PLASMA MOTION IN AN INCREASING STRONG DIPOLAR MAGNETIC FIELD

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A solution of the nonlinear problem of plasma flow in an increasing dipolar magnetic field is obtained in the magneto-hydrodynamic approximation for a strong field. The distributions of plasma velocities, displacements, and density are calculated. It is shown that in an increasing dipolar magnetic field the plasma becomes more concentrated near the dipole axis. Possible applications of this effect are discussed.

THE problem of plasma motion in the strong magnetic field of a variable dipole has been formulated in[1]. The analytic solution obtained there for the linearized problem is valid subject to the condition that changes of the magnetic dipole moment, and therefore the displacements and density changes of the plasma, are small. This solution indicates the direction of change of the different quantities and leads to the conclusion, which is important for applications, that plasma concentrations are found near the axis of a growing magnetic dipole.

For concrete applications to astrophysics and to the physics of laboratory plasmas it is important to have exact nonlinear solutions corresponding to arbitrary (but not small) changes of the magnetic moment, when we may expect the appearance of dense concentrations or, in astrophysical terminology, condensations of plasma.

We shall here derive and analyze the appropriate nonlinear equations and shall present their numerical solutions.

1. STATEMENT OF THE PROBLEM

The magnetohydrodynamic equations for an axially symmetric compressible ideal fluid have the following dimensionless form in the spherical coordinates r, θ, φ:

$$\partial\Phi / \partial t + \delta \mathbf{v} \nabla\Phi = 0, \tag{1}$$

$$\partial\rho / \partial t + \delta \operatorname{div} \rho\mathbf{v} = 0, \tag{2}$$

$$\frac{\varepsilon^2}{\delta}\frac{\partial \mathbf{v}}{\partial t} + \varepsilon^2 \mathbf{v}\nabla\mathbf{v} = -s^2\frac{\nabla p}{\rho} + \frac{j_\varphi}{\rho r \sin\theta}\nabla\Phi. \tag{3}$$

Here $\Phi = \Phi(r, \theta, t)$ is the dimensionless "stream function," which is by definition related to the single nonvanishing φ component of the vector potential **A** by

$$\Phi(r, \theta, t) = r\sin\theta A_\varphi(r, \theta, t); \tag{4}$$

the quantity

$$j_\varphi = -\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial^2\Phi}{\partial r^2} + \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\Phi}{\partial\theta}\right)\right] \tag{5}$$

is the φ component of the dimensionless current density vector **j**,

$$\delta = \frac{V\tau}{L}, \quad \varepsilon = \frac{V}{V_A}, \quad s^2 = \frac{p_0}{\rho_0 V_A{}^2} \tag{6}$$

are dimensionless parameters formed by combining characteristic values of length L, time τ, velocity V, the stream function Φ_0, density ρ_0, and pressure p_0; $V_A^2 = \Phi_0^2/4\pi\rho_0 L^4$ is the square of the Alfvén velocity.

As in[1], we shall assume:

1) In the investigated region the Alfvén velocity greatly exceeds the velocity of sound; we may therefore regard s^2 as small and may neglect the first right-hand term of (3).

2) Plasma motions induced by changes in the magnetic field are sufficiently slow to give us the small parameter

$$\varepsilon^2 = V^2 / V_A{}^2 \ll 1. \tag{7}$$

In contrast with[1] we shall not assume that magnetic field changes and the associated plasma motions are small. Instead, we shall assume the parameter

$$\delta = V\tau / L = 1, \tag{8}$$

which will subsequently determine the choice of the time unit τ.

On the basis of the foregoing assumptions the initial system of equations assumes the form

$$\partial\Phi / \partial t + \mathbf{v}\nabla\Phi \equiv d\Phi / dt = 0, \tag{9}$$

$$\partial\rho / \partial t + \operatorname{div}\rho\mathbf{v} = 0,$$

$$\varepsilon^2 d\mathbf{v} / dt = K(r, \theta, t)\nabla\Phi, \tag{11}$$

where $K(r, \theta, t) = j_\varphi/\rho r \sin\theta$.

The solutions of these equations will be sought as series in the small parameter ϵ^2:

$$\Phi(r, \theta, t) = \Phi^{(0)}(r, \theta, t) + \varepsilon^2\Phi^{(1)}(r, \theta, t) + \ldots \tag{12}$$

etc.

The zeroth approximation in ϵ^2 gives us $K^{(0)}(r, \theta, t) = 0$, so that

$$j_\varphi^{(0)}(r, \theta, t) = 0, \tag{13}$$

which corresponds to a time-dependent potential magnetic field described by the stream function $\Phi^{(0)}(r, \theta, t)$.

From the equation of motion (11) the next approximation in ϵ^2 gives

$$d\mathbf{v}^{(1)} / dt = K^{(1)}(r, \theta, t)\nabla\Phi^{(0)}. \tag{14}$$

Thus in first order with respect to ϵ^2 the acceleration is perpendicular to the lines of force of the zeroth-approximation potential magnetic field, which is assumed to be known. Equations (9), (10), and (14) completely determine the unknown quantities of the first approximation.

2. FIELD OF A MAGNETIC DIPOLE

A dipolar magnetic field will be assumed in zeroth approximation. The corresponding stream function is

$$\Phi^{(0)}(r, \theta, t) = \frac{m(t)\sin^2\theta}{r}, \quad (15)$$

where m(t) is the time-dependent magnetic moment. For example, in the case of a uniformly magnetized gaseous sphere (star) of radius R(t) with a frozen-in magnetic field $H_i(t)$ we have

$$m(t) = {}^1\!/_2 H_i(t) R^3(t) = {}^1\!/_2 H_0 R_0^2 R(t), \quad (16)$$

where H_0 and R_0 are the values of $H_i(t)$ and R(t) at the initial time t = 0.

From the equation (9) for the frozen-in field we find that the stream function $\Phi(r, \theta, t)$ has the advantageous character of an integral of motion:

$$\Phi(r, \theta, t) = \Phi(r_0, \theta_0). \quad (17)$$

The subscript 0 designates the initial values of the Lagrangian coordinates $r(r_0, \theta_0, t)$ and $\theta(r_0, \theta_0, t)$ of a fluid particle. Thus for the dipole field we obtain as the first integral

$$\frac{m(t)\sin^2\theta}{r} = \frac{m_0 \sin^2\theta_0}{r_0}. \quad (18)$$

After inserting the stream function (15) into (14) and dividing the radial component of the equation by the angular component, we eliminate $r = r(r_0, \theta_0, t)$ by means of (18). The resulting ordinary differential equation for $\theta(r_0, \theta_0, t)$ is

$$ma(\theta)\ddot{\theta} + mb(\theta)\dot{\theta}^2 + 2\dot{m}a(\theta)\dot{\theta} + \ddot{m}c(\theta) = 0, \quad (19)$$

where

$$a(\theta) = \sin\theta(1 + 3\cos^2\theta), \quad b(\theta) = 2\cos\theta(3\cos^2\theta - 1)$$
$$c(\theta) = 2\sin^2\theta\cos\theta. \quad (20)$$

The dots denote differentiation with respect to time.

We shall assume that the gas was initially at rest. Therefore the solution $\theta(t)$ of (19) should satisfy the initial conditions

$$\theta(0) = \theta_0 \qquad \dot{\theta}(0) = \dot{\theta}_0 = 0. \quad (21)$$

We note that the solution of (19) does not depend on r_0. The function $r(r_0, \theta_0, t)$ is determined from the solution $\theta(t, \theta_0)$ by means of (18):

$$r(r_0, \theta_0, t) = r_0 \frac{m(t)\sin^2\theta(t, \theta_0)}{m_0 \sin^2\theta_0}. \quad (22)$$

This similarity property of the solution with respect to the initial radius vector r_0 of a fluid particle results from the simple geometry of the magnetic field—the lines of force of the variable dipole have the same form at all times.

The similarity can easily be interpreted on the basis of dimensional theory.[2] Of the five independent parameters ρ, m, r, θ, t the first three have independent dimensions. We can therefore form only two dimensionless combinations as independent variables. One of these is the polar angle θ; it is convenient to choose a dimensionless time as the second variable.

Thus, if a fluid particle was initially at rest at a point having the coordinates r_0, θ_0, then, as shown by (22), at a time t its coordinates and velocity components are given by

$$\theta = \theta(t, \theta_0), \quad (23)$$

$$r = r_0\mathcal{R}(t, \theta), \quad v_r = r_0\mathcal{V}_r(t, \theta), \quad v_\theta = r_0\mathcal{V}_\theta(t, \theta). \quad (24)$$

In (24) we have

$$\mathcal{R}(t, \theta) = \frac{m(t)\sin^2\theta}{m_0 \sin^2\theta_0},$$
$$\mathcal{V}_r(t, \theta) = \dot{\mathcal{R}}(t, \theta) = \frac{\partial\mathcal{R}}{\partial t} + \frac{\partial\mathcal{R}}{\partial\theta}\dot{\theta}, \quad \mathcal{V}_\theta(t, \theta) = \dot{\theta}\mathcal{R}(t, \theta) \quad (25)$$

and θ is the material angular coordinate as defined by (23).

The continuity equation (10) in the Lagrangian form $\rho d\vartheta = \rho_0 d\vartheta_0$ (where $d\vartheta_0$ is the initial volume of the fluid particle and $d\vartheta$ is its volume at the time t) yields the change of density:

$$\rho(r, \theta, t) = \rho_0(r_0, \theta_0)\frac{r_0^2 \sin\theta_0}{r^2\sin\theta}\frac{D(r_0, \theta_0)}{D(r, \theta)} = \rho_0(r_0, \theta_0)\mathcal{P}(t, \theta), \quad (26)$$

where $\rho_0(r_0, \theta_0)$ is the initial density distribution,

$$\frac{D(r_0, \theta_0)}{D(r, \theta)} = \frac{\partial r_0}{\partial r}\frac{\partial\theta_0}{\partial\theta} - \frac{\partial r_0}{\partial\theta}\frac{\partial\theta_0}{\partial r}, \quad (27)$$

and

$$\mathcal{P}(t, \theta) = \frac{r_0^2\sin\theta_0}{r^2\sin\theta}\frac{\partial r_0}{\partial r}\frac{\partial\theta_0}{\partial\theta} = \left(\frac{m_0}{m(t)}\right)^3 \frac{\sin^7\theta_0(\theta, t)}{\sin^7\theta}\frac{\partial\theta_0(\theta, t)}{\partial\theta}. \quad (28)$$

In deriving (28) we have taken into account the similarity property of the solution [the second term in the Jacobian (27) vanishes] and the relation (18). The function $\theta_0(\theta, t)$ is the inverse of the solution $\theta(t, \theta_0)$ of (19) at a given time t.

If the initial density distribution is homogeneous: $\rho_0(r_0, \theta_0) \equiv \rho_0$ = const, then, in accordance with (26), we have

$$\rho(\theta, t) = \rho_0\mathcal{P}(t, \theta). \quad (29)$$

Therefore, for ρ_0 = const the plasma density distribution at a time t is independent of the radius r, so that constant density surfaces are conical surfaces (θ = const). This feature of the problem also results from the aforementioned similarity property.

3. ASYMPTOTIC SOLUTIONS

We shall now consider the asymptotic solution of the problem represented by (19) and (21). For small changes of the moment m[1] and the corresponding small changes of $\theta(t)$, in the linear approximation we find that (19) assumes the form

$$m_0 a^{(0)}\ddot{\theta}^{(1)} + \ddot{m}^{(1)}c^{(0)} = 0, \quad (30)$$

where $a^{(0)} = a(\theta_0)$, $c^{(0)} = c(\theta_0)$ are the coefficients of (20) at $\theta = \theta_0$ (all quantities are represented in the form $\theta = \theta_0 + \lambda\theta^{(1)} + \ldots$, where λ is a small parameter characterizing a small change in the dipole moment $m = m_0 + \lambda m^{(1)} + \ldots$). Integrating (30), we obtain

$$\Delta\theta \equiv \theta - \theta_0 = -\frac{\sin 2\theta_0}{1 + 3\cos^2\theta_0}\frac{\Delta m}{m_0}, \quad (31)$$

where $\Delta m = m - m_0$.

Utilizing (24) and (26), we obtain

$$\frac{\Delta r}{r_0} = \frac{\sin^2\theta_0}{1 + 3\cos^2\theta_0}\frac{\Delta m}{m_0}, \quad (32)$$

$$\frac{\Delta\rho}{\rho_0} = \frac{15\cos^4\theta_0 + 6\cos^2\theta_0 - 5}{(1 + 3\cos^2\theta_0)^2}\frac{\Delta m}{m_0}. \quad (33)$$

The asymptotic solution (31)—(33) agrees with that obtained in[1].

Since, as shown by the equation of motion (14), the acceleration is perpendicular to the field lines, and the plasma was initially at rest, plasma motion does not occur on the dipole axis at any time t. Therefore, for small θ we can assume that $\dot{\theta}$ is small, and from (19) we obtain the asymptotic equation

$$2m\theta\ddot{\theta} + 2m\dot{\theta}^2 + 4\dot{m}\theta\dot{\theta} + \ddot{m}\theta^2 = 0,$$

or

$$\frac{d^2}{dt^2}(m\theta^2) = 0. \tag{34}$$

Taking (21), (24), and (26) into account and assuming

$$\dot{m}(0) = 0 \tag{35}$$

(which is required for continuity of the solution at t = 0), the integration of (34) yields an asymptotic solution for small polar angles and all changes of the dipole moment:

$$\frac{\Delta\theta}{\theta_0} = \left(\frac{m_0}{m(t)}\right)^{1/2}, \quad \frac{r(t)}{r_0} = 1, \quad \frac{\rho(t)}{\rho_0} = \frac{m(t)}{m_0}. \tag{36}$$

For small changes of the moment we have

$$\frac{\Delta\theta}{\theta_0} = -\frac{1}{2}\frac{\Delta m}{m_0}, \quad \frac{\Delta r}{r_0} = 0, \quad \frac{\Delta\rho}{\rho_0} = \frac{\Delta m}{m_0}. \tag{37}$$

The same equations are obtained from the asymptotic solution (31)–(33) in the case of small θ_0.

The asymptotic solutions permit certain useful conclusions. First, it is shown by (33) that $\Delta\rho$ reverses sign at $\cos\theta \approx 0.64$, $\theta \approx 50°$. This angle corresponds to the boundary between the region of compression and the region of density reduction for small changes of the moment. When the dipole moment is reduced the plasma density grows in the equatorial plane. On the other hand, an increase of the moment is accompanied by plasma motion toward the dipole axis. Secondly, (36) shows that change of the plasma density on the dipole axis is proportional to change of the dipole moment for any magnitude of this change.

4. RESULTS OF NUMERICAL INTEGRATION

For m(t) we choose a nearly linear increasing function of time:

$$m(t) = 1 + t^{n+1}/(\alpha + t)^n. \tag{38}$$

This function satisfies (35). The initial value of the dipole moment is unity. The parameter α governs the rate at which the function approaches a linear growth law.

Equation (19), subject to the initial conditions (21) and (38) with n = 1 and α = 0.1, was integrated on a digital computer. In addition to the solution for $\theta(t)$ or $\theta(m)$, which is shown in Fig. 1, we calculated the functions $\mathcal{R}(t, \theta)$, $\mathcal{V}_r(t, \theta)$, $\mathcal{V}_\theta(t, \theta)$, and $\mathcal{P}(t, \theta)$ [see (25) and (28)]. The results are shown in Figs. 1–6. The scale of the horizontal axis in Figs. 2–5 is $\sin^{1/2}\theta$; here the

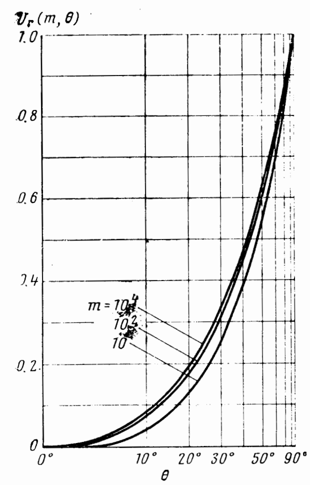

FIG. 3

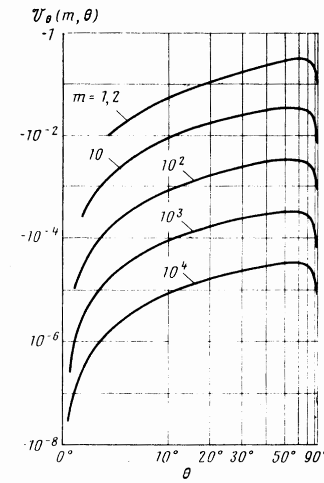

FIG. 4

FIG. 3. Radial velocity of the plasma as a function of the dipole moment and the material angular coordinate $\theta(m, \theta_0)$.

FIG. 4. Angular velocity of the plasma as a function of the dipole moment and the material angular coordinate $\theta(m, \theta_0)$.

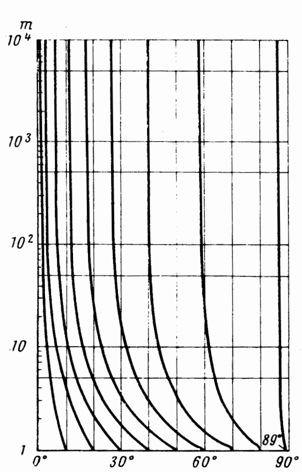

FIG. 1

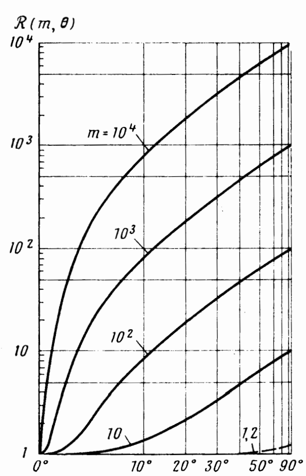

FIG. 2

FIG. 1. Solution of Eq. (19)–the dependence of the material (Lagrangian) angular coordinate θ on its initial value (at m = 1) and on the magnetic dipole moment m.

FIG. 2. Ratio of the radius vector r of a fluid particle for given values of m to the initial radius vector r_0 as a function of the material angular coordinate $\theta = \theta(m, \theta_0)$.

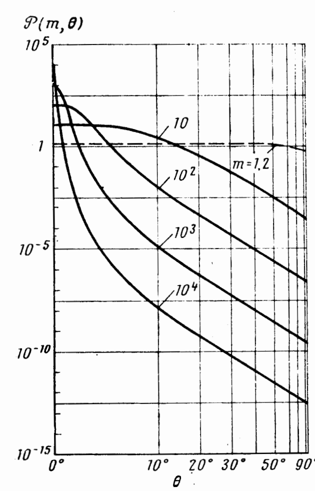

FIG. 5

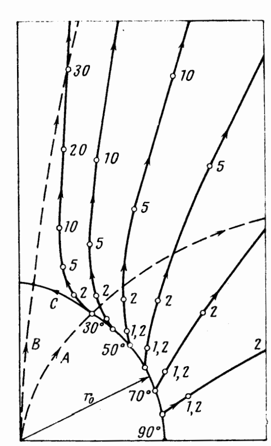

FIG. 6

FIG. 5. Relative condensation of the plasma as a function of the dipole moment and the material angular coordinate θ.

FIG. 6. Trajectories of particles having the initial coordinates r_0 and θ_0 = 30°, 40°, ... 80°. The dashed curve A corresponds to a line of force passing through the point r_0, θ_0 = 30° at the initial instant. The location of the same field line for m = 30 is shown by the dashed line B. The trajectory C corresponds to an asymptotic solution [1] for small changes of the dipole moment.

curves are labeled with the values of the magnetic dipole moment m (instead of the corresponding values of t).

Figure 1 demonstrates the reduction of the polar angle θ as the dipole moment is increased. From Fig. 2 we determine the radius vector r of a fluid particle at the instants when m = 1.2, 10, ..., 10^4. For this purpose, with a given value of θ_0 we obtain $\theta(m, \theta_0)$ in Fig. 1; the value of $\mathscr{R}(m, \theta)$ obtained in Fig. 2 is then multiplied by r_0 in accordance with (24). Similarly, Figs. 3 and 4 are used to determine the radial and angular components of fluid particle velocity. Both components increases with the dipole moment and both vanish on the dipole axis.

The curves in Fig. 5 show how the density distribution depends on the polar angle θ for given values of m when ρ_0 = const. In the general case $\rho_0 = \rho_0(r_0, \theta_0, \varphi_0)$ we can determine the density change of a given fluid particle. For example, on the dipole axis the density always increases in proportion to m, thus agreeing with the asymptotic solution (36). For small changes of the moment (m = 1.2) the solution agrees with (33): $\Delta\rho$ reverses sign at $\theta \sim 50°$. This result is shown more precisely by the following data for the dependence of the angular width θ_{comp} of the compression region on the relative growth of the dipole moment:

m: ~1.0	1.2	1.5	3.0	5.0	10	30	50	10^2	10^3	10^4
θсж: ~50°	46°42′	40°6′	26°56′	20°7′	13°52′	6°46′	5°6′	3°12′	36′26″	7′13″

Figure 6 illustrates the "raking together" of the plasma toward the dipole axis simultaneously with its acceleration along the lines of force.

5. DISCUSSION

Our solution is valid within an axially symmetric region whose inner boundary depends on the size of the region outside of which we can postulate a dipole field. The existence and form of the outer boundary are derived from the fact that at certain distances from the dipole smallness of the parameters s^2 and ϵ^2 cannot be assumed. Thus, at some distance R_1 the magnetic energy density, which decreases as r^{-6}, equals the initial density n_0kT_0 of plasma thermal energy, or, equivalently, the Alfvén velocity equals the velocity of sound. On the dipole axis, where the plasma density is higher because of the "raking together" effect, the parameter s^2 equals unity at $r = R_2 < R_1$. In the equatorial region an additional limitation is imposed by the increasing velocity of plasma motion with increasing distance from the dipole; as a result, beginning at some distance (7) is not fulfilled. Therefore, for the application of our solution to concrete problems we require the fulfillment of quite severe conditions which can ultimately be reduced to the requirement of a sufficiently high magnetic field.

In astrophysics the aforementioned conditions can be realized in the atmospheres of stars (and of some planets), where the energy density of the magnetic field generated by intrastellar (intraplanetary) processes can be much greater than the internal energy density of the rarefied atmospheric plasma. In this case the behavior of the plasma is completely controlled by the magnetic field; this leads to peculiarities of the dynamics. For example, on the sun coronal condensations, ejecta, spicules, several types of protuberances, and other structural elements of the chromosphere and corona are formed.

Similar processes, but on a considerably larger scale, should occur in the atmospheres of magnetically variable stars and exploding stars when the latter possess a sufficiently strong magnetic field. Thus, when a magnetic star explodes the growing magnetic moment of the expanding envelope should, in accordance with the foregoing solution, cause the rarefied gas surrounding the star to form condensations in the vicinity of the magnetic poles. This process may be one cause of the approximately identical structure observed in the envelopes of novae.[3]

Under laboratory conditions the described process of magnetic "raking together" can be utilized to produce dense hot plasma concentrations. This process is essentially a variant of the familiar magnetic compression of plasma,[4] except that in our case the effect of gas pressure is negligibly small.

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