NONLINEAR PROBLEM OF THE TURBULENT DYNAMO

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The problem of generation of a magnetic field in the presence of a turbulent velocity field is considered. In contrast to the usual kinematic statement of the problem, the problem is stated dynamically: the term in the equation of motion which is nonlinear in the magnetic field strength is taken into account. It is shown that the very presence of an inhomogeneous magnetic field causes the turbulence to become gyrotropic (such gyrotropy has been termed "magnetic"). An equation is derived for the gyrotropy parameter in a certain turbulence model. The ordinary gyrotropy due to rotation (detected by M. Steenbeck) induces, as is well known, a magnetic field. Magnetic gyrotropy cancels the ordinary gyrotropy, thus blocking the increase of the field strength and causing nonlinear stabilization to set in. A class of stationary fields that should be encountered in nature is indicated. The order of magnitude of the stationary field is evaluated for the case when the field energy is still far from that corresponding to equipartition of magnetic and kinetic energies. Nonlinear stabilization may thus be sufficiently effective for weak fields.

As is well known, the problem of generation of a magnetic field (the "dynamo" problem) involves a search for growing solutions of the equation

$$\partial \mathbf{H} / \partial t = \operatorname{rot} [\mathbf{vH}] + \nu_m \Delta \mathbf{H}, \qquad (1)^*$$

where v is the velocity field and ν_m is the magnetic viscosity. Only a kinematic formulation of the problem is known at present, namely, the velocity field v is assumed specified (independent of H), and therefore one can dispense with the use of the equation of motion. Equation (1) is linear in v, so that the problem is linear in this case. In a different formulation of the problem one seeks the field v that maintains the magnetic field, i.e., $\partial \mathbf{H}/\partial t = 0$, and the problem reduces to a solution of the equation

$$\operatorname{rot} [\mathbf{vH}] + \nu_m \Delta \mathbf{H} = 0$$

with respect to v[1]. In such a formulation, however, the reaction of the magnetic field on the motion is not taken into account.

In turbulent-dynamo problems, the field v is random and specified by its characteristics (see, for example,[2-4]). It is shown in[2,3] that gyrotropic turbulence, i.e., turbulence in which the velocity probability distribution density is not invariant against the reflection group, is capable of causing generation of a large-scale field (field scale $L \gg l$, where l is the scale of the pulsations). Naturally, such a formulation of the problem, in which the statistical characteristics of the field v are specified, is limited. In fact, Eq. (1) can yield, generally speaking, a field that grows to infinity, which is meaningless, since nonlinear effects should come into play in the presence of intense large-scale fields, and the magnetic field will suppress the turbulence.

Let us resolve H into slowly-varying and rapidly-varying components: H = B + h, where $\langle h \rangle = 0$. It is clear from simple physical considerations that the upper limit of the intensity B is determined from the condition

$$\beta_{st} = 4\pi\rho v^2 / B^2 \approx 1, \qquad (2)$$

whereas at t = 0 we have $\beta \gg 1$.

One can assume, however, that a situation is possible wherein no equipartition of the kinetic and magnetic energies (2) is reached. In fact, in the linear problem, the gyrotropic part of the spectral tensor of the field v

$$iA_1(k)\varepsilon_{ijf}k_f$$

(ϵ_{ijf} is an antisymmetrical tensor of third rank) is responsible for the generation of the field. One can therefore expect the magnetic field to "suppress" primarily just the gyrotropy, i.e., the term A_1, and consequently $\beta_{st} \gg 1$ (the stationary state corresponds to weak nonlinearity). This idea was advanced by Ya. B. Zel'dovich and stimulated the present work. It will be shown below that such a steady state is quite possible.

Steenbeck[2] has shown that gyrotropy arises in a rotating body in the presence of a density gradient and

$$a_* = \langle \mathbf{v} \operatorname{rot} \mathbf{v} \rangle = -2 \int A_1 k^2 dk \sim \mathbf{q}\omega, \qquad (3)$$

where $\mathbf{q} = \nabla \ln \rho$, ρ is the density, and ω is the angular velocity of rotation. Such a gyrotropy will henceforth be called rotational. In the present paper we shall show that the very presence of the magnetic field leads to gyrotropy, but one acting in opposition to the rotational gyrotropy.

Such a gyrotropy will be called magnetic, since it can arise also without rotation, and only as a result of the action of the electromagnetic force curl $\mathbf{H} \times \mathbf{H}/4\pi\rho$. We shall elucidate in what follows some of the most important properties of magnetic gyrotropy, which cancels out the rotational gyrotropy and stops the growth of the field at the weakly-linear stage.

*[vH] ≡ v × H.

1. DERIVATION OF THE EQUATION FOR THE MAGNETIC GYROTROPY a_M

In the presence of a large-scale magnetic field **B**, the turbulence is anisotropic. The general form of the spectral tensor of the field **v** for an anisotropic distribution of the velocity probability and for div **v** = 0 is:

$$\langle u_i(\mathbf{k},\ t)u_j^*(\mathbf{k}',\ t')\rangle = T_{ij}(\mathbf{k},\ s)\delta(\mathbf{k}-\mathbf{k}'),$$
$$T_{ij}(\mathbf{k},s) = \Big[A_2k_ik_j + k^2\frac{k^2A_2+A_3}{(\lambda\mathbf{k})^2}\lambda_i\lambda_j + A_3\delta_{ij} - \frac{k^2A_2+A_3}{\lambda\mathbf{k}}(\lambda_ik_j+\lambda_jk_i)\Big], \qquad s = t-t' \tag{4}$$

(see[3]). Here $A_2 = A_2(k,\ \mathbf{k}\lambda,\ s)$, $A_3 = A_3(k,\ \mathbf{k}\lambda,\ s)$, and λ is a vector parallel to the selected direction. We separate a certain volume, much larger than l^3 but smaller than L^3, which we call elementary. It is natural to assume that $\lambda = \overline{\mathbf{B}}$, where the superior bar denotes averaging over our volume. $\mathbf{u}(\mathbf{k},\ t)$ is the Fourier transform of **v** in the elementary volume, i.e.,

$$\mathbf{v}(\mathbf{r},t) = \sum_k \mathbf{u}(\mathbf{k},t)e^{i\mathbf{kr}},$$

and the δ function in (4) corresponds to homogeneity of the field over this volume. One can expect the influence of **B** on the tensor (4) to be small when $\beta \gg 1$, i.e., A_2 and A_3 depend weakly on $\mathbf{k}\lambda$, and we can therefore expand A_2 and A_3 in terms of the second argument:

$$A_2 = A_2'(k) + A_2''(k)(\mathbf{k}\overline{\mathbf{B}})^2 + \ldots,$$
$$A_3 = A_3'(k) + A_3''(k)(\mathbf{k}\overline{\mathbf{B}})^2 + \ldots, \tag{5}$$

There are no first-degree terms in the expansion (5), since **B** is a polar vector, and substitution of such a term in (4) would yield a pseudotensor, whereas T_{ij} is a true tensor. When substituting (5) in (4), we recognize that T_{ij} should not have any singularities at $\mathbf{k}\cdot\lambda = 0$ (which has no physical meaning), whence

$$T_{ij} = [A(k,s) + C(k,s)(\mathbf{k}\overline{\mathbf{B}})^2]\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right) + D(k,s)[(k_i\overline{B}_j + k_j\overline{B}_i)\mathbf{k}\overline{\mathbf{B}} - (\mathbf{k}\overline{\mathbf{B}})^2\delta_{ij} - k^2\overline{B}_i\overline{B}_j] + iA_1(k,s)\varepsilon_{ijf}k_f. \tag{6}$$

The last term in (6) was added to take the sought gyrotropy into account. We obtain an equation for a_M by using the equation of motion

$$\frac{\partial\mathbf{v}}{\partial t} - [\mathbf{v}\,\mathrm{rot}\,\mathbf{v}] + \frac{1}{2}\nabla v^2 = -\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}[\mathrm{rot}\,\mathbf{H},\mathbf{H}] + \nu\Delta\mathbf{v},$$
$$a_M = \langle\mathbf{v}\,\mathrm{rot}\,\mathbf{v}\rangle, \tag{7}$$

where p is the plasma pressure and ν the viscosity. We take the curl of (7) and then the scalar product of the resultant expression by **v**; we take the scalar product of (7) by curl **v**. We add the resultant expressions and average over the elementary volume. We leave $\partial a_M/\partial t$ on the left side. It is remarkable that all the triple correlations with respect to the velocity vanish in this case:

$$[\mathrm{rot}\,\mathbf{v},\ \mathbf{v}]\ \mathrm{rot}\,\mathbf{v} = 0,$$
$$\mathrm{rot}\,\mathbf{v},\nabla\left(\frac{v^2}{2}+\frac{p}{\rho}\right) = \mathrm{div}\left[\mathbf{v}\nabla\left(\frac{v^2}{2}+\frac{p}{\rho}\right)\right],$$
$$\mathbf{v}\,\mathrm{rot}\,[\mathbf{v},\ \mathrm{rot}\,\mathbf{v}] = \mathrm{div}\,[\mathbf{v}[\mathbf{v}\,\mathrm{rot}\,\mathbf{v}]].$$

In the case of spatial averaging, the divergence makes no contribution. We now calculate the term with the magnetic field:

$$M = \frac{1}{4\pi\rho}\langle[\mathrm{rot}\,\mathbf{v}\,\mathrm{rot}\,\mathbf{H}]\mathbf{H}\rangle + \frac{1}{4\pi\rho}\langle\mathbf{v},\mathrm{rot}[\mathrm{rot}\,\mathbf{H},\mathbf{H}]\rangle$$
$$= 2\{\langle[\mathrm{rot}\,\mathbf{v}\,\mathrm{rot}\,\mathbf{h}]\rangle\mathbf{B} + \langle[\mathbf{h}\,\mathrm{rot}\,\mathbf{v}]\rangle\mathrm{rot}\,\mathbf{B} + \langle[\mathrm{rot}\,\mathbf{v}\,\mathrm{rot}\,\mathbf{h}]\mathbf{h}\rangle\}. \tag{8}$$

To substitute **h** in (8), we express it in terms of $\mathbf{B}(\mathbf{k},\ 0)$ (the largescale field at t = 0) with the aid of the perturbation-theory series (which can be obtained from (1)) for $\mathbf{h}(\mathbf{k},\ t)$ (the Fourier transform of $\mathbf{h}(\mathbf{r},\ t)$):

$$\mathbf{h}(\mathbf{k},t) = \sum_{n=1}^{\infty}\mathbf{h}^{(n)},$$
$$\mathbf{h}^{(1)}(\mathbf{k},t) = i\int_0^t e^{-\nu_m k^2(t-t_1)}dt_1\int d\mathbf{k}_1[\mathbf{k}[\mathbf{u}(\mathbf{k}-\mathbf{k}_1,t_1)\mathbf{B}(\mathbf{k},0)]], \tag{9}$$
$$\mathbf{h}^{(n)}(\mathbf{k},t) = i\int_0^t e^{-\nu_m k^2(t-t_1)}dt_1\int d\mathbf{k}_1[\mathbf{k}[\mathbf{u}(\mathbf{k}-\mathbf{k}_1,t_1)\mathbf{h}^{(n-1)}(\mathbf{k},t_1)]].$$

We indicate first the results of the calculation of (8) for small fluctuations $h \ll B$. In this case the third term in the right-hand side of (8) can be neglected and we can assume $\mathbf{h} = \mathbf{h}^{(1)}$. The most interesting situation occurs when the magnetic Reynolds number $R_m \gg 1$ (ν_m is small), in which case $\exp[-\nu_m k^2(t-t_1)] \approx 1$. The expressions in the angle brackets in (8) are best calculated in Fourier space, after which we transform again to **r**-space. The resultant cumbersome expression becomes simpler if it is recognized that $k_iT_{ij}(\mathbf{k}) = 0$ (the incompressibility condition),

$$\int k_jT_{ij}(\mathbf{k})\,d\mathbf{k} = 0, \qquad \int k_ik_jA(k)\,d\mathbf{k} = \frac{1}{3}\delta_{ij}\int A(k)k^2\,d\mathbf{k},$$
$$\int k_ik_jk_fk_pC(k)\,d\mathbf{k} = \frac{1}{15}(\delta_{ij}\delta_{fp} + \delta_{if}\delta_{jp} + \delta_{ip}\delta_{jf})\int C(k)k^4\,d\mathbf{k}.$$

We write it out for $t \gg \tau$, where τ is the correlation time, but $t \ll \tau_B$, where τ_B is the characteristic time of the variation of **B** (it is meaningful to consider (8) precisely at such values of t):

$$M = M_aB^2\mathbf{B}\,\mathrm{rot}\,\mathbf{B} + \frac{1}{3}\frac{B^2}{4\pi\rho}A_4,$$
$$M_a = \frac{1}{15}\frac{1}{4\pi\rho}\int[2C(k,s) + D(k,s)]k^4\,d\mathbf{k}\,ds, \tag{10}$$
$$A_4 = \int A_1(k,s)k^4\,d\mathbf{k}\,ds.$$

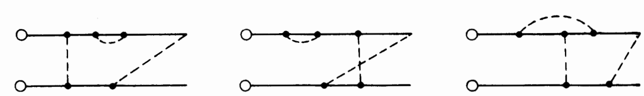
Since $\overline{\mathbf{B}}$ differs little from **B** in the elementary volume, we no longer distinguish between the two in (10). Indeed, at $R_m \gg 1$ we have $h \gg B$ and inclusion of only the first term of the series (9) is insufficient; we therefore use the selective summation proposed by Kazantsev[5] for the perturbation-theory series. We employ here the following turbulence model:

1) The distribution of the velocity probabilities is Gaussian;

2) we assume that

$$A(k,s) = w_1(k)\delta(s), \quad C(k,s) = w_2(k)\delta(s),$$
$$D(k,s) = w_3(k)\delta(s), \quad A_1(k,s) = w_4(k)\delta(s).$$

It is convenient to use a diagram representation of the terms of the series:



a b c

The figure shows examples of the diagrams of third order in T_{ij} (and of sixth order in $\mathbf{u}(\mathbf{k}, t)$). The circles correspond to $\mathbf{B}(\mathbf{k}, 0)$. The set of diagrams represents the product of a series, namely, the product of the expansion of curl v × curl h by the expansion of h, with the upper straight line corresponding to some term of the series of curl v × curl h, and the lower to the series h. The points correspond to the variables t_n of integration with respect to time; if it follows from the integration limits that $t_n \le t_m$, then the point t_n is placed to the left of t_m. Therefore the straight lines of the diagrams can be interpreted as the time axes—zero to the left and t to the right. The dashed lines correspond to $T_{ij}(q, t_p - t_m)$. From condition (1) of the model it follows that only diagrams with odd numbers of points remain; the dashed line on the extreme right corresponds to $T_{ij}(q, t - t_m)$ and is due to the presence of curl $\mathbf{v}(\mathbf{r}, t)$. From condition (2) it follows that diagrams in which the dashed lines intersect (Fig. 1b) are equal to zero. Further, for the same reason, diagrams in which a dashed line joins two points on a line do not vanish only when these are two neighboring points, and therefore the diagram of Fig. 1c is equal to zero.

We note that these diagrams are topologically completely analogous to the Kazantsev diagram[5], and therefore the summation can be carried out by the same method. A calculation shows that the resultant expression is similar to (10), except that in place of $\mathbf{B}(\mathbf{k}, 0)$ in (10) we have

$$\mathbf{B}(\mathbf{k},0)\exp\left(-tk_ik_j\int T_{ij}(\mathbf{p},s)\,d\mathbf{p}\,ds-\nu_mk^2t\right)\mathrm{ch}\left(tk\int A_1k^2\,d\mathbf{k}\,ds\right).$$

Since this multiplier of B is close to unity when $t \ll \tau_B$, it follows that the result of the summation gives a small contribution to (10). This is precisely why we neglect the third term in (8) and why a δ-like correlation is used in this turbulence model.

We now write down the equation for a_M:

$$\frac{\partial a_M}{\partial t}=M_aB^2\mathbf{B}\,\mathrm{rot}\,\mathbf{B}+4\nu\int A_1k^4dk+\frac{1}{3}\frac{B^2}{4\pi\rho}\int A_1(k,s)\,k^4\,d\mathbf{k}\,ds,$$
$$a_M=-2\int A_1k^2\,d\mathbf{k}. \qquad (11)$$

If M_a is known, then (11) is an equation with respect to A_1. The first term of the right-hand side of (11) describes the generation of magnetic gyrotropy, while the second describes its viscous diffusion and the third describes the suppression of the gyrotropy as a result of the magnetic stresses produced by helical twisting.

We note that the observed magnetic gyrotropy is already essentially nonlinear (unlike rotational gyrotropy), and appears only when the velocity field is already distorted by the magnetic field (M_a contains no contribution from $A(k, s)$).

To take into account the generation of rotational gyrotropy when $\omega \neq 0$, it is necessary to add the Coriolis force to (7) and a term that takes $\nabla\rho$ into account to (6).

2. CALCULATION OF THE GENERATION COEFFICIENT α UNDER CERTAIN ASSUMPTIONS

Equation (11) describes the establishment of gyrotropy if at $t = 0$ we have $a_M = 0$. In the steady state $\partial a_M/\partial t = 0$. We shall not stop to calculate the steady-state a_M, since the important quantity for field generation is not a_M but α:

$$\alpha=-\frac{1}{3}\int_{-\infty}^{+\infty}\langle\mathbf{v}(t)\,\mathrm{rot}\,\mathbf{v}(t')\rangle\,dt'=\frac{2}{3}\int A_1(k,s)\,k^2\,d\mathbf{k}\,ds \qquad (12)$$

and the equation for B takes the form[3]:

$$\partial\mathbf{B}/\partial t=\mathrm{rot}\,\alpha\mathbf{B}+\nu_0\Delta\mathbf{B},$$
$$\nu_0=\frac{1}{3}\int A(k,s)\,d\mathbf{k}\,ds. \qquad (13)$$

We shall henceforth use the following turbulence model. We assume that the spectral functions A, C, and D are described as follows: $A(k, 0)$, $C(k, 0)$, $D(k, 0)$, $A_1(k, 0)$ contain the energy interval—these functions grow in the interval $(0, 2\pi/l)$, while the interval $k_\nu > k > 2\pi/l$ corresponds to the inertial subregion and the functions decrease in power-law fashion

$$A\sim k^{-\gamma},\quad A=\beta(2C+D)k^2B^2, \qquad (14)$$

with $3 < \gamma < 5$ (this means that the energy of the pulsations is concentrated at $k \approx 1/l$, and the dissipation region corresponds to $k \approx k_\nu$; this is the normal situation in hydrodynamic turbulence). When $k \ge k_\nu$ the spectrum is cut off as a result of the viscosity. We assume a simple correlation time dependence, for example an exponential one, with the correlation time having the following dependence on k:

$$\tau(k)\sim k^{(\gamma-5)/2}. \qquad (15)$$

Such a model corresponds to the Kolmogorov turbulence at $\gamma = {}^{11}/_3$. Let us elucidate briefly the physical meaning of C and D in formula (14). It is clear that in the presence of B, the hydrodynamic turbulence excites random Alfven waves. The vortical motions serve as the source, whereas the decay time of the waves is determined by their collisions with the vortices. It can be assumed that the source corresponds in Fourier space to a term

$$\frac{1}{4\pi\rho}i[[\mathbf{kh}]\mathbf{B}]-\frac{1}{4\pi\rho}\frac{\mathbf{k}}{k^2}[[\mathbf{kh}]\mathbf{B}]\mathbf{k}$$

in the equation of motion; the first term is the electromagnetic force, while the second corresponds to $-\nabla p/\rho$ and is obtained from the requirement div v = 0. It can easily be seen that the spectral tensor of such a source, assuming that h is isotropically distributed (in the first approximation), coincides in form with (6) (at $A_1 = 0$) and C = D. Thus, the nonisotropic part of (6) corresponds to new excited degrees of freedom—Alfven waves. Further, if it is assumed that $\langle h^2\rangle/8\pi = \rho\langle v^2\rangle/2$, i.e., equipartition in pulsation scales (in the presence of acoustic turbulence, equipartition actually sets in[4] and this is why we make such an assumption), then the problem has a single small parameter β^{-1}, and this gives rise to (14).

Finally, it is natural to assume that the time of establishment of stationary α (and also of its disintegration at B = 0) is τ.

In such a model, Eq. (11) is replaced by the following equation for α:

$$\frac{\partial\alpha}{\partial t}=-2M_aB^2\mathbf{B}\,\mathrm{rot}\,\mathbf{B}-2\frac{\alpha}{\tau}, \qquad (16)$$

$$M_\alpha = \frac{1}{15}\frac{1}{4\pi\rho}\int_0^\infty s\,ds\int(2C(k,s)+D(k,s))k^4\,dk$$
$$= n\frac{\ln(k_\nu l)}{5\langle v^2\rangle(4\pi\rho)^2}, \quad n\approx 1,$$

whence (17)

$$\alpha_M = -\mathbf{B}\,\mathrm{rot}\,\mathbf{B}\Phi, \quad \Phi = 2\tau n\ln(k_\nu l)/5\beta 4\pi\rho.$$

3. MOST IMPORTANT PROPERTIES OF MAGNETIC GYROTROPY

We proceed now to determine certain properties of α_M, which are connected with the generation of magnetic fields. Let $\omega = 0$ (there is no rotation) and assume that at $t = 0$ we have $\mathbf{B}$ curl $\mathbf{B} \neq 0$. We turn now to Eq. (13). For simplicity, we neglect diffusion ($v_0 = 0$), and then we obtain readily from (13)

$$\tfrac{1}{2}\partial B^2/\partial t = -(\mathbf{B}\,\mathrm{rot}\,\mathbf{B})^2\Phi. \tag{18}$$

We note that $\Phi > 0$, and therefore

$$B^2(t) - B^2(0) = -2\int_0^t dt_1(\mathbf{B}\,\mathrm{rot}\,\mathbf{B})^2\Phi.$$

We let $t \to \infty$, in which case the integral in the right-hand side converges, since it is monotonic in t and is bounded by the quantity $B^2(0)/2$. Hence

$$(\mathbf{B}\,\mathrm{rot}\,\mathbf{B})^2 \to 0.$$

It is thus clear that if we have $\mathbf{B}$ curl $\mathbf{B} \neq 0$ at $t = 0$, then magnetic gyrotropy sets in and brings about $\mathbf{B}$ curl $\mathbf{B} = 0$, after which the gyrotropy itself vanishes, as follows from (16).

Another property of magnetic gyrotropy is that it acts in opposition to the rotational gyrotropy. In fact, in the presence of rotational gyrotropy α_ω and under the condition that $\mathbf{B}$ curl $\mathbf{B} = 0$ when $t = 0$, we have

$$\frac{d}{dt}\int \mathbf{B}\,\mathrm{rot}\,\mathbf{B}\,dr = \frac{1}{2}\alpha_\omega\int(\mathrm{rot}\,\mathbf{B})^2\,dr$$

where the integration is over the entire volume. We see that $\mathbf{B}$ curl $\mathbf{B}$ acquires the same sign as α_ω, and therefore if $\alpha_\omega > 0$ then it follows from (17) that $\alpha_M < 0$, but $\alpha = \alpha_M + \alpha_\omega \geq 0$, for a stationary state arises at $\alpha = 0$. If $\alpha_\omega < 0$, then $\alpha_M > 0$. In both cases $|\alpha| = |\alpha_\omega + \alpha_M| < |\alpha_\omega|$, i.e., the generation coefficient decreases in the presence of magnetic gyrotropy. It is precisely the last circumstance which makes nonlinear stabilization of the field $\mathbf{B}$ possible in the case of weak fields.

4. POSSIBILITY OF NONLINEAR STABILIZATION OF A WEAK MAGNETIC FIELD

The equation for the steady-state field is

$$\mathrm{rot}\,(\alpha_M + \alpha_\omega)\mathbf{B} = v_0\,\mathrm{rot}^2\,\mathbf{B}. \tag{19}$$

Taking the scalar product of (19) with $\mathbf{B}$, we obtain

$$-(\mathbf{B}\,\mathrm{rot}\,\mathbf{B})^2\Phi + \alpha_\omega\mathbf{B}\,\mathrm{rot}\,\mathbf{B} = v_0\,\mathrm{rot}^2\,\mathbf{B}, \tag{20}$$

$$\int\alpha_\omega\mathbf{B}\,\mathrm{rot}\,\mathbf{B}\,dr - \int(\mathbf{B}\,\mathrm{rot}\,\mathbf{B})^2\Phi\,dr = v_0\int(\mathrm{rot}\,\mathbf{B})^2dr. \tag{21}$$

It follows from (21) that the first integral on the left is positive. Equations (19)–(21) are valid if $\beta_{st} \gg 1$ in the steady state. To estimate β_{st} we use (21); $\alpha_\omega \approx \omega l^2/L$, L is the dimension of the body, and $v_0 = \tau\langle v^2\rangle/6$. In addition, we assume that the scale of the field $\mathbf{B}$ is also of the order of L, i.e., we are dealing with the harmonic having the largest scale. We then obtain

$$\beta_{st}^2 = \frac{2}{5}\ln(k_\nu l)\left(\frac{l\omega}{v} - 1\right)^{-1}, \quad v = (\langle v^2\rangle)^{1/2}. \tag{22}$$

Naturally, all the equations written out in the present section are meaningful if field generation actually took place during the initial period, and then one can speak of a steady state, i.e., the growth increment of the field

$$\gamma = \alpha_\omega/L - v_0/L^2 \tag{23}$$

(see[3]) should be positive. It follows therefore, as can be readily seen, that

$$N = l\omega/v > 1, \tag{24}$$

i.e., the expression in the right-hand side of (22) is positive, as it should be. It follows from (22) that the most effective stabilization occurs at the "threshold" of the instability, i.e., when $N \gtrsim 1$. (We note that $\ln(k_\nu l) \approx \ln \mathrm{Re}$, where Re is the Reynolds number, is usually quite large, so that $\ln(k_\nu l)$ can be of the order of 10.) On the other hand, a situation wherein $N \gg 1$ is apparently not realistic physically, since this would mean that the frequency of rotation of the object is much higher than the frequency of rotation of the turbulent element.

Thus, nonlinear stabilization of a weak field is realistic, so that the class of fields satisfying Eq. (19) should become particularly important, since it is precisely such fields that should be encountered in nature.

There is one more field-stabilization possibility that can find application in astrophysics. Let us imagine a conducting sphere, inside which turbulence is excited, and in addition, there is differential rotation of the sphere (the model of a star), i.e., ω depends on r: $\omega = \omega(r)$ (r, θ, φ are spherical coordinates). At certain values of the parameters, field generation is also possible here, but the excitation conditions are less stringent than in the preceding example, and the growth increment is larger[6]. This is natural, since the differential rotation itself is an effective additional field generator. In particular, (24) may not be satisfied. The field growth increment

$$\gamma = \sqrt{-\frac{\partial\omega}{\partial r}\alpha} - \frac{v_0}{L^2} \tag{25}$$

should be positive. We assume that $\partial\omega/\partial r \approx -\omega/L$; in the stationary state $\gamma = 0$, from which we can obtain $\alpha^{st} = \alpha_M^{st} + \alpha_\omega$ and β_{st}:

$$\beta_{st}^2 = \frac{2}{5}\ln(k_\nu l)\left[N - \frac{l^2}{L^2}\frac{1}{N}\right]^{-1}. \tag{26}$$

In this case, as above, the expression the square brackets in (26) is positive, and the stabilization is very effective at the "threshold" of the excitation: $N \gtrsim l/L$. Here, however, there is another possibility of stabilizing the field, namely, when $1 > N \gg l/L$, i.e., not at the excitation threshold, but in a wider range, and then

$$\beta_{st}^2 = \frac{2}{5}\frac{\ln(k_\nu l)}{N}. \tag{27}$$

Usually $N < 1$, since the rotation frequency of the star is smaller than the rotation frequency of the turbulent element. The steady-state field should satisfy the equation

$$\text{rot}\,\{[\mathbf{vB}] + (\alpha_\omega + \alpha_M)\mathbf{B}\} = \nu_0\,\text{rot}^2\,\mathbf{B}. \tag{28}$$

Thus, under certain perfectly realistic conditions, the stabilized field satisfies the condition $\beta_{st} \gg 1$, i.e., a state is established which is sufficiently far from equipartition of the magnetic and kinetic energies ($\beta_{st} \approx 1$).

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