BEAM INSTABILITY IN GRAVITATING MEDIA

A. B. MIKHAĬLOVSKIĬ and A. M. FRIDMAN

Novosibirsk State University

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It is shown that beam (two-stream) instability may develop in gravitating systems besides the Jeans instability.

1. INTRODUCTION

ONE of the remarkable properties of a many-particle system with Coulomb interaction is the possibility of a spontaneous buildup of small fluctuations in the system when the distribution of the particles over velocity is non-Maxwellian. The simplest example of a non-Maxwellian particle distribution is the totality of two particle beams moving relative to each other. The phenomenon of spontaneous buildup of fluctuations in such a system is called "beam (two-stream) instability."

Beam instability has been thoroughly studied in the case when the particles participating in the Coulomb interaction are electrons and ions; in other words, in the case of an electron-ion plasma. It is now known that, depending upon the concrete conditions, the development of quite a large family of beam instabilities in a plasma (see, for example, the monograph[1] about this) is possible. Beam instabilities play in many cases a decisive role in plasma dynamics[2]. Taking formally into consideration the analogy between the Coulomb law for charged particles and Newton's law for gravitating particles, it is natural to conjecture that certain types of instabilities, which are characteristic of a plasma medium, should occur in gravitating media. In particular, the question of the possibility of beam instabilities developing in gravitating media is an interesting one. This question has not yet been cleared up, although it has been repeatedly posed (see, for example,[3-6]).

One of the main difficulties encountered in the theoretical investigation of beam instabilities in a gravitating medium is connected with the fact that here, in contrast to a plasma, there is only attraction between the particles. This, in the final analysis, makes it necessary to consider a medium of finite dimensions smaller than some critical dimensions. When these dimensions are exceeded, either the macroscopic equilibrium is destroyed and a Jeans condensation (collapse) occurs, or a Jeans instability arises which leads to a breakup of the system into spatially separated subsystems. In the case of a plasma, however, we can use the approximation of an unbounded spatially homogeneous medium, and this simplifies the problem considerably. In the first papers[3,4] devoted to the study of beam instabilities in gravitating media these media were assumed to be unbounded. Therefore, the conclusion drawn in these papers that a beam instability is possible is not convincing, especially as the conditions for instability given there are fulfilled only in the absence of macroscopic equilibrium or under the conditions of a Jeans instability.

A convenient (from the point of view of the theory of instabilities) model of a spatially inhomogeneous gravitating medium may be a gravitating cylinder of finite radius and infinite length. Radial equilibrium of such a system is attained owing to the revolution of the particles around the axis of the cylinder, while the time of relaxation collapse in the longitudinal direction is proportional to the length L of the cylinder. In this case the boundary-value problem is quite easy to solve and the dispersion equation turns out to be similar to the one obtained for a plasma in a magnetic field.

The dispersion equation for a rotating gravitating cylinder has been derived before in[5,6] 1). However, incorrect boundary conditions were used in these derivations (continuity of the derivative of the perturbed potential at the boundary of the cylinder was assumed). Therefore, the dispersion equation obtained in[6] is, generally speaking, incorrect (although, as will be shown below, it leads in some limiting cases to correct results). In particular, the view expressed in[6], that the development of a beam instability in a gravitating medium is impossible, is incorrect.

In Sec. 2 a dispersion equation for the perturbation of the rotating gravitating cylinder is obtained. The law of particle distribution over the longitudinal velocities is, as in[6], assumed to be arbitrary. The boundary conditions used here are those obtained by integrating Poisson's equation over the transition layer; this does not reduce to the condition assumed in[6] for continuity of the derivative of the perturbed potential.

Section 3 is devoted to the study of the dispersion equation for the case of a Maxwellian particle distribution over the longitudinal velocities. We determine in this section the conditions under which the Maxwellian distribution can be considered quasi-stationary (the condition for an exponentially small negative damping of the Jeans instability). We also show that, besides perturbations of the Jeans type, there is a branch of undamped perturbations ($\mathrm{Im}\,\omega = 0$), connected with the azimuthal revolution of the particles. We consider in Sec. 4 the kinetic buildup of this "orbital" branch of the oscillations by a low-density beam of fast particles, while in Sec. 5 we consider the hydrodynamic buildup. These processes correspond to different types of beam instability of a gravitating medium. In Sec. 6 we investigate the stability of a system consisting of two clashing beams of equal density with a Jackson type distribution

1)The hydrodynamic approximation was used in [5]; in [6] kinetic effects connected with the longitudinal motion of the particles–in particular, beam effects–were taken into account.

over the longitudinal velocities[7]. It turns out that a beam instability, not noted in[6], occurs in this case also. Section 7 is devoted to a discussion of the results.

2. THE STEADY STATE AND THE DISPERSION EQUATION

1. The Steady State

Radial equilibrium of a cylindrically symmetric system of gravitating particles is realized if by chance the particles possess an azimuthal velocity $V_{0\varphi}$ such that

$$V_{0\varphi}^2/r = -\partial\Psi_0/\partial r. \tag{2.1}$$

Here Ψ_0 is the potential of the steady-state gravitational field satisfying the Poisson equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi_0}{\partial r}\right) = 4\pi G\rho_0 \tag{2.2}$$

and ρ_0 is the steady-state mass density of the particles. By prescribing the profile of the density $\rho_0(r)$, we can, with the help of (2.1) and (2.2), find the functions $\Psi_0(r)$ and $V_{0\varphi}(r)$. Let

$$\rho_0 = \begin{cases} \text{const for } 0 < r < R, \\ 0 \quad \text{for} \quad r > R. \end{cases} \tag{2.3}$$

Thus, we consider a homogeneous cylinder with a sharp boundary.

It must be borne in mind that a sharp boundary is an idealization of a thin transition layer between the gravitating medium and vacuum. In the following analysis of the perturbations of the cylinder we shall have to consider the phenomena occurring in this transition layer. Therefore, we shall consider below the thickness δ of the transition layer to be finite albeit small in comparison with the radius of the cylinder R, $\delta \ll R$. As for the influence of the finiteness of the parameter δ/R on the radial dependence of $\Psi_0(r)$ and $V_{0\varphi}(r)$, it is insignificant, as can be seen from (2.1) and (2.2). From these equations follow to within small terms of the order of δ/R:

$$\Psi_0 = {}^1/_4\omega_0^2 r^2, \qquad V_{0\varphi} = \pm\Omega_0 r.$$

Here $\omega_0 = \sqrt{4\pi G\rho_0}$ is the Jeans frequency and Ω_0 is the angular velocity of the particles along the circular orbits. The relation between the characteristic frequencies[5] $\omega_0^2 = 2\Omega_0^2$ is easily established from the above equalities. We shall henceforth assume that all the particles revolve in the same direction, i.e., that $V_{0\varphi} = r\Omega_0$. Therefore, the homogeneous cylinder rotates as a rigid body[5].

The velocities of the particles along the axis of the cylinder are taken to be arbitrary. In the investigation below of the instabilities we shall assume the length L of the cylinder to be infinitely large ($L \to \infty$), in comparison with all other dimensions (the radius R of the cylinder and the characteristic wavelengths of the perturbations). For large L the longitudinal gravitational field is weak, its magnitude going like 1/L. Therefore, the longitudinal "collapse" time connected with this field can be assumed to be infinitely large in comparison with all the times which are of interest to us.

2. The Dispersion Equation

When the potential Ψ deviates from its equilibrium value, each group of particles initially possessing a velocity $\mathbf{V}_0 = \mathbf{e}_\varphi V_{0\varphi} + \mathbf{e}_z V_{0z}$ will move with a velocity $\mathbf{V}$ satisfying the equation

$$d\mathbf{V}/dt = -\nabla\Psi. \tag{2.4}$$

If the mass density of this group of particles was initially equal to $\rho_0(V_{0z})$, then, for $\Psi \neq \Psi_0$, these particles will be characterized by the density $\rho(V_{0z})$ such that

$$\partial\rho(V_{0z})/\partial t + \operatorname{div}(\rho\mathbf{V}) = 0. \tag{2.5}$$

Let us linearize Eqs. (2.4) and (2.5), denoting deviations by the index one. Let us choose the spatial and time dependence of the perturbations in the form $f_1(r)\exp(-i\omega t + ik_z z)$, restricting ourselves to the analysis of only axially symmetric perturbations, i.e., of those for which $\partial/\partial\varphi = 0$. We find, as a result, the density perturbation $\rho_1(V_{0z})$ of the group of particles with the unperturbed velocity V_{0z}:

$$\rho_1(V_{0z}) = -\rho_0(V_{0z})\left(\frac{\Delta_r\Psi_1}{\omega'^2 - 4\Omega_0^2} - \frac{k_z^2}{\omega'^2}\Psi_1\right) - \frac{\partial\rho_0(V_{0z})}{\partial r}\frac{\partial\Psi_1/\partial r}{\omega'^2 - 4\Omega_0^2};$$

$$\omega' = \omega - k_z V_{0z}, \qquad \Delta_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right).$$

Integrating over all V_{0z}, we obtain the total value of the perturbed density

$$\rho_1(\mathbf{r}, t) = \int_{-\infty}^{\infty} \rho_1(\mathbf{r}, t, V_{0z})\,dV_{0z}.$$

Substituting this result into the Poisson equation, we arrive at the following dispersion equation for Ψ_1 2):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\varepsilon_\perp\frac{\partial\Psi_1}{\partial r}\right) - k_z^2\varepsilon_\parallel\Psi_1 = 0. \tag{2.6}$$

Here we use the notation

$$\varepsilon_\perp = 1 + \omega_0^2\int_{-\infty}^{\infty}\frac{f(V_z)\,dV_z}{\omega'^2 - 4\Omega_0^2}, \quad \varepsilon_\parallel = 1 + \omega_0^2\int_{-\infty}^{\infty}\frac{f(V_z)\,dV_z}{\omega'^2}. \tag{2.7}$$

In the integration with respect to the longitudinal velocity we used the following notation $\rho_0 = \int\rho_0(V_{0z})dV_{0z}$, where ρ_0 = const.

Equation (2.6) has been written in the form which is characteristic of the problem of plasma oscillations in a magnetic field. If, however, we replace in (2.7) the square of the Jeans frequency by minus the square of the plasma frequency, $\omega_0^2 \to -\omega_p^2 \equiv -4\pi n_0 e^2/m$, and twice the angular velocity of the gravitating particles by the cyclotron frequency of charged particles (with charge e and mass m in a magnetic field B_0), $2\Omega_0 \to \omega_B \equiv eB_0/mc$, then it turns out that the expressions $\epsilon_\perp$ and $\epsilon_\parallel$ respectively coincide with the transverse and longitudinal components of the dielectric susceptibility tensor of an electron plasma. Equation (2.6) coincides in such a transition with the differential equation for the electric potential of the electrostatic (potential) oscillations of a plasma cylinder of uniform density situated in a uniform magnetic field $\mathbf{B}_0 \parallel \mathbf{z}$.

2) For more details see an analogous derivation in [1] (problem on Sec. 2.1).

The procedure for obtaining the dispersion equation with the aid of (2.6) is similar to the case of the plasma. Inside the cylinder, where the steady-state density is uniform, the quantities $\epsilon_\perp$ and $\epsilon_\parallel$ do not depend on the radius, so that (2.6) reduces to the Bessel equation and has the solution

$$\Psi_1^{(1)} = C_1 J_0(k_\perp r), \quad r < R, \tag{2.8}$$

where $k_\perp$ is defined by the relation

$$k_\perp^2 = -k_z^2 \varepsilon_\parallel^{(0)} / \varepsilon_\perp^{(0)}, \tag{2.9}$$

and the superscript zero indicates that $\epsilon_\parallel$ and $\epsilon_\perp$ pertain to the interior region of the cylinder. Outside the cylinder (in the vacuum) the solution to (2.6) is the Macdonald function

$$\Psi_1^{(2)} = C_2 K_0(k_z r), \quad r > R. \tag{2.10}$$

At the boundary of the cylinder, the solutions of (2.8) and (2.10) are connected by two relations. One of them is the continuity condition for the potential

$$\Psi_1^{(1)}(R) = \Psi_1^{(2)}(R). \tag{2.11}$$

The other is obtained by integrating (2.6) along the transition layer $(R - \delta, R + \delta)$ and then allowing $\delta \to 0$. It has the form

$$\varepsilon_\perp^{(0)} \left(\frac{\partial \Psi_1^{(1)}}{\partial r}\right)_{r=R} = \left(\frac{\partial \Psi_1^{(2)}}{\partial r}\right)_{r=R} \tag{2.12}$$

and corresponds to the electrodynamic condition for continuity of the normal component of the electric induction D. Notice that in[6], instead of (2.12), the condition for continuity of the derivatives of the perturbed potential is used, which, of course, is not correct. With the aid of (2.8)—(2.12) we obtain the dispersion equation

$$\varepsilon_\perp^{(0)} \frac{k_\perp J_0'(k_\perp R)}{J_0(k_\perp R)} = \frac{k_z K_0'(k_z R)}{K_0(k_z R)}. \tag{2.13}$$

The dispersion equation given in[6] does not contain on the left hand side the factor $\epsilon_\perp^{(0)}$ and is therefore incorrect.

Equation (2.13) assumes a simpler form in the limiting cases of perturbations—of long and short wavelengths along z, i.e. for $k_z R \ll 1$ and $k_z R \gg 1$. In the first case, $k_z R \ll 1$, $K_0(x) \sim \ln(1/x)$, so that from (2.13) follows:

$$\varepsilon_\perp^{(0)} \frac{k_\perp R J_0'(k_\perp R)}{J_0(k_\perp R)} = -\frac{1}{\ln(1/k_z R)}. \tag{2.14}$$

Owing to the smallness of the right hand side of the equality, the numerator of the left hand side of the equality should be close to zero. This is possible for $k_\perp R \ll 1$ and, then, using (2.9), we can reduce the dispersion equation to the form

$$1 + \frac{k_z^2 R^2}{2} \ln\left(\frac{1}{k_z R_0}\right) \varepsilon_\parallel^{(0)} = 0. \tag{2.15}$$

Furthermore, Eq. (2.14) is approximately satisfied if

$$k_\perp R = k_{\perp n} R \equiv \lambda_n, \tag{2.16}$$

where λ_n is a nontrivial root of the equation

$$J_1(\lambda_n) = 0. \tag{2.17}$$

Computing λ_n and $k_{\perp n}$ and using (2.9), we can in this case represent the dispersion equation in the form

$$k_{\perp n}^2 \varepsilon_\perp^{(0)} + k_z^2 \varepsilon_\parallel^{(0)} = 0. \tag{2.18}$$

For $k_z R \gg 1$, we have on the right hand side of (2.13) a large quantity $\sim k_z$. Therefore the dispersion equation is approximately satisfied if $k_\perp$ satisfies the condition (2.16) and λ_n is determined from the equation

$$J_0(\lambda_n) = 0. \tag{2.19}$$

The dispersion equation is also reduced in this case to the form (2.18), but with some other value of $k_{\perp n}$.

3. BRANCHES OF THE OSCILLATIONS OF A ROTATING CYLINDER WITH A MAXWELLIAN PARTICLE DISTRIBUTION OVER THE LONGITUDINAL VELOCITIES

We shall henceforth assume that the distribution of the particles over the longitudinal velocities $f(V_z)$ can be split up into two parts

$$f(V_z) = f^{(0)}(V_z) + f^{(1)}(V_z), \tag{3.1}$$

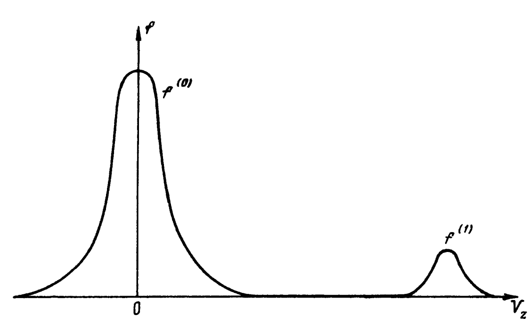
where $f^{(0)}(V_z)$ is a Maxwellian function with a thermal velocity V_T:

$$f^{(0)}(V_z) = (\pi V_T^2)^{-1/2} \exp(-V_z^2/V_T^2),$$

while $f^{(1)}(V_z)$ is some function which is different from zero when $V_z \gg V_T$. An example of such a function is shown in the figure. In other words, we are assuming that the system of gravitating particles being considered consists of two subsystems: a slow subsystem and a fast subsystem. We shall assume that the slow subsystem has a larger mass density than the fast one, so that

$$\int_{-\infty}^{\infty} f^{(1)}(V_z)\,dV_z \Big/ \int_{-\infty}^{\infty} f^{(0)}(V_z)\,dV_z = \alpha \ll 1.$$

Using the fact that the parameter α is small, we can find the solution of the dispersion equation (2.18) by the method of successive approximations. The spectrum of the oscillations of a cylinder with a Maxwellian distribution over the longitudinal velocities, i.e., the branches of the oscillations of such a cylinder, is found in the zeroth approximation in α. The increments (negative damping constants) or the decrements for these oscillations are determined in the next order; the growth or attenuation of the oscillations is due to the interaction of the oscillations with the particles of the fast component. Such a formulation of the problem is characteristic of the theory of the interaction of a beam of charged particles with a dense plasma.



Here, we analyze the zeroth approximation in α and take into account terms of the order of α in the following paragraphs.

When $\alpha = 0$, i.e., when $f(V_z) = f^{(0)}(V_z)$ the expressions for $\epsilon_\perp^{(0)}$ and $\epsilon_\parallel^{(0)}$ have the form (the zero superscript is henceforth dropped):

$$\varepsilon_\perp = 1 + \frac{i\sqrt{\pi}}{4}\frac{\omega_0^2}{\Omega_0|k_z|V_T}\left[W\left(\frac{\omega+2\Omega_0}{|k_z|V_T}\right) - W\left(\frac{\omega-2\Omega_0}{|k_z|V_T}\right)\right], \quad (3.2)$$

$$\varepsilon_\parallel = 1 - 2\frac{\omega_0^2}{k_z^2V_T^2}\left[1 + i\sqrt{\pi}\frac{\omega}{|k_z|V_T}W\left(\frac{\omega}{|k_z|V_T}\right)\right].$$

Using these expressions, we obtain, in the limiting cases $k_zR \ll 1$ and $k_zR \gg 1$, the following.

1. Long-wavelength Oscillations, $k_zR \ll 1$

a. Large-scale perturbations, $k_\perp R \ll 1$. For $k_zR \ll 1$ and $k_\perp R \ll 1$ the basic equation is the dispersion equation (2.18). Neglecting in it terms of the order of $k_z^2R^2$ compared with unity and using (3.2), we reduce it to the form:

$$1 - \frac{R^2\omega_0^2}{V_T^2}\beta\left[1 + i\sqrt{\pi}\frac{\omega}{|k_z|V_T}W\left(\frac{\omega}{|k_z|V_T}\right)\right] = 0, \quad (3.3)$$

where $\beta = \ln(1/k_zR)$. Considering the limiting cases of large and small ω/k_zV_T, we can verify that Eq. (3.3) does not have roots $\omega(k_z)$ corresponding to slowly decaying oscillations. Indeed, for $|\omega| \gg |k_z|V_T$, from (3.3) follows:

$$\omega^2 = -\beta k_z^2V_0^2,$$

where $V_0 \equiv V_0\varphi(R)$ is the linear velocity of border particles. This solution describes aperiodically growing or aperiodically damped perturbations

$$\operatorname{Re}\omega = 0, \quad \operatorname{Im}\omega = \pm\beta^{1/2}|k_z|V_0. \quad (3.4)$$

The solution with $\operatorname{Im}\omega > 0$ corresponds to a Jeans instability. It is valid (the condition $\omega \gg |k_z|V_T$) if the thermal spread is not too large, i.e., $V_T^2 \ll \beta^2V_0^2$. The increment of the instability decreases with increase in the thermal spread. Indeed, in the opposite limiting case, when $|\omega| \ll k_zV_T$, from (3.3) follows the expression for the frequency

$$\omega = -\frac{i}{\sqrt{\pi}}|k_z|V_T\left(\frac{1}{2\beta}\frac{V_T^2}{V_0^2} - 1\right), \quad (3.5)$$

which is valid (the condition $|\omega| \ll k_zV_T$) if

$$\left|\frac{1}{2\beta}\frac{V_T^2}{V_0^2} - 1\right| \ll 1.$$

For a sufficiently large thermal spread $V_T^2 = 2\beta V_0^2$, the increment of the Jeans instability investigated above vanishes. At still larger V_T the perturbations, as can be seen from (3.5), decay aperiodically.

Thus, the Jeans instability of large-scale perturbations with $k_zR \ll 1$ is suppressed if the longitudinal thermal spread is sufficiently large: $V_T^2 > 2\beta V_0^2$. It follows from the definition of β that for any arbitrarily large but finite value of V_T the last condition cannot be fulfilled for all k_z. Perturbations for which

$$k_zR < \exp(-V_T^2/2V_0^2) \quad (3.6)$$

remain unstable. This means that an infinitely long cylinder is unstable for any arbitrarily large but finite thermal scatter of the particles over the longitudinal velocities. The opposite conclusion drawn in[5,6] is due to an error in the computations[3)].

Using the condition (3.6) and the expressions (3.4) and (3.5), we arrive at an estimate for the increment

$$\gamma \approx \varepsilon e^{-\varepsilon}\Omega_0,$$

where $\epsilon = V_T^2/V_0^2 \gg 1$. It can be seen that for large ϵ the increment of the Jeans instability is exponentially small.

The buildup of large-scale perturbations can be understood as the result of an increase in the effective Jeans frequency for small k_z and k_T. Indeed, as can be seen from (3.4) or (3.5), the role of characteristic frequency of the collective motion is played by the quantity $\sqrt{\beta}\omega_0$, and not simply by ω_0 as is the case in small-scale perturbations (see the following subsection). The effect of an increase in the frequency of the collective oscillations in perturbations with small wave numbers is well known in the theory of plasma oscillations (see, for example, the review by Faĭnberg[2]).

b. Small-scale perturbations, $k_\perp R \gtrsim 1$. Perturbations with $k_zR \ll 1$, and $k_\perp R \gtrsim 1$ are described by the dispersion equation (2.18). When $\epsilon_\perp$ and $\epsilon_\parallel$ have the form (3.2) this equation implies:

$$1 + \frac{i\sqrt{\pi}}{4}\frac{k_\perp^2}{k_z^2}\frac{\omega_0^2}{\Omega_0|k_z|V_T}\left[W\left(\frac{\omega+2\Omega_0}{|k_z|V_T}\right) - W\left(\frac{\omega-2\Omega_0}{|k_z|V_T}\right)\right] \quad (3.7)$$
$$- 2\frac{\omega_0^2}{k^2V_T^2}\left[1 + i\sqrt{\pi}\frac{\omega}{|k_z|V_T}W\left(\frac{\omega}{|k_z|V_T}\right)\right] = 0.$$

Here, $k^2 = k_z^2 + k_\perp^2 \approx k_\perp^2$ is the square of the total wave number, while $k_\perp$, as follows from (2.16) and (2.17), runs through a series of values, the smallest of which is $k_\perp^{(1)} = 3.8/R$.

Allowing for the smallness of $k_z/k_\perp$, we notice that Eq. (3.7) has two kinds of solutions: with $\omega \ll \Omega_0$ and $\omega \approx \Omega_0$. Solutions of the first kind correspond to the Jeans perturbations, a particular case of which ($k_\perp R \ll 1$) was considered in subsection a. When $k_zV_T \ll \Omega_0$ we obtain from (3.7) a dispersion equation for perturbations with $\omega \ll \Omega_0$ which is analogous to (3.3):

$$\frac{1}{2} - \frac{2\omega_0^2}{k_\perp^2V_T^2}\left[1 + i\sqrt{\pi}\frac{\omega}{|k_z|V_T}W\left(\frac{\omega}{|k_z|V_T}\right)\right] = 0.$$

The perturbations described by this equation attenuate aperiodically if

$$V_T^2 > 4\Omega_0^2/k_\perp^2. \quad (3.8)$$

Since $k_\perp \gtrsim k_\perp^{(1)} = 3.8/R$ the condition (3.8) is satisfied for all small-scale perturbations if $V_T > V_0/2$. When this condition is met, only the large-scale perturbations considered in subsection a remain unstable.

If $\omega \approx \Omega_0$ the terms connected with the azimuthal revolution of the particles are the principal terms in Eq. (3.7). For sufficiently small k_z such that $|k_z|V_T \ll |\omega|$ and $|\omega \pm 2\Omega_0|$, Eq. (3.7) takes the simple form:

$$1 + \omega_0^2/(\omega^2 - 4\Omega_0^2) = 0 \quad (3.9)$$

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3)The authors of [5,6] did not notice that in the case of long wavelength perturbations, $k_zR \ll 1$, the dispersion equation can be satisfied not only when $k_\perp R \gtrsim 1$, but also when $k_\perp R \ll 1$.

which describes oscillations with the real frequency

$$\omega^2 = -\omega_0^2 + 4\Omega_0^2 \equiv \omega_0^2. \quad (3.10)$$

This branch of the oscillations may be called the orbital branch. It is analogous to the branch of plasma electron oscillations in a magnetic field with a frequency ω given by $\omega^2 = \omega_p^2 + \omega_B^2$.

Taking into account the imaginary terms in Eq. (3.7), which are exponentially small when $|\omega \pm 2\Omega_0|$ and $|\omega| \gg |k_z| V_T$, we find the damping constant:

$$\operatorname{Im}\omega = -\sqrt{\pi}\frac{\omega_0^2}{|k_z| V_T}\left(\frac{\chi_1}{4\sqrt{2}} + \frac{\omega_0^2}{k^2 V_T^2}\chi_2\right). \quad (3.11)$$

Here χ_1 and χ_2 are positive and are defined by the relations

$$\chi_1 = \exp\left[-\left(0.6\frac{\Omega_0}{k_z V_T}\right)^2\right] - \exp\left[-\left(3.4\frac{\Omega_0}{k_z V_T}\right)^2\right], \quad \chi_2 = \exp\left[-\frac{2\Omega_0^2}{k_z^2 V_T^2}\right]. \quad (3.12)$$

The addend on the right hand side of (3.11) that is proportional to χ_1 is the damping constant which characterizes the damping of the orbital oscillations caused by the resonance interaction of the revolving particles with the waves (a resonance of the type $\omega = k_z V_z \pm 2\Omega_0$). This is the analog of the cyclotron damping of plasma oscillations in a magnetic field. The addend on the right hand side of (3.11) proportional to χ_2 is the damping constant which characterizes the damping of the orbital oscillations owing to Cerenkov interaction with resonance particles (resonance of the type $\omega = k_z V_z$).

It can be seen from (3.11) that the ratio $\operatorname{Im}\omega/\operatorname{Re}\omega$ exponentially decreases with decrease of $|k_z|$. Therefore, for sufficiently small $|k_z|$ these oscillations can be considered undamped.

2. Short-wavelength perturbations, $k_z R \gg 1$

As follows from Sec. 2, perturbations with $k_z \ll k_\perp$ are described formally by the same dispersion equation as for $k_z \ll k_\perp$, with only a slightly different set of discrete $k_\perp$. Therefore, in the analysis of perturbations of a Maxwellian (with respect to the longitudinal particle velocities) medium, we can start with Eq. (3.7) given above, assuming now, however, that $k_z \gg k_\perp$.

For $k_z \gg k_\perp$ and small values of V_T, $k_z V_T \ll \Omega$, Eq. (3.7) (as in the case when $k_z \ll k_\perp$) describes perturbations of two kinds: the Jeans and orbital perturbations. If, however, the thermal velocity is not small compared to the azimuthal velocity ($V_T \gtrsim V_0$), then the conditions $k_\perp R \gtrsim 1$ ($k_\perp^{(1)} R \approx 2.4$, see Eqs. (2.16) and (2.19)) and $k_z \gg k_\perp$ lead to the inequality $k_z V_T \gg \Omega_0$. When $k_z V_T \gg \Omega_0$ and $k_z \gg k_\perp$, all the solutions to Eq. (3.1) correspond to rapidly decaying perturbations. The explicit form of these solutions can be found in the same manner as in Landau's paper[8].

4. KINETIC BEAM INSTABILITY

As follows from Sec. 3, a Jeans instability develops with an exponentially small increment if $V_T \gg V_0$. Therefore, a rearrangement of the initial spatial distribution of the particles (the formation of constrictions) will proceed extremely slowly. Under these conditions we can speak of a quasistationary state of the Maxwellian subsystem and consider the building up of undamped oscillations of this subsystem by a group of fast particles.

According to Sec. 3, a branch of the undamped oscillations of the Maxwellian subsystem exists when $k_z \ll k_\perp$ and $k_\perp R \gtrsim 1$. The frequency of these oscillations are determined by Eq. (3.7). The interaction of these oscillations with the group of fast particles is described by Eq. (2.18) with $\epsilon_\perp$ and $\epsilon_\parallel$ containing addents from both subsystems. Substituting in the expression (2.7) for $\epsilon_\perp$ and $\epsilon_\parallel$ the distribution function in the form of the sum (3.1) and assuming $\Omega \gg k_z V_T$, we obtain with the aid of (2.18) the following generalization of Eq. (3.9):

$$1 + \frac{\omega_0^2(1-\alpha)}{\omega^2 - 4\Omega_0^2} + \alpha\omega_0^2\left[\frac{k_z^2}{k^2}\int_{-\infty}^{\infty}\frac{f(V_z)\,dV_z}{(\omega - k_z V_z)^2} + \int_{-\infty}\frac{f(V_z)\,dV_z}{(\omega - k_z V_z)^2 - 4\Omega^2}\right] = 0. \quad (4.1)$$

Here and below the subscript one is dropped from the function f_1.

To obtain the imaginary correction to the frequency (3.10) of the oscillations, it is sufficient to take into account only the imaginary terms of this equation. We then obtain:

$$\operatorname{Im}\omega = \frac{\pi}{2}\frac{\omega_0^3}{k^2}\left\{f'\left(\frac{\omega}{k_z}\right)\operatorname{sign}k_z - \frac{1}{4}\frac{k^2}{|k_z|\Omega_0}\left[f\left(\frac{\omega - 2\Omega_0}{k_z}\right) - f\left(\frac{\omega + 2\Omega_0}{k_z}\right)\right]\right\}. \quad (4.2)$$

We must substitute in the right-hand side of the equation the expression (3.10) for ω, i.e., put $\omega = \omega_0$.

Let us analyze this general result in the following limiting cases.

1. Beam with a Maxwellian Velocity Distribution. Let

$$f = \frac{\alpha}{\pi^{1/2} V_{T1}}\exp\left[-\left(\frac{V_z - V}{V_{T1}}\right)^2\right]. \quad (4.3)$$

Then

$$\operatorname{Im}\omega = -\sqrt{\pi}\,\alpha\frac{\omega_0^4}{|k_z| k^2 V_{T1}^3}\left\{\left(1 - \frac{k_z V}{\omega_0}\right)\exp\left[-\left(\frac{\omega_0 - k_z V}{k_z V_{T1}}\right)^2\right] + \frac{1}{4\sqrt{2}}\frac{k_\perp^2 V_{T1}^2}{\omega_0^2}\left(\exp\left[-\left(\frac{0.4\omega_0 + k_z V}{k_z V_{T1}}\right)^2\right] - \exp\left[-\left(\frac{1.6\omega_0 - k_z V}{k_z V_{T1}}\right)^2\right]\right)\right\} \quad (4.4)$$

The first term in the braces in (4.4) corresponds to a resonance of the type $\omega = k_z V_z$, while the two others correspond to the resonances $\omega \pm 2\Omega_0 = k_z V_z$. It can be seen that a resonance of the type $\omega = k_z V_z$ (a Cerenkov resonance) can lead to an instability if $k_z V > \omega_0$. An orbital resonance of the type $\omega + 2\Omega_0 = k_z V_z$ also leads to an instability with the maximum increment

$$\operatorname{Im}\omega \approx \alpha\omega_0 V / V_{T1}. \quad (4.5)$$

The expression (4.5) is valid for $V_{T_1} \gg V$ as well as for $V_{T_1} \ll V$. The ratio V_{T_1}/V should, however, be assumed greater than $\alpha^{1/2}$ since otherwise a stronger hydrodynamic beam instability, discussed in Sec. 5, develops.

2. Beam with a Distribution in the Form of a Step.

A beam instability is connected first and foremost with asymmetry in the velocity distribution of the fast particles ($f(V_z) \neq f(-V_z)$) and not with the presence of a second maximum in the general distribution function. To verify this, let us consider an asymmetric distribution of the particles of a beam in the form of a step:

$$f = \frac{\alpha}{4V_1}\begin{cases} 1, & 0 < V_z < V_1, \\ 0, & V_z < 0, \ V_z > V_1. \end{cases}$$

In this case the resonance of the type $\omega + 2\Omega_0 = k_z V_z$ makes the sole contribution to (4.2), so that

$$\operatorname{Im}\omega = \frac{\pi}{8\sqrt{2}}\alpha\frac{\omega_0^2}{|k_z|V_1}.$$

This expression is valid when $|k_z|V_1 > 2.4\Omega_0$, from which follows the estimate $\operatorname{Im}\omega \lesssim \alpha\omega_0$ which coincides with (4.5) when $V \approx V_{T_1}$.

5. HYDRODYNAMIC BEAM INSTABILITY

If the ordered velocity of the beam is high in comparison with the scatter of the beam particles over the longitudinal velocities, i.e., if $V \gg V_{T_1}$, then, besides the kinetic instability considered in Sec. 4, a hydrodynamic beam instability can develop in a two-beam gravitating medium. Let us show this, assuming that the thermal scatter of the beam is nevertheless finite, $V_{T_1}^2 \gg \alpha V_0^2$, so that the development of a Jeans instability in the beam is not possible.

Let us proceed from Eq. (4.1) with the function f of the form (4.3), and let us assume that $|\omega + 2\Omega_0 - k_z V| \gg |k_z| V_{T_1}$. Then, from (4.1) follows

$$1 + \frac{(1-\alpha)\omega_0^2}{\omega^2 - 4\Omega_0^2} - \frac{\alpha\omega_0^2}{4\Omega_0(\omega - k_z V + 2\Omega_0)} = 0. \qquad (5.1)$$

Assuming that $k_z V = \omega_0 + 2\Omega_0$, we find that oscillations with $\operatorname{Re}\omega = \omega_0$ grow with the increment

$$\operatorname{Im}\omega = 2^{-5/4}\alpha^{1/2}\omega_0. \qquad (5.2)$$

This instability is analogous to the cyclotron buildup by a monoenergetic beam of charged particles of the cyclotron oscillations of a plasma in a magnetic field.

6. INSTABILITY OF TWO CLASHING BEAMS OF EQUAL DENSITY

The following example of velocity distribution was considered in[6]:

$$f(V_z) = \frac{\Delta}{2\pi}\left[\frac{1}{(v+V)^2 + \Delta^2} + \frac{1}{(v-V)^2 + \Delta^2}\right]. \qquad (6.1)$$

It is a characteristic of such a distribution that the distribution function of every beam, upon deviation from the mean velocity of this beam, should decrease rather slowly (in comparison with the Maxwellian distribution). It is therefore necessary to take into account the resonance particles of both beams. It then turns out that the damping of the orbital branch of the oscillations of one or the other beam due to the interaction with the resonance particles of the same beam is stronger than the buildup of the oscillations by the particles of the other beam. This may explain why the kinetic beam instability was not discovered in[6]. In the case of two Maxwellian beams of the same density, however, the damping of the oscillations is exponentially small (see the formulas (3.11) and (3.12)). Therefore, such beams can be kinetically unstable.

However, as follows from Sec. 5, in a two-beam system, besides the kinetic instability, the development of a hydrodynamic beam instability is possible. Such an instability may also develop in the case of a distribution of the type (6.1). This fact went unnoticed in[6]. Let us therefore discuss it in greater detail.

For an f of the form (6.1) with $V \gg \Delta$ and $\omega \approx \omega_0 \gg k_z\Delta$, the dispersion equation (2.18) reduces to the form

$$1 + \frac{\omega_0^2}{2}\left[\frac{1}{(\omega - k_z V)^2 - 4\Omega_0^2} + \frac{1}{(\omega + k_z V)^2 - 4\Omega_0^2}\right] = 0. \qquad (6.2)$$

This is a particular case of the dispersion equation of[6]. The solutions of Eq. (6.2) are:

$$\omega_\pm^2 = (k_z V)^2 + 3\Omega_0^2 \pm \sqrt{\Omega_0^4 + 12\Omega_0^2(k_z V)^2}. \qquad (6.3)$$

It follows from this that $\omega_-^2 < 0$ if $|(k_z V)^2 - 3\Omega_0^2| \leq \Omega_0^2$. When $(k_z V)^2 = 3\Omega_0^2$ the increment of the perturbations is equal to

$$\operatorname{Im}\omega \approx 0.3\Omega_0. \qquad (6.4)$$

Thus, it is shown that when $V \gg \Delta$ a beam distribution of the form (6.1) is unstable. A Jeans instability is insignificant here if $\Delta \gg V_0$.

7. DISCUSSION OF THE RESULTS

We have considered axially-symmetric perturbations of a gravitating cylinder rotating about its own axis with a constant angular velocity. Such perturbations can grow in time and can be responsible for two kinds of instabilities: the Jeans and beam instabilities.

The increment of the Jeans instability is exponentially small if the thermal scatter of the particles over the longitudinal velocities is large compared with the azimuthal velocities. We can, under these conditions, speak of a quasistationary state of the cylinder.

A beam instability occurs if a group of fast particles with an asymmetric distribution over the longitudinal velocities exists in the gravitating system. It is connected with a buildup of the normal modes of the cylinder due to the azimuthal revolution of the particles. Then, in contrast to the Jeans instability, which leads to the growth of the large-scale (along the radius) perturbations, the beam instability leads to the buildup of small-scale (along the radius) perturbations, although these perturbations are greatly extended along the axis of the cylinder.

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