mathrm{h.c.}\right). \quad (42)$$

Here $1 \rightleftarrows 2$ denotes a term which differs from the preceding one only by a permutation of indices.

For $\delta r(1|2)$ the condition $\mathrm{Sp}_{(1)}(\delta r) = \mathrm{Sp}_{(2)}(\delta r)$ holds, and Eq. (42) agrees with this condition.

By its form (42) reminds us of the equation for the correlation function in plasma theory,[13] but differs from it in a fairly significant manner. If in a plasma for $kv_0 \gg 1/\tau$ (v_0 is the thermal velocity of the particles) one can let $1/\tau \to 0$, one cannot do so here, since the effective interaction of two particles with a relative velocity $\Delta\mathbf{v}$ occurs for $k\Delta\mathbf{v} \lesssim 1/\tau$. In other words, the interaction between particles is of a resonance character.

In the general case Eq. (42) is very complicated, and therefore we shall restrict ourselves to a consideration of the simplest case of a traveling wave for low energies of the radiation.

Traveling Wave Laser

In this case $g(\mathbf{r}) = g\exp(i\mathbf{k}\mathbf{r})$. We shall also assume that the generation is stationary. Then ρ depends only on ξ, while the probability flux vanishes: $v_+ = v_- = 0$.

We note that in the case of a traveling wave there is no shot-effect noise, owing to the modulation of the emitting medium.[14]

For low energies $\mathcal{H}(\mathbf{r}) \ll 1/\tau$ and the effect of the field can be taken into account utilizing perturbation theory. In (38) and (42) r and δr in the case of a traveling wave depend effectively only on $\omega = \mathbf{kv}$ and $\omega_1 = \mathbf{kv}_1$, $\omega_2 = \mathbf{kv}_2$. Integrating over the velocity components perpendicular to $\mathbf{k}$ one can assume that f depends also only on ω.

We further restrict ourselves to the case of strong inhomogeneous broadening

$$kv_0 \gg 1/\tau. \tag{43}$$

The situation in this case is greatly simplified since the Maxwellian exponential in $f(\omega)$ can be replaced by unity:

$$f = \frac{1}{\sqrt{\pi}\, v_0} \frac{N}{V}, \tag{44}$$

where N/V is the density of the excited atoms.

Solving first the classical problem we neglect S and δS and we expand r in a series in terms of the field up to the third order:

$$r = r_0 + r_1 + r_2 + r_3 + \ldots;\quad r_0 = {}^1/_2(1+\sigma_3)f,$$
$$r_1 = -gf\left(\frac{\bar{z}\sigma_- e^{i\mathbf{kr}}}{\omega - i/\tau} + \text{h.c.}\right) \tag{45}$$

etc. We now substitute into (42) $r \approx r_0$, $\partial r/\partial z \approx \partial r_1/\partial z$ and we omit $\mathcal{H}(\mathbf{r})$. Setting

$$\delta r(1|2) = e^{i\mathbf{k}(r_1 - r_2)} \sigma_-^{(1)} \sigma_+^{(2)}\, \delta R(1|2) + \text{h.c.} \tag{46}$$

we obtain the following equation for the scalar function $R(\omega_1 \omega_2)$:

$$R(\omega_1\omega_2) + \frac{\nu}{\omega_2 - \omega_1 + 2i/\tau} \int_{-\infty}^{+\infty} d\omega \left[\frac{R(\omega_1\omega)}{\omega_2 + i/\tau} + \frac{\bar{R}(\omega_2\omega)}{-\omega_1 + i/\tau}\right]$$
$$= \frac{g^2 f^2}{(\omega_2 + i/\tau)(-\omega_1 + i/\tau)}. \tag{47}$$

Here we have utilized the fact that now the generation parameter is

$$\eta = \pi g^2 f V / \nu \text{ and } \eta - 1 \ll 1.$$

The solution of this integral equation in view of the simple analytical properties of the kernal and of the right hand side is obtained in an elementary fashion:

$$R(\omega_1\omega_2) = \frac{g^2 f^2}{(\omega_2 + i/\tau)(-\omega_1 + i/\tau)} \left(1 - \frac{\nu}{\omega_2 - \omega_1 + 2i/\tau}\right). \tag{48}$$

The first term in (48) has a factorized form, and this corresponds to the statistical independence in processes of spontaneous emission. The second term takes into a account the correlation effects, but it does not given an integral contribution to $\delta S(\omega)$. It is now not difficult to find the additional quantum term to r, which contains the gradient of ρ and which we denote by Δr:

$$\left(\nu \frac{\partial}{\partial \mathbf{r}} + \frac{1}{\tau}\right) \Delta r = S + \delta S. \tag{49}$$

The correction to the radiation flux (37) obtained with the aid of Δr has the form

$$\text{Sp}(\sigma_+ \Delta r) = -\frac{g\bar{z}f}{\omega + i/\tau}\left(1 - \frac{i\nu}{\omega + i/\tau}\right)\frac{\delta\xi}{d\bar{n}}. \tag{50}$$

In this expression we have utilized the explicit form of the distribution function (18). Integrating (50) over ω and substituting the result (together with the classical part of the radiation flux) into the condition $v_+ = 0$ we obtain the equation for the dispersion parameter d. The second term in brackets in (50) comes from δS and also gives no integral contribution to v_+. The absence of a contribution is associated with approximations due to the conditions (43) and to the smallness of the field. At the same time the local contribution from the second term in (50) can be large for large values of $\nu\tau$.

In order to obtain the contribution from the self-consistent field we consider the following approximation in terms of the parameter $\eta - 1$. Of course, this contribution to the fluctuations will be appreciable only for sufficiently large $\nu\tau$, when the parameter

$$\eta' = (\eta - 1)\nu\tau \tag{51}$$

is not small.

Without dwelling on the details of the calculations which are carried out according to perturbation theory in terms of the parameter $\eta - 1$, we quote the final equation which serves to determine the dispersion for the photons:

$$\frac{1}{d(\eta - 1)} = \frac{4 + 2\varepsilon - 3\varepsilon^2}{(1 + 2\varepsilon)(2 + \varepsilon)(2 + 3\varepsilon^2)},\quad \varepsilon = \frac{\nu\tau}{d}. \tag{52}$$

From this it follows that for $\eta' \ll 1$

$$d = 1/(\eta - 1), \tag{53}$$

i.e., we have the usual expression for the dispersion of the photons near the threshold; the effects due to the self-consistent field play no role here. For $\eta' \gg 1$, setting the right hand side of (52) to zero we obtain

$$d = \nu\tau(\sqrt{13} - 1) \approx 0.66\,\nu\tau. \tag{54}$$

Thus, for not too weak a radiation field the fluctuations are determined by collective processes.

DISCUSSION

We now consider the physical interpretation of the results obtained above.

The pair correlation of spins can be graphically interpreted as a mutual effect of two classic dipoles situated with respect to each other at a distance of a photon mean free path c/ν.

If the atoms are fixed and are in a condition of resonance with the radiation, then in the case of coherent interaction of wave trains ($\gamma = 1/\tau$) the correlation effects reduce the level of fluctuations: near the generation threshold $d \sim 1/\nu\tau$.[7] Indeed, if one dipole has emitted a wave train of phase φ, then the second dipole under the action of this wave train will acquire a phase of oscillation $\varphi + \pi/2$, while the wave-train emitted by it will have the phase $\varphi + \pi$. The resulting radiation will be weakened to a degree which is the greater, the greater is the degree of overlapping of the wave trains, i.e., the larger is the parameter $\nu\tau$. It is clear that in a weak radiation field the principal role is played by the process of a single exchange of wave-trains.

In a sufficiently strong field one must take into account the multiple processes of reradiation. After an even number of reradiations the wave trains add and coherent amplification of fluctuations occurs, after an odd number of reradiations the fluctuations are weakened. This leads to the fact that for a degree of overlapping of the wave trains greater than critical ($\nu\tau > 2$)

there always exists a sufficiently strong field for which the fluctuations can become anomalously large and lead to instability.

It is easy to understand in a similar manner the effect of thermal motion of the atoms on the correlation of dipoles. Since here we have discussed the case of a weak field, then we are concerned with a single exchange of wave trains.

If $k\Delta v \lesssim 1/\tau$ (Δv is the relative velocity of the two dipoles), then the fluctuations produced by them weaken one another (resonance interaction). For $k\Delta \mathbf{v} \gtrsim 1/\tau$ the fluctuations add (nonresonant interaction). In this case, if $kv_0 \gg 1/\tau$, then the two effects compensate one another and only in the next order in the field the latter effect becomes more important.

The case of a field which is not weak is difficult to discuss. However, one can show that in a strong radiation field and under the conditions of the inequality (43) the stationary generation becomes unstable also for $\nu\tau > 2$.

We now discuss the problem of the noncoherent interaction of wave trains which occurs for $\gamma \gg 1/\tau$. In this case an atom undergoing spontaneous emission emits a wave train of length $c\tau$, which consists of short wave-trains c/γ which are not correlated among themselves in phase. Therefore for $\gamma \gg \nu$ one can speak only of incoherent interaction of wave trains. For the dispersion parameter of the number of photons we have the expression (32). Near the threshold $d = 1/(\eta - 1)$, i.e., interaction of the wave trains gives no contribution to the fluctuation of photons. Far from the threshold in the case of a strong overlapping of wave trains ($\nu\tau \gg 1$) we have $\overline{\Delta n^2}/\bar{n} = \nu\tau$. This relation can be interpreted as a diffusion expression for the square of independent displacements, where τ plays the role of the displacement time.

It is of interest to note that the dispersion parameter d determined from Eq. (52) has approximately the same value as the one determined by means of formula (32) although the physical content of these two cases is different.

Thus, thermal motion of the atoms converts coherent interaction of wave trains into an incoherent interaction. But this occurs only near the generation threshold. For large energies in the case $\gamma = 1/\tau$ (coherent interaction) for $\nu\tau > 2$ the generation becomes unstable; in the case $\gamma \gg 1/\tau$, $\gamma \gg \nu$ (incoherent interaction) in a strong field the fluctuations have approximately the same level as in the threshold region ($\overline{\Delta n^2}/\bar{n} \sim \nu\tau$). Consequently one can say that the single-mode generation can become unstable only in the case of coherent interaction of wave trains.

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Translated by G. Volkoff
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