PROPERTIES OF THE MASS TENSOR

I. G. FIKHTENGOL'TS

Leningrad Institute of Precision Mechanics and Optics

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Integral relations applicable to the motion of a small lumped mass are derived for the mass tensor. A Lagrangian function yielding correct expressions for the momentum of a given mass and for the force acting on the mass is derived in integral form.

WE shall show that when motions of a small concentrated mass are considered, the following relations hold true

$$\frac{\partial}{\partial a_i}\int_{(a)}\rho\sqrt{-g}(dx)^3=\int_{(a)}\Gamma_{i\nu}{}^{\mu}T_{\mu}{}^{\nu}\sqrt{-g}(dx)^3 \tag{1}$$

$$\frac{\partial}{\partial \dot a_i}\int_{(a)}\rho\sqrt{-g}(dx)^3=\int_{(a)}T_i{}^0\sqrt{-g}(dx)^3. \tag{2}$$

Here ρ is the invariant mass density, $T_{\mu\nu}$ is the mass tensor, g the term made up of the components of the fundamental tensor $g_{\mu\nu}$, $\Gamma^{\alpha}_{\mu\nu}$ are Christoffel brackets of the second kind, a_i is the coordinate of the considered small mass a, and $(dx)^3 = dx_1dx_2dx_3$. The integration is carried out over the region (a) containing the mass a. A superior dot denotes the derivative with respect to the time $x_0 = t$. Greek indices assume the values 0, 1, 2, and 3, and summation from 0 to 3 over repeated Greek indices is stipulated. Latin indices assume the values 1, 2, and 3.

To prove the foregoing statement, we make use of the fact that when the motion of a small mass is considered we can put (see the book[1])

$$T^{\mu\nu}=c^{-2}\rho u^{\mu}u^{\nu}, \tag{3}$$

where c is the speed of light in vacuum and u^{μ} is the four-dimensional contravariant velocity vector normalized in accordance with the formula

$$g_{\mu\nu}u^{\mu}u^{\nu}=c^2.$$

It is shown in the cited book[1] that, as applied to the motion of a concentrated mass a we have

$$\int_{(a)}\rho u^0\sqrt{-g}(dx)^3=m_ac \tag{4}$$

and

$$\int_{(a)}\rho\sqrt{-g}(dx)^3=m_a\mathcal{R}|_{x_i=\alpha_i}, \tag{5}$$

where

$$\mathcal{R}=(g_{00}+2g_{0i}\dot x_i+g_{ik}\dot x_i\dot x_k)^{1/2};$$

with, of course,

$$u^0=c/\mathcal{R}$$

and m_a is the rest mass of the mass a under consideration. Summation from 1 to 3 over repeated Latin indices is stipulated.

Starting from (3) and (4), we can readily show that in the limiting case of a small concentrated mass a we have

$$\int_{(a)}\Gamma_{i\nu}{}^{\mu}T_{\mu}{}^{\nu}\sqrt{-g}(dx)^3=\frac{m_a}{2\mathcal{R}}\left(\frac{\partial g_{00}}{\partial x_i}+2\frac{\partial g_{0k}}{\partial x_i}\dot x_k+\frac{\partial g_{kl}}{\partial x_i}\dot x_k\dot x_l\right)\Big|_{x_i=a_i} \tag{6}$$

and

$$\int_{(a)}T_i{}^0\sqrt{-g}(dx)^3=m_a\frac{g_{0i}+g_{ik}x_k}{\mathcal{R}}\Big|_{x_i=a_i}. \tag{7}$$

The validity of (1) and (2) follows from Eqs. (5), (6), and (7).

When considering the motions of a small concentrated mass a in a specified gravitational field, the Lagrangian function $\mathcal{L}_a$ can be represented in the form

$$\mathcal{L}_a=-c\int_{(a)}\rho\sqrt{-g}(dx)^3. \tag{8}$$

However, the Lagrangian function (8) does not lead to correct expressions for the momentum of the mass a in question and for the force acting on this mass[1)] (see[2]).

We obtain correct expressions for the momentum and for the force if we choose the Lagrangian for the considered case of motion of a small concentrated mass in the form

$$L_a=-c\int_{(a)}\rho\sqrt{-g}(dx)^3-\frac{1}{c}\frac{d}{dt}\int_{(a)}\rho\,\varphi u^0\sqrt{-g}(dx)^3, \tag{9}$$

where φ is a function satisfying the equation

$$u^0u^{\mu}\frac{\partial\varphi}{\partial x_{\mu}}=J_{\mu}{}^0u^{\mu}+g^{0\mu}\Gamma_{0\mu}{}^{\nu}V_{\nu}u^0. \tag{10}$$

Here (see[2])

$$J_{\nu}{}^{\mu}=\nabla^{\mu}V_{\nu} \tag{11}$$

is the tensor of the gravitational potential and V_{α} is a four-dimensional vector satisfying the equations

$$\rho g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}V_{\alpha}=c^2\Pi_{\alpha\nu}{}^{\mu}T_{\mu}{}^{\nu}, \tag{12}$$

where

$$\Pi_{\mu\nu}{}^{\alpha}=\Gamma_{\mu\nu}{}^{\alpha}-(\Gamma_{\mu\nu}{}^{\alpha})_0 \tag{13}$$

and $(\Gamma^{\alpha}_{\mu\nu})_0$ are Christoffel brackets of the second kind not for the investigated space-time, but for an auxiliary one, against the background of which the real physical space-time is considered. In the foregoing formulas ∇_{μ}, as usual, denotes the covariant derivative in the investigated space-time.

As applied to the astronomical problem of an isolated system of masses, the solution of (10) in first approximation is the function

$$\varphi=c^{-2}V_0 \tag{14}$$

[1)] "Correct" expressions should lead to equations of motion having the usual form (the time derivative of the momentum is equal to the acting force, and the resultant of the forces for an isolated system is equal to zero).

or, which is the same in our approximation (see[2]), the function

$$\varphi = \frac{1}{c^2}\frac{\partial W}{\partial t}, \qquad (15)$$

where

$$W = \frac{1}{2}\gamma \int_{(\infty)} \rho' |\mathbf{r} - \mathbf{r}'| (dx')^3. \qquad (16)$$

Here γ is the Newtonian gravitational constant and $\mathbf{r}$, as usual, is the radius vector of the point with coordinates x_1, x_2, x_3. The integration is over all of infinite space.

Thus, in the case of motion of a small concentrated mass a in a gravitational field of n other finite (not small) masses, the Lagrangian function (9) can be represented, accurate to quantities of the order U/c^2, where U is the Newtonian gravitational potential, in the form

$$L_a = -c \int_{(a)} \rho \sqrt{-g}(dx)^3 - \frac{1}{c^2}\frac{d}{dt}\int_{(a)} \rho \frac{\partial W}{\partial t}(dx)^3. \qquad (17)$$

Under the indicated assumptions with respect to the mass a, the Lagrangian function (17) coincides, apart from the constant factor $m_a c^2$, with the Lagrangian function L^* obtained under the same assumptions in[3]. The change-over from the function $\mathscr{L}_a$ (8) to the function L_a (9), which satisfies the additional condition formulated above, leads to discarding of the derivative

$$\frac{1}{c}\frac{d}{dt}\int_{(a)} \rho \varphi u^0 \sqrt{-g}(dx)^3.$$

which is contained in the function $\mathscr{L}_a$ (8) as a term.

It can be shown that formula (9) and the approximate formula (17) that follows from it are valid for the motion of not only a small mass but also of a finite (not small) mass.

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[1] V. A. Fock, Teoriya prostranstva, vremeni i tyagoteniya (Theory of Space, Time, and Gravitation), Fizmatgiz, 1961.

[2] I. G. Fikhtengol'ts, Zh. Eksp. Teor. Fiz. **60**, 1206 (1971) [Sov. Phys.-JETP **33**, 652 (1971)].

[3] I. G. Fikhtengol'ts, Zh. Eksp. Teor. Fiz. **32**, 1098 (1957) [Sov. Phys.-JETP **5**, 898 (1957)].

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