ON THE THEORY OF THE ABRIKOSOV VORTEX LATTICE IN SUPERCONDUCTORS WITH $\kappa \gg 1$

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A method is presented for calculating the lattice sums in superconductor vortex states for field strengths $H_{c1} \ll H_0 \ll H_{c2}$ and arbitrary lattices. It is shown that minimal energy is attained for a triangular lattice. The numerical coefficient in the logarithmic De Gennes induction law, the nuclear resonance line width, and the vacancy-formation energy in an Abrikosov lattice are calculated. It is shown that in a strong magnetic field parallel to the plate surface the thickness of the plate does not affect the nature of the vortex state.

1. INTRODUCTION

According to Abrikosov's theory[1], a magnetic field H_0 penetrates in the interior of a type II superconductor ($\kappa > 1/\sqrt{2}$) in the form of vortex filaments, each of which carries one quantum of flux $\Phi_0 = ch/2e$. The interaction of the vortex filaments with one another leads to the occurrence of a two-dimensional periodic lattice. Calculations have shown that the minimum energy apparently corresponds in the entire region of H_0 to a triangular lattice[2].

For superconductors with large values of the parameter $\kappa = \lambda/\xi \gg 1$, the vortex state in the region $H_{c1} \leq H_0 \ll H_{c2}$ can be described with the aid of the modified London equations[1]

$$\mathbf{H} + \lambda^2 \operatorname{rot}\operatorname{rot}\mathbf{H} = \Phi_0 \mathbf{n} \sum_i \delta(\mathbf{r}-\mathbf{r}_i), \tag{1}$$

where $\mathbf{n}$ is the direction of the external field H_0, parallel to the z axis, and $\mathbf{r}_i$ is a two-dimensional vector (in the plane z = 0) characterizing the position of the i-th filament. For an isolated filament, the field H decreases exponentially at large distances and diverges logarithmically near the filament at $r \ll \lambda$.

The lattice energy and the magnetic moment can be calculated exactly in two limiting cases.

1) The Abrikosov case: $H - H_{c1} \ll H_{c1}$; in this region, the period of the structure a is large compared with the penetration depth $\lambda \ll a$, and therefore only the nearest neighbors take part in the interaction of the vortex filaments.

2) The de Gennes case[2]: $H_{c1} \ll H_0 \ll H_{c2}$. This case corresponds to a large vortex-filament density ($\xi \ll a \ll \lambda$). In the calculation of the energy, de Gennes replaced summation over the lattice by integration, and obtained as a result a logarithmic dependence of the induction B on the external field:

$$B \equiv n_L \Phi_0 = H_0 - \frac{\Phi_0}{4\pi\lambda^2} \ln\left(\beta' \frac{a}{\xi}\right) \tag{2}$$

where n_L is the density of the vortex filaments. For a quadratic lattice $n_L = 1/a^2$, and for a triangular one $n_L = 2/\sqrt{3}a^2$, where a is the distance between neighboring sites. To find the numerical coefficient β' in formula (2), it is necessary to be able to calculate the lattice sums exactly¹⁾.

An investigation of lattice sums of the type (8) for an infinite superconductor at high densities ($a \ll 1$) was carried out by Fetter, Hohenberg, and Pincus[4]. These authors used an approach similar to Ewald's method in crystal theory, and obtained results in the form of rapidly converging series. This procedure, however, is quite cumbersome and cannot be applied directly to more complicated problems with allowance for the boundary of the sample.

The purpose of the present investigation was to calculate the magnetization curve of a plate of thickness d in a longitudinal magnetic field. The thickness is assumed to be arbitrary (compared with the depth of penetration $\lambda(T)$), but to satisfy the condition $d \gg \xi(T)$. We also assume that the field H_0 is sufficiently strong (but weak compared with H_{c2}), so that the period is small compared with the thickness of the plate ($a \ll d$) and with the penetration depth ($a \ll \lambda$).

The method of calculating the lattice sums is described in Sec. 2 using an infinite superconductor as an example. Mathematically, the problem is analogous to the summation of series in the method of images for a plate (see, for example,[6]). This method makes it possible to obtain a solution of the problem in closed form at small values of a for arbitrary plate thicknesses.

2. VORTEX LATTICE IN INFINITE SPACE

For an infinite superconductor, the solution of Eq. (1) has the well known form

$$\mathscr{H}(x,y) = \frac{\Phi_0}{2\pi\lambda^2} \sum_i K_0(|\mathbf{r}-\mathbf{r}_i|/\lambda) = \frac{\Phi_0}{\lambda^2} \sum_i \int \frac{e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}_i)/\lambda}}{k^2+1} \frac{d^2k}{(2\pi)^2}, \tag{3}$$

where the summation is carried ouj over all the sites of the lattice $\mathbf{r}_i$. According to[1,2], the free energy $\mathscr{F}$ for a system of filaments can be written in the form

$$\mathscr{F} = N\left\{\varepsilon_0 + \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \sum_i{}' K_0(|\mathbf{r}_i|/\lambda)\right\}, \tag{4}$$

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¹⁾The value $\beta_\Delta' = 0.3815$, given in [2] and obtained on the basis of a numerical calculation in [3], is incorrect. The correct value is contained in Fetter's paper [5].

where N is the total number of vortex filaments, and the prime at the summation sign denotes that the term with $r_i = 0$ has been omitted. The first term in the curly brackets is the self-energy ϵ_0 of an isolated vortex filament. It is connected with the critical field H_{c1} by the relation

$$H_{c1} = \frac{4\pi}{\Phi_0}\varepsilon_0. \tag{5a}$$

The value of H_{c1} can be determined approximately by cutting off the divergence in ϵ_0 at small distances for $|\mathbf{r}| = \xi$:

$$\varepsilon_0 = \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 K_0\left(\frac{\xi}{\lambda}\right) \approx \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln\frac{2\varkappa}{\gamma}, \quad \varkappa = \frac{\lambda}{\xi} \gg 1 \tag{5b}$$

($\ln\gamma = C = 0.577$). This value is in good agreement with the exact result obtained for ϵ_0 in[1] by numerical integration of the Ginzburg-Landau equations.

For a periodic vortex lattice, in which there is one filament per unit cell, the induction B is equal to

$$B = \Phi_0/s, \tag{6}$$

where s is the area of the unit cell. Thus, the Gibbs potential G (per unit volume) can be represented in the form

$$G = F - \frac{BH_0}{4\pi} = \frac{1}{s}\left\{\left(\frac{\Phi_0}{4\pi\lambda}\right)^2 J - \frac{(H - H_{c1})\Phi_0}{4\pi}\right\}, \tag{7}$$

where

$$J = \sum_i{}' K_0(|\mathbf{r}_i|/\lambda). \tag{8}$$

For the general case, when the unit cell is a parallelogram, the sum J is calculated in appendix A. We denote by a and b the sides of the parallelogram and by α the angle between them. Then in the limit of large densities (a, b $\ll \lambda$) we have in accordance with (A.6a)

$$J(a, b, \alpha) = \ln\left(\frac{\gamma b}{4\pi\lambda}\right) + \frac{2\pi\lambda^2}{s} + \frac{\pi c \sin\alpha}{6} + 2\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{1}{n(e^{-2\pi i n\zeta} - 1)}, \tag{9}$$

where $s = ab\sin\alpha$, $c = a/b$, $\zeta = ce^{i\alpha}$.

Minimizing G with respect to B for specified H_0, c, and α, we obtain with the aid of (6), (7), and (9)

$$B = H_0 - H_{c1} - \frac{\Phi_0}{4\pi\lambda^2}\left\{\ln\left(\frac{\gamma b}{4\pi\sqrt{e}\lambda}\right) + \frac{\pi c\sin\alpha}{6} + 2\,\mathrm{Re}\sum_{n=1}^{\infty}\frac{1}{n[\exp(-2\pi n i\zeta) - 1]}\right\}, \quad b^2 = \frac{\Phi_0}{Bc\sin\alpha}. \tag{10}$$

This is the sought logarithmic dependence of the induction on the field

$$B = H_0 - H_{c1} - \frac{\Phi_0}{8\pi\lambda^2}\ln\left(\beta\frac{\Phi_0}{\lambda^2 B}\right), \tag{11a}$$

$$\frac{1}{2}\ln\beta = \frac{1}{2}\ln\left(\frac{\gamma^2}{16\pi^2 e}\right) - \frac{1}{2}\ln(c\sin\alpha) + \frac{\pi c\sin\alpha}{6} - \sum_{n=1}^{\infty}\ln|1 - e^{2\pi n i\zeta}|^2. \tag{11b}$$

It follows from (7) and (11a) that for an arbitrary lattice the Gibbs potential is equal to

$$G = -\frac{B^2}{8\pi} + \frac{\Phi_0 B}{32\pi^2\lambda^2}. \tag{12}$$

We see therefore that in the case under consideration ($H_0 \gg H_{c1} > \Phi_0/4\pi\lambda^2$), the minimum of G corresponds to the largest possible value of the induction B, i.e., according to (11a) the smallest value of β.

The coefficient β does not depend on the parameters of the superconductor and is determined exclusively by the structure of the vortex lattice. We shall show here that the minimum value of β is realized for a triangular lattice. The proof is analogous to that given in the book of St. James et al.[7] for the region of fields $H_{c2} - H_0 \ll H_{c2}$. We write the parameter ζ in the form

$$\zeta = \rho + i\sigma, \quad \rho = c\cos\alpha, \quad \sigma = c\sin\alpha.$$

Minimizing expression (11b) with respect to ρ, we find immediately that $\rho = n/2$ (n = 0, 1, ...). Since the function β is periodic in ρ with a period 1 (i.e., $\beta(\rho + 1) = \beta(\rho)$), it suffices to consider the values $\rho = 0$ and $\rho = 1/2$. Minimization with respect to σ yields the equation

$$\frac{\pi}{6} - \frac{1}{2\sigma} = 4\pi\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n(\sigma + i\rho)} - 1}, \quad \rho = 0,\ 1/2. \tag{13}$$

The case $\rho = 0$ corresponds to a rectangular lattice ($\alpha = 90^\circ$, $\sigma = c$). It is easily seen that at $\rho = 0$ Eq. (13) has the unique solution c = 1 (quadratic lattice).

At $\rho = 1/2$, the unit cell obviously can be chosen in the form of a rhombus with diagonals $2b\sigma$ and b. Consequently, at $\rho = 1/2$, there should exist among the solutions (13) the solution $\sigma = 1/2$ corresponding to a quadratic lattice turned 45° relative to $\rho = 0$, $\sigma = 1$. The other two solutions of (13) at $\rho = 1/2$ correspond to a regular triangular lattice ($2\sigma = \sqrt{3}$, $1/\sqrt{3}$).

The equivalence of the solutions $\zeta = i$ and $\zeta = (1 + i)/2$ (and also of $\zeta = (1 + i\sqrt{3})/2$ and $\zeta = (1 + i/\sqrt{3})$) can be verified formally with the aid of the identities obtained in Appendix A:

$$\beta(\zeta) \equiv \beta(1/\zeta^*), \quad \beta(\rho + 1, \sigma) \equiv \beta(\rho, \sigma).$$

An investigation of the spectrum of the oscillations shows[4] that a quadratic lattice is unstable for small perturbations of definite symmetry, i.e., it corresponds not to an absolute energy minimum but to a saddle point.

We can now calculate the value of the coefficient β' in the de Gennes law (2). Using (5) for H_{c1}, we have for a quadratic lattice (c = 1, $\alpha = 90°$)

$$\ln\beta'_{\square} = \frac{\pi}{6} - \frac{1}{2} - \ln(2\pi) + 2\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n} - 1)} \approx -1.810, \tag{14a}$$

and for a triangular lattice (c = 1, $\alpha = 60°$)

$$\ln\beta_{\triangle}' = \frac{\pi}{4\sqrt{3}} - \frac{1}{2} - \ln(2\pi) + 2\sum_{n=1}^{\infty}\frac{1}{n[e^{\pi n i}e^{\pi n\sqrt{3}} - 1]} \approx -1.893. \tag{14b}$$

In a given field H_0, the energy gain for a triangular lattice compared with a quadratic lattice is

$$G_{\triangle} - G_{\square} = -\frac{\Phi_0 H_0}{16\pi^2\lambda^2}\ln\left(\frac{a_{\square}\beta_{\square}'}{a_{\triangle}\beta_{\triangle}'}\right) \approx -0.10\,\frac{\Phi_0 H_0}{16\pi^2\lambda^2}.$$

We present also the energy f per vortex filament in the lattice:

$$f = \frac{\mathcal{F}}{N} = \frac{\Phi_0 H_0}{8\pi} + \frac{\Phi_0^2}{32\pi^2\lambda^2}\ln\left(\frac{\beta' e a}{\xi}\right). \tag{15}$$

Knowing f, we can calculate the energy that must be consumed in the formation of a vacancy in an Abrikosov lattice when the total number of filaments is decreased by unity. The vacancy energy Δ is equal to

$$\Delta = -2f + \varepsilon_0 + \frac{\Phi_0 H_0}{4\pi}$$

where the last term describes the loss of magnetic energy. With the aid of (15) and (5) we get

$$\triangle=\left(\frac{\Phi_0}{4\pi\lambda}\right)^2\ln\left(\frac{2}{\gamma\beta' e}\frac{\lambda}{a}\right)\ll\varepsilon_0.$$

This formula is approximate in the sense that no account is taken in it of the lattice deformation in the vicinity of the vacancy (see[8]).

With the aid of formulas (A.5) given in the Appendix for the sum, we can find the nuclear-resonance line width ΔH in the vortex state (c = 1):

$$(\Delta H)^2=\overline{H^2}-\overline{H}^2=n_L{}^2\Phi_0{}^2\left\{\frac{1}{4\pi n_L\lambda^2}\left[1-\frac{a}{\lambda}J'\left(\frac{a}{\lambda}\right)\right]-1\right\},J'(x)=\frac{dJ}{dx}.$$

For a quadratic lattice at large densities ($a \ll 1$) we have

$$(\Delta H)^2_{\square}=\frac{\Phi_0{}^2}{16\pi^3\lambda^4}\left[\zeta(3)+\frac{\pi^3}{45}+2\sum_{n=1}^{\infty}\frac{1}{n^3[\exp(2\pi n)-1]}+\pi\sum_{n=1}^{\infty}\frac{1}{n^2\operatorname{sh}^2(\pi n)}\right]\approx 1{,}92\frac{\Phi_0{}^2}{16\pi^3\lambda^4}.$$

This exceeds by 1.92 times the value given in the literature[2,7]. The reason for this discrepancy is that at large densities it is incorrect to replace the sum (over the reciprocal lattice) by an integral.

For a triangular lattice the broadening is equal to

$$(\Delta H)^2_{\triangle}=\frac{\Phi_0{}^2}{16\pi^3\lambda^4}\left[\frac{2\zeta(3)}{\sqrt{3}}+\frac{\pi^3}{60}+\frac{4}{\sqrt{3}}\sum_{n=1}^{\infty}\frac{1}{n^3[(-1)^n\exp(\pi n\sqrt{3})-1]}+\frac{\pi}{4}\sum_{n=1}^{\infty}\frac{1}{n^2\operatorname{sh}^2(\pi n\sqrt{3})}-\pi\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2\operatorname{ch}^2(^1/_2\pi\sqrt{3}(2n-1))}\right]\approx 1.83\frac{\Phi_0{}^2}{16\pi^3\lambda^4}.$$

3. SUPERCONDUCTING PLATE IN A PARALLEL FIELD

We now consider the more complicated case of a superconducting plate in a longitudinal field ($H_{c1} \ll H_0 \ll H_{c2}$). At first glance it might seem that at thicknesses $d \sim \lambda$ the vortex structure should experience strong changes (in analogy with the situation in the vicinity of the field H_{c1}[9]). This is not the case, however. If the filament density is large ($a \ll \lambda$), then, as will be shown below, the lattice parameters do not depend on the plate thickness even in the limit $d \ll \lambda$[2]). The point is that at large lattice densities the filament images due to the plate boundaries practically cancel one another. As a result the magnetic moment of the plate (per unit volume) coincides with the value calculated above for an unbounded sample.

To simplify the formulas, we shall use below the reduced units of the Ginzburg-Landau theory

$$\lambda\tilde{r}=r,\quad H=\frac{\Phi_0}{2\pi\lambda\xi}\tilde{H}=\sqrt{2}H_c\tilde{H},\quad G=\lambda^2\frac{H_c{}^2}{4\pi}\tilde{G},\quad \varkappa=\frac{\lambda}{\xi}.$$

In terms of these variables, the solution of (1) satisfying the boundary conditions $H(0, y) = H(d, y) = H_0$ in the presence of a single vortex filament at the point (x_0, y_0) is given by

$$H(x,y)=H_0\frac{\operatorname{ch}(x-d/2)}{\operatorname{ch}(d/2)}+H_v(x,y),$$

where

$$H_v(x,y)=\frac{4\pi}{\varkappa d}\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}\frac{\sin(\pi nx/d)\sin(\pi nx_0/d)}{k^2+(\pi n/d)^2+1}e^{ik(y-y_0)}\frac{dk}{2\pi}$$
$$=\frac{1}{\varkappa}\sum_{m=-\infty}^{+\infty}\{K_0(\sqrt{(2md+|x_0-x_0'|)^2+(y-y_0)^2})-K_0(\sqrt{(2md+x+x_0)^2+(y-y_0)^2})\}. \tag{16}$$

According to a theorem proved by Shmidt[10], the field $H_v(\mathbf{r}_0)$ is directly connected with the energy for the given configuration of the vortex filaments by the relation

$$\mathcal{F}=\frac{2\pi}{\varkappa}\sum_{\mathbf{r}_0,\mathbf{r}_0'}H_v(\mathbf{r}_0-\mathbf{r}_0'), \tag{17}$$

where the summation is carried out over the positions of all the filaments. This connection is an exact analog of expression (4) for an infinite superconductor and makes it possible to simplfiy greatly the calculation of the energy. The divergence in (17) as $\mathbf{r}_0 \to \mathbf{r}_0'$ is eliminated with the aid of cutoff at distances $|\mathbf{r}_0 - \mathbf{r}_0'| = 1/\kappa$, just as in the case of an isolated filament.

Starting from the statements made at the beginning of this section, we assume that a regular lattice of vortex filaments exists in the plate. We present here a derivation for a rectangular lattice, and at the conclusion we shall formulate the results for an arbitrary lattice. Let a and b denote the periods of the structure in the x and y directions, respectively. Then the free-energy density F, in accord with (16) and (17), can be written in the form

$$F=\frac{2\pi}{\varkappa^2ab}\left\{K_0\left(\frac{1}{\varkappa}\right)+\sum_{m,n}{}'K_0(\sqrt{\varepsilon^2m^2+n^2b^2})\right\}+\frac{2\pi}{\varkappa^2bd}\left\{\sum_{m,n}\sum_{x_0\neq x_0'}K_0(\sqrt{\varepsilon^2(m+\mu)^2+b^2n^2})-\sum_{m,n}\sum_{x_0,x_0'}K_0(\sqrt{\varepsilon^2(m+\mu')^2+b^2n^2})\right\}, \tag{18}$$

where $\epsilon = 2d$, $\mu = |x_0 - x_0'|/2d$, $\mu' = (x_0 + x_0')/2d$. The first sum was already calculated by us in the case of an infinite superconductor; it coincides with the function $J(2d, b; \alpha = 90^\circ)$ and is given by expression (A.5) with $b \ll 1$. The last term is calculated in Appendix B. With the aid of (A.5) and (B.4) we obtain ultimately

$$F_{\square}=\frac{2\pi}{\varkappa^2ab}\ln(\tilde{\alpha}\varkappa b)+\left(\frac{2\pi}{\varkappa ab}\right)^2\left(\frac{a}{2}\operatorname{cth}\frac{a}{2}\right)\left[1-\delta\frac{a}{2}\operatorname{cth}\frac{a}{2}\right],$$
$$\ln\tilde{\alpha}_{\square}=-\ln(2\pi)+2\sum_{n=1}^{\infty}\frac{1}{n[e^{2\pi nc}-1]},\quad c=\frac{a}{b},\quad \delta=\frac{\operatorname{th}(d/2)}{d/2}. \tag{19}$$

The induction B can easily be calculated with the aid of (16), by starting from the definition

$$B=\frac{1}{dL_y}\int_0^d dx\int_{-L_y/2}^{L_y/2}dyH(x,y),\qquad L_y\to\infty, \tag{20}$$

where L_y is the length of the plate in the y direction. The result is

$$B_{\square}=H_0\delta+\frac{2\pi}{\varkappa ab}\left(1-\delta\frac{a}{2}\operatorname{cth}\frac{a}{2}\right). \tag{21}$$

From (19) and (21) there follows an expression for the Gibbs potential

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2) A qualitative theory of the vortex structure in a thin plate ($d \ll \lambda$) was proposed in [10]. We thank V. V. Shmidt for acquainting us with his work prior to publication.

$$G = F - 2BH_0 = \frac{2\pi}{\varkappa ab}\left[\frac{1}{\varkappa}\ln(\tilde{a}\varkappa b) + \left(1 - \delta\frac{a}{2}\operatorname{cth}\frac{a}{2}\right)\left(\frac{2\pi}{\varkappa ab}\frac{a}{2}\operatorname{cth}\frac{a}{2} - 2H_0\right)\right]. \quad (22)$$

In the last expression we have omitted an additive term not connected with the presence of vortex filaments. For larger densities, expanding the cotangent in powers of a, we ultimately obtain

$$G_\square = \frac{h}{2\varkappa}\ln\left(\frac{2\pi\tilde{a}^2\varkappa}{hc}\right) + h^2(1-\delta) - 2H_0h(1-\delta) + \frac{\pi}{6\varkappa}(1-2\delta)hc + \frac{\pi}{3}\frac{H_0}{\varkappa}\delta c, \quad (23a)$$

$$B_\square = H_0\delta + h(1-\delta) - \frac{\pi}{6}\frac{\delta}{\varkappa}c, \quad h = \frac{2\pi}{\varkappa ab}. \quad (23b)$$

As seen from the definition, the field h becomes equal to the induction if the plate thickness tends to infinity ($\delta = 0$). A generalization of expressions (23) to the case of a unit cell in the form of a parallelogram, as shown by the calculations, reduces to a redefinition of the quantities h, c, and $\tilde{\alpha}$. To this end it is necessary to make everywhere in (23) the substitution $c \rightarrow \sigma = c \sin\alpha$, and the field h must be expressed in terms of the area of the unit cell, i.e., $h = 2\pi/ab\sin\alpha$. The parameter $\tilde{\alpha}$ then coincides with the corresponding value for an infinite superconductor

$$\ln\tilde{a} = -\ln(2\pi) - \sum_{n=1}^{\infty}\ln|1 - e^{2\pi n\zeta}|^2.$$

Since such a generalization yields nothing new compared with the infinite superconductor, we shall not stop to discuss it in detail.

From the condition that G be an extremum with respect to h (i.e., the induction) and c, in accordance with (23), there follows the system of equations

$$\ln\left(\frac{2\pi\tilde{a}^2(c)\varkappa}{ehc}\right) = 4\varkappa(H_0 - h)(1-\delta) - \frac{\pi}{3}(1-2\delta)c,$$
$$\frac{2\pi}{3}\frac{H_0}{h}\delta c = 1 - \frac{\pi}{3}(1-2\delta)c + 2c(\ln\tilde{a})'. \quad (24)$$

For a thick plate, the thickness of which is large compared with the depth of penetration, we can put in (24) $\delta = 0$, and we return naturally to the earlier results (11) and (13) at $\alpha = 0°$, $c = 1$.

We shall now show that the corrections to the moment, due to the thickness of the plate, are small under the condition that the external field is sufficiently large ($a, b \ll d$). To this end, we seek a solution of (24) in the form

$$h = H_0 + \tilde{h}, \quad \tilde{h} \ll H_0. \quad (25)$$

Linearization of the first equation yields

$$\tilde{h} = -\frac{1}{4\varkappa(1-\delta)}\ln\left(\frac{2\pi\tilde{a}^2\varkappa}{ecH_0}\right) - \frac{\pi}{12\varkappa}\frac{1-2\delta}{1-\delta}c. \quad (26)$$

Substitution of (25) and (26) in (23b) again leads to a value of the induction for a bulky sample (11a) at arbitrary thicknesses d. Since in (11a) the last two terms are small compared with the first at $\kappa \gg 1$, we can use the zero-order approximation for the "anisotropy" coefficient. Putting in the second equation of (24) $h = H_0$, we find that the parameter c is likewise independent of the thickness of the plate and coincides with its value in an unbounded sample.

The applicability of the results follows from the inequality $\tilde{h} \ll H_0$, which yields in accordance with (26) the condition

$$(\varkappa H_0)(1-\delta) \gg \ln(\varkappa/H_0). \quad (27)$$

At small thicknesses ($d \ll 1$) this condition (i.e., $d^2 \gg a^2\ln(\kappa a)$) is somewhat stronger than the condition for the presence of a large number of vortex filaments in the thickness of the film ($d \gg a$), a condition used in the derivation of (24).

In the region $a^2 \ll d^2 \ll a^2\ln(\kappa a)$ we can expect considerable changes in the lattice parameters, but actually this region is quite narrow, and we therefore do not present the corresponding formulas.

In conclusion, the authors thank V. V. Shmidt for useful discussions.

APPENDIX A

We consider the series

$$J(a,b,\alpha) = \sum_i{}' K_0(|\mathbf{r}_i|), \quad (A.1)$$

where the summation is over all the points of the vortex lattice (with the exception of $\mathbf{r}_i = 0$). We choose as the unit cell a parallelogram with sides a and b and an angle α between them. Here, unlike in (8), the quantities a and b pertain to the depth of penetration λ. With the aid of the Fourier representation (3) we rewrite J in the form

$$J(a,b,\alpha) = 2\sum_{m=1}^{\infty}K_0(mb) + \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}{}' \int d^2k \frac{e^{ik_x na\sin\alpha}e^{ik_y(mb+na\cos\alpha)}}{k_x^2+k_y^2+1}. \quad (A.2)$$

Using the identity

$$\sum_{-\infty}^{+\infty} e^{ik_y mb} = 2\pi\sum_{-\infty}^{+\infty}\delta(k_y b - 2\pi m)$$

and then integrating in (A.2) with respect to k_x and k_y, we find

$$J(a,b,\alpha) = 2\sum_{m=1}^{\infty}K_0(mb) + \pi\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}{}'\frac{e^{2\pi nmic\cos\alpha}e^{-|n|uc\sin\alpha}}{\sqrt{(2\pi m)^2+b^2}}, \quad (A.3)$$

where $c = a/b$, $u \equiv \sqrt{(2\pi m)^2 + b^2}$. A convenient expression for the series in the first term is given in[11]:

$$2\sum_{m=1}^{\infty}K_0(mb) = \ln\left(\frac{\gamma b}{4\pi}\right) + \frac{\pi}{b} + 2\pi\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{(2\pi n)^2+b^2}} - \frac{1}{2\pi n}\right]. \quad (A.4)$$

Finally, summing in the second term in (A.3) with respect to n and using (A.4), we obtain ultimately

$$J(a,b,\alpha) = \ln\left(\frac{\gamma b}{4\pi}\right) + \frac{\pi}{b}\operatorname{cth}\left(\frac{a\sin\alpha}{2}\right) + 2\pi\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{(2\pi n)^2+b^2}} - \frac{1}{2\pi n}\right] + 4\pi\operatorname{Re}\sum_{m=1}^{\infty}\frac{1}{\sqrt{(2\pi m)^2+b^2}}\{\exp[c\sin\alpha\sqrt{(2\pi m)^2+b^2} - 2i\pi mc\cos\alpha] - 1\}^{-1}. \quad (A.5)$$

This is the exact formula. In the limit of large densities ($a, b \ll 1$) we have

$$J(a,b,\alpha) = \ln\left(\frac{\gamma b}{4\pi}\right) + \frac{2\pi}{ab\sin\alpha} + \frac{\pi c\sin\alpha}{6} + 2\operatorname{Re}\sum_{n=1}^{\infty}\frac{1}{n[\exp(-2\pi in\zeta) - 1]} + O(a^2, b^2), \quad (A.6a)$$

where $\zeta = (a/b)e^{i\alpha}$. The last term in this expression can also be transformed into

$$2\mathrm{Re}\sum_{n=1}^{\infty}\frac{1}{n[e^{-2\pi in\zeta}-1]}=-\sum_{n=1}^{\infty}\ln|1-e^{2\pi in\zeta}|^2,\quad \mathrm{Im}\,\zeta>0. \tag{A.6b}$$

Expression (A.6a) can be recast in a convenient form by introducing the unit-cell area s = ab sin α:

$$J(s,\sigma,\rho)=\ln\left(\frac{\gamma}{4\pi}\right)+\frac{1}{2}\ln\left(\frac{s}{\sigma}\right)+\frac{2\pi}{s}+\frac{\pi\sigma}{6} - \sum_{n=1}^{\infty}\ln|1-e^{2\pi in\zeta}|^2,\quad \sigma>0, \tag{A.7}$$

where $\zeta = \rho + i\sigma$. From this we see immediately that $J(\zeta)$ is invariant against the transformation $\rho \to \rho + 1$ (at a given s). Another important property of J is its invariance under inversion: $J(\zeta) = J(1/\zeta^*)$. Such an invariance follows from the equivalence of the sides a and b in the analysis.

Thus,

$$J(s,\sigma,\rho)=J(s,\sigma,\rho+1),\qquad J(s,c,\alpha)=J\left(s,\frac{1}{c},\alpha\right). \tag{A.8}$$

APPENDIX B

We are interested in the difference between the last two sums in the expression (16):

$$I=\sum_{x_0\neq x_0'}\sum_{m,n}K_0\left(\sqrt{\varepsilon^2(m+\mu)^2+b^2n^2}\right)- -\sum_{x_0,x_0'}\sum_{m,n}K_0\left(\sqrt{\varepsilon^2(m+\mu')^2+b^2n^2}\right). \tag{B.1}$$

With the aid of the Fourier representation for K_0 and the Poisson formula, we first carry out the summation with respect to m. This yields

$$I=\pi\sum_n\sum_{x_0\neq x_0'}\frac{\exp[\varepsilon\mu u_n/b]+\exp[\varepsilon u_n(1-\mu)/b]}{[\exp(\varepsilon u_n/b)-1]u_n} -\pi\sum_n\sum_{x_0,x_0'}\frac{\exp[\varepsilon\mu' u_n/b]+\exp[\varepsilon u_n(1-\mu')/b]}{[\exp(\varepsilon u_n/b)-1]u_n}, \tag{B.2}$$

where $u_n \equiv \sqrt{(2\pi n)^2 + b^2}$. The summation over x_0 and x_0' then entails no difficulty. After straightforward but cumbersome calculations we obtain

$$I=2\pi\sum_{n=-\infty}^{+\infty}\frac{1}{u_n}\left\{\left(\frac{d}{a}-1\right)\frac{\exp(2du_n/b)-\exp(au_n/b)}{[\exp(au_n/b)-1][\exp(2du_n/b)-1]} \right. \tag{B.3}$$

$$\left. -\frac{[\exp(du_n/b)-\exp(au_n/b)][\exp(au_n/b)+1]}{[\exp(au_n/b)-1]^2[\exp(du_n/b)+1]}\right\}.$$

Assuming a, b $\ll 2\pi$, we can put $u_n = 2\pi n$ in all the terms of this series, with the exception of u_0. As a result we find

$$I=\frac{2\pi}{b}\left\{\left(\frac{d}{a}-1\right)\frac{e^{2d}-e^a}{(e^a-1)(e^{2d}-1)}-\frac{(e^d-e^a)(e^a+1)}{(e^a-1)^2(e^d+1)}\right\} +\frac{2d}{a}\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi nc}-1)},\qquad c=\frac{a}{b}. \tag{B.4}$$

The first term of this expression has a singularity at $a \to 0$, and we have therefore retained the unity in the factor (d/a − 1), unlike the last term. The expansion in powers of a, which is of interest to us, is more conveniently carried out in the final expression for the free energy.

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