CRITICAL CURRENT OF AN IDEAL TYPE II SUPERCONDUCTOR IN THE MIXED STATE

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We consider a perfectly uniform film of a type II superconductor ($\kappa \gg 1$, $\xi \ll d \ll \delta_0$, d is the film thickness and δ_0 the penetration depth) and a perfectly uniform plate ($d \gg \delta_0$). We show that in both cases, when the external magnetic field H_0 is parallel to the surface of the film or plate, the mixed state is a triangular vortex lattice which is uniform in density, similar to the lattice of a bulk sample. The interaction between the vortex lattice and the Meissner currents causes its stability against small displacements, i.e., pinning. We find the pinning force and the critical current at right angles to the field, which violates the stability of the vortex lattice. In the case of a film the current is determined by Eq. (20) and in the case of a plate by Eq. (30). The latter formula is similar to the one proposed by Campbell, Evetts, and Dew-Hughes.[6]

We have found earlier[1] the dependence of the critical current on the external magnetic field for a perfectly uniform film of a type II superconductor when the external magnetic field which is parallel to the surface of the film and at right angles to the current changes from zero to $\sim H_{c1}(d)$, the first critical field of the film.[2]

In the present paper we calculate the critical field of a perfectly uniform superconductor in the mixed state. We consider two limiting cases: the case of a thin film, the thickness of which d satisfies the inequalities $\xi \ll d \ll \delta_0$, where δ_0 is the penetration depth, and the case of a thick film (plate), the thickness of which $d \gg \delta_0$.

We consider first the structure of the mixed state of the film when the external magnetic field H_0 ($H_{c1}(d) \ll H_0 \ll H_{c2}$) is parallel to the film surface while there is no transport current. It turns out that in a film, as in a bulk superconductor, the vortices are distributed with a constant density. However, the surfaces of the film are by their very nature pinning centers[1], and this leads to stability of the vortex structure with respect to a transverse transport current. When a transverse transport current is switched on the whole vortex structure moves as a unit in the direction of the Lorentz force over a distance proportional to the transport current.

All these results turn out to be valid also for the mixed state of a plate.

We call the transport current for which the vortex instability develops the critical one. In the case of a film the critical current turned out to be independent of the external magnetic field.

1. STRUCTURE OF THE MIXED STATE OF A FILM WHEN THERE IS NO TRANSPORT CURRENT

Let us to start with define more precisely the problem. We consider a perfectly uniform film of thickness $d(\kappa^{-1} \ll d \ll 1)$ of a type II superconductor, $\kappa \gg 1$. Here and below we use the units introduced in the GL paper[3]: the unit length is the quantity $\delta_0(T)$ and the unit magnetic field the quantity $\sqrt{2H_{cm}}$, where H_{cm} is the critical thermodynamic field.

The film is in an external uniform magnetic field H_0 parallel to the z-axis, and $H_{c1}(d) \ll H_0 \ll H_{c2}$. According to[2]

$$H_{c1}(d) = \frac{4}{\varkappa d^2} \ln\left(\frac{\gamma \varkappa d}{\pi} + 0.081\right), \quad \gamma = e^C \approx 1.78.$$

The surfaces of the film coincide with the planes $x = \pm d/2$. We shall assume that the vortex structure is a two-dimensional triangular lattice consisting of rows of vortices. All vortex lines are parallel to the z-axis. Each row lies in a plane $x = x_l$, where l is an integer which takes on $2L + 1$ values from $-L/2$ to $+L/2$ and which numbers the rows ($L \gg 1$). The row in the plane $x = 0$ has the index $l = 0$. The distance between the nearest vortices in each row (along the y-axis) is taken to be equal to $a \ll d$ and assumed to be independent of the row number l. The coordinates of the vortices along the y-axis for any two adjacent rows differ by a/2 (triangular lattice). The distance between rows (along the x-axis) is initially assumed to depend on the number l. Calculation shows that it is independent of the number l and equal to $b \ll d$. The problem of the present paper is to find the equilibrium values of a and b for a given external field H_0.

To find the equilibrium values of a and b we must express the Gibbs free energy $\mathcal{G} = \mathcal{F} - 2\int_{V_S} \mathbf{H}\cdot\mathbf{H}_0 dV_S$ of our system in terms of the parameters a and x_l and then minimize it with respect of these parameters.

The free energy of a film layer of unit height along the z-axis is equal to[4]

$$\mathcal{F} = \int_{V_s} (\mathbf{H}^2 + (\operatorname{rot}\mathbf{H})^2) dV_s,$$

V_S is the volume of the film layer. According to the theorem proved in the Appendix (see (A.2))

$$\mathcal{F} = \frac{2\pi}{\varkappa} \sum_{lm} H_{v,lm}, \tag{1}$$

where $H_{v,lm}$ is the field produced by the whole vortex system (and only by the vortex system) at the center of the vortex with index (l, m), where l is the number of the row in which the vortex is situated, and m the number of the vortex in that row. We take as the zero of the energy (1) the film energy which is independent of the vortices. As all vortices with given l are in the same circumstances, the energy density F can be written in the form

$$F = \frac{2\pi}{\varkappa ad}\sum_l H_{vl}, \tag{2}$$

where H_{vl} is the field produced by the whole vortex system at the center of one of the vortices in the l-th row. The problem is thus reduced to evaluating H_{vl} and summing (2). Calculations given in the Appendix give

$$H_{vl} = \frac{\pi d}{2\varkappa a}(L+1) - \frac{2\pi}{\varkappa ad}\sum_{l'} x_l x_{l'} - \frac{\pi}{\varkappa a}\sum_{l'} |x_l - x_{l'}| - \frac{1}{\varkappa}\ln\frac{2\pi}{\varkappa a} \tag{3}$$
$$-\frac{1}{\varkappa}\sum_{l'}{}' \exp\left\{-\frac{2\pi}{a}|x_l - x_{l'}|\right\} + \frac{1}{\varkappa}\sum_{l'(\neq l)}{}'' \exp\left\{-\frac{2\pi}{a}|x_l - x_{l'}|\right\}$$

where the Σ' sign indicates summation over all rows which are odd in relation to the vortex $(x_l, 0)$, and the Σ'' sign summation over the even rows.

Substituting this result into (2) we find the free energy density F:

$$F = \frac{2\pi^2}{\varkappa^2 a^2 d}\left[\frac{d}{2}(L+1)^2 - \frac{2}{d}\sum_{ll'} x_l x_{l'} - \sum_{ll'} |x_l - x_{l'}| - \frac{a(L+1)}{\pi}\ln\frac{2\pi}{\varkappa a}\right.$$
$$\left. - \frac{a}{\pi}\sum_{l,l'}{}' \exp\left\{-\frac{2\pi}{a}|x_l - x_{l'}|\right\} + \frac{a}{\pi}\sum_{l\neq l'}{}'' \exp\left\{-\frac{2\pi}{a}|x_l - x_{l'}|\right\}\right]. \tag{2'}$$

The summation in (2′) is over all values of l and l' from $-L/2$ to $L/2$.

In the state of thermodynamic equilibrium the Gibbs free energy reaches a minimum; its density is

$$G = F - 2BH_0, \tag{4}$$

where $B = \overline{H}$ is the average magnetic field in the film. We obtain the magnitude of B by using the result of the evaluation of B in[1]:

$$B = \frac{2H_0}{d}\,\mathrm{th}\,\frac{d}{2} + \frac{2\pi}{\varkappa ad}\sum_l\left(1 - \frac{\mathrm{ch}\, x_l}{\mathrm{ch}(d/2)}\right). \tag{5}$$

Substituting (2′) and (5) into (4) and solving the equation

$$\partial G / \partial x_l = 0,$$

we find the equilibrium values of x_l for fixed a:

$$x_l = bl + \frac{b}{d}\sum_{l'} x_{l'},$$

where

$$b = 2\pi / \varkappa a H_0. \tag{6}$$

We get at once the solution of the equation for the x_l by summing it over l:

$$x_l = bl. \tag{7}$$

We have dropped here a vanishingly small contribution given by the last two terms in (2′).

We found thus that the equilibrium vortex distribution in the film corresponds to a uniform distribution of the vortex rows, i.e., the distance between adjacent rows is equal to b given by (6) and independent of l.

The next step in the study of the structure of the mixed state of a film will be the determination of the parameter a, the distance between neighboring vortices in one row (along the y-axis). To do this we substitute the equilibrium value (7) of x_l into (2′) and (5) and find G as function of a.

We first make one preliminary remark. We have just found that the vortex rows are separated from one another by a distance b. On the other hand, it follows from physical considerations that the boundary rows with $x_l = \pm d/2$ are completely cancelled by their own image[5] and should therefore be assumed to contribute nothing to G. It is thus natural to assume that the row with index $l = L/2$ has the coordinate $x_{L/2} = d/2 - b$, i.e., the boundary rows lie at a distance b from the film boundaries; it thus follows from (7) that

$$L + 1 = d/b - 1. \tag{8}$$

Substituting now (7) and (6) into (2′) and (5) and using (8) we find

$$F(a) = \mathrm{const} - \frac{\pi^2}{3\varkappa^2 a^2} + \frac{H_0}{\varkappa}\ln a - \frac{2H_0}{\varkappa}\exp\left\{-\frac{4\pi^2}{\varkappa a^2 H_0}\right\}. \tag{9}$$

The constant term here includes all terms independent of a. Moreover, we neglected here terms of order $(\kappa^2 ad)^{-1}$. Since (as we shall show) $a \sim b$, we have from (6) $H_0 \sim (\kappa a^2)^{-1}$. Using this estimate we see easily that terms of order $(\kappa^2 ad)^{-1}$ are indeed smaller than the other ones by a factor d/a.

Substituting (7) into (5), summing and bearing in mind that $d \ll 1$, we find B:

$$B = H_0 - \pi^2 / 3\varkappa^2 a^2 H_0.$$

Substituting this expression for B and (9) into (4) and minimizing G with respect to a we find an equation for a:

$$a^2 = \frac{2\pi^2}{3\varkappa H_0} + \frac{16\pi^2}{\varkappa H_0}\exp\left\{-\frac{4\pi^2}{\varkappa H_0 a^2}\right\}.$$

Solving this by iteration, we find

$$a = 2.69 / \sqrt{\varkappa H_0}. \tag{10}$$

Substituting this result in (6) we find

$$b = 2.34 / \sqrt{\varkappa H_0}. \tag{11}$$

These are thus the equilibrium parameters of our vortex lattice. The mixed state of the film considered is thus a triangular vortex lattice which does not differ from the one which would occur in the bulk matter.[1)] Indeed, in our case $b/a = 0.87$ while in the case of an equilateral triangle this ratio is equal to $\sqrt{3/2} \approx 0.87$.

2. CRITICAL CURRENT OF THE FILM

All considerations of the preceding section referred to the case when there was no transport current.

Let there now be in the film a transport current in the positive y-direction which produces at the surfaces of the film a magnetic field H_I:

$$H(\pm d/2) = H_0 \mp H_I.$$

We find the equilibrium vortex distribution for given H_0 and H_I. In[1] we showed that when there is a transport current present the potential Φ reaches a minimum in the thermodynamic equilibrium state:

$$\Phi = G + U, \qquad U = \frac{8\pi H_I}{\varkappa a d^2}\sum_l \mathrm{sh}\, x_l.$$

1)It was shown in [9] that for films of arbitrary thickness in a strong magnetic field a regular triangular lattice is realized.

The expression for U is obtained by a simple generalization of the quantity U for a single row, given in[1], to the case of many rows. The equilibrium values of x_l are determined from the solution of the equation $\partial\Phi/\partial x_l = 0$:

$$x_l = bl + \Delta, \qquad \Delta = \frac{b}{d}\sum_l x_l - \frac{2H_I}{H_0 d}. \qquad (12)$$

It follows from (12) that when a transport current is present the equilibrium vortex distribution will be the vortex structure which is rigidly shifted as a whole. The magnitude of the shift Δ depends on the transport current and vanishes when $H_I = 0$.

We study how G changes when the whole lattice as a complete unit shifts by a distance $\Delta \ll b$. We therefore split off in (3) and (5) the terms that depend on x_l:

$$G = \text{const} - \frac{2\pi^2}{\varkappa^2 a^2 d}\sum_{ll'}|x_l - x_{l'}| - \frac{4\pi^2}{\varkappa^2 a^2 d^2}\sum_{ll'} x_l x_{l'} + \frac{4\pi H_0}{\varkappa a d}\sum_l \operatorname{ch} x_l, \qquad (13)$$

where the constant term contains the terms which are independent of x_l. The last two terms in (3), which are unimportant for the present calculation, have been omitted. Since the whole lattice moves under the action of a transport current as a complete unit, this means that each vortex undergoes the same displacement Δ. We obtain the change in G under the displacement by substituting in (13) $x_l = x_l^0 + \Delta$ where $x_l^0 = bl$ is the position of the rows when there is no transport current.

One sees easily that

$$\sum_{ll'} x_l x_{l'} = \sum_{ll'} (x_l^0 + \Delta)(x_{l'}^0 + \Delta) = \Delta^2 (L+1)^2. \qquad (14)$$

Moreover, we have

$$\sum_l \operatorname{ch} x_l = \sum_l \operatorname{ch} x_l^0 + \frac{1}{2}\Delta^2 \sum_l \operatorname{ch} x_l^0.$$

Substituting here $x_l^0 = bl$ and summing, using (8), we get

$$\sum_l \operatorname{ch} bl = \frac{d}{2}\left(\frac{2}{b} + \frac{b}{6}\right) - 1. \qquad (15)$$

Using (14) and (15) we have finally

$$G = \text{const} + \Delta^2 2\pi H_0 / \varkappa a d. \qquad (16)$$

It follows from this equation that the vortex system in a film in an external field H_0 is in a potential well, i.e., pinning takes place.

The pinning-force (or restoring-force) density is easily determined:

$$f_p = -\frac{\partial G}{\partial \Delta} = -\frac{4\pi H_0}{\varkappa a d}\Delta. \qquad (17)$$

In the equilibrium state this force is balanced by the Lorentz force caused by the transport current. The Lorentz force acting on one line is according to[1] equal to $8\pi H_I/\kappa d$. One sees easily that the Lorentz force density will then be

$$f_L = \frac{8\pi H_I}{\varkappa a d^2}(L+1)$$

or, using (8), (10), and (11)

$$f_L = 4 H_I H_0 / d. \qquad (18)$$

Equating (17) and (18) we get a connection between the transport current H_I and the equilibrium displacement Δ of the vortex structure:

$$H_I = \pi\Delta / \varkappa a. \qquad (19)$$

It is now clear that when Δ reaches the magnitude b then at least in this case the lattice really ceases to be stable. Indeed, a shift of the lattice by b would change it into the earlier, unshifted state (b is the translation period of the lattice), but the transport current continues to run and again causes a shift of the lattice, again by b, and so on. The vortex structure will flow and energy will be dissipated. If then $H_0 \parallel Oz$ and the transport current flows in the y-direction, the vortex rows will be generated at the surface $x = -d/2$ and be annihilated at the surface $x = d/2$.

Without claiming any accuracy for the numerical constant, it is then natural to introduce the following definition of the critical current: a transport current is critical when it leads to an equilibrium displacement of the whole vortex lattice by an amount $\Delta = b/2$. Substituting $\Delta = b/2$ into (19) and using (10) and (11) we find finally

$$H_{I_c} = 1.37\varkappa^{-1}.$$

Changing to absolute Gaussian units, we get

$$H_{I_c} = \frac{1.94 H_{cm}}{\varkappa} = \frac{2\pi}{c} j_c d,$$

where j_c is the critical current density. Hence we get

$$j_c = 0.31 c H_{cm} / \varkappa d. \qquad (20)$$

3. STRUCTURE OF THE MIXED STATE OF A SUPERCONDUCTING PLATE

So far we have considered a film ($d \ll 1$). We now turn to the consideration of a large perfectly uniform film with $\kappa \gg 1$ and $d \gg 1$ which in what follows we shall call a plate. We study first the state that arises in that plate when the external field H_0 is applied parallel to its surface and $H_{c1} \ll H_0 \ll H_{c2}$.

Since $d \gg 1$ the mixed state in the middle of the plate will clearly be the same as in a bulk superconductor.[4] On the other hand, the mixed state at its boundary must be the same as on the boundary of a semi-infinite superconductor. To simplify the calculations for the time being we digress from our plate and consider a semi-infinite superconductor.

Let the boundary of the superconductor coincide with the $x = 0$ plane and let the region occupied by the superconductor correspond to $x > 0$, and let the external magnetic field H_0 be parallel to the z-axis. If $H_0 > H_{c1}$ the mixed state will occur in the superconductor. The vortex lines form a triangular lattice. As in the case of the film, we assume that the vortices are arranged in rows, that the l-th row lies in the plane $x = x_l$, that all vortices in a row are parallel to the external field and lie at a distance a from one another. We do not assume that the distances between the rows are the same.

First of all we find the free energy of our vortex system. It is convenient to use for the semi-infinite superconductor the method of images[5], i.e., we consider an infinite superconductor but such that for each vortex with coordinates (x, y) there exists its "image," a vortex with the opposite sense with coordinates (−x, −y).

We consider some vortex in the l-th vortex row. The field produced in its center by some other (l'-th) row is equal to (see Appendix (A.11)) $(\pi/\kappa a)\exp\{-x_{ll'}\}$ where $x_{ll'}$ is the distance between the rows l and l'. Using (1) we find easily the free energy of a strip of unit width along the y-axis and unit height along the z-axis:

$$\mathcal{F} = \frac{2\pi^2}{\varkappa^2 a^2}\sum_{ll'}\left(\exp\{-|x_l - x_{l'}|\} - \exp\{-(x_l + x_{l'})\}\right) + \text{const.} \tag{21}$$

The constant term includes here all terms independent of x_l and $x_{l'}$. In the thermodynamic-equilibrium state the Gibbs free energy will be a minimum:

$$\mathcal{G} = \mathcal{F} - 2H_0 \int H\, dS. \tag{22}$$

The integral in this formula is taken over that part of the x, y-plane which is restricted by the inequalities $x \geq 0$, $0 \geq y \leq 1$.

The second term in (22) we denote by W:

$$W = 2H_0 \int H\, dS.$$

According to the formula (A.17) obtained in the Appendix, we can write the expression for W in the form

$$W = \text{const} - \frac{4\pi H_0}{\varkappa a}\sum_l e^{-x_l}. \tag{23}$$

Again, the constant term contains the terms independent of x_l. Substituting (21) and (23) into (22) we find the condition for equilibrium of the vortex lattice:

$$\partial\mathcal{G} / \partial x_l = 0.$$

After simple transformations the condition for equilibrium takes the form

$$\frac{\pi}{\varkappa a}\left[e^{x_l}\sum_{\mu > l} e^{-x_\mu} - e^{-x_l}\sum_{\mu < l} e^{x_\mu} + e^{-x_l}\sum_{\mu} e^{-x_\mu}\right] = H_0 e^{-x_l}. \tag{24}$$

Changing from a sum to an integral, i.e., taking x to be a continuous function of a continuous parameter l and twice differentiating (24) with respect to l we get

$$d^2 x_l / dl^2 = 0,$$

i.e.,

$$x_l = bl + \bar{x}, \tag{25}$$

where b is the distance between neighboring rows for $x_l \gg 1$, and $\bar{x}$ an integration constant which just now is unimportant for us. In the case of a triangular lattice $b = a\sqrt{3/2}$. Therefore, the presence of a boundary in the semi-infinite superconductor does not affect the structure of the mixed state, not even close to that boundary. There is here no vortex density gradient whatever. A change in H_0 changes a, b, and $\bar{x}$, i.e., it leads to a change in the vortex density and to a shift of the lattice as a complete unit along the x-axis.

We now turn to our plate. It follows from the considerations just given that the mixed state in the plate does not differ from the mixed state in an infinite superconductor and there is no vortex density gradient even at the boundaries of the plate.

We study the stability of the mixed state in the plate. We shift thereto the whole vortex lattice as a complete unit in the x-direction over a distance $\Delta \ll b$ (the surfaces of the plate coincide with the planes $x = \pm d/2$).

We can easily find the free energy $\mathcal{G}$ of a segment of the plate ($0 \leq y \leq 1$, $0 \leq z \leq 1$, $-d/2 \leq x \leq d/2$) using Eqs. (21), (22), and (23) and the condition $a \ll 1$ and $d \gg 1$:

$$\mathcal{G} = -\frac{16\pi^2}{3\varkappa^2 a^4}\operatorname{ch} 2\Delta + \frac{16\pi}{\sqrt{3}\varkappa a^2} H_0 \operatorname{ch}\Delta + \text{const}, \tag{26}$$

where the constant terms contains the terms independent of Δ.

We now check easily that $\partial^2\mathcal{G}/\partial\Delta^2|_{\Delta=0} > 0$, i.e., the mixed state is stable.

4. THE CRITICAL CURRENT OF A PLATE IN THE MIXED STATE

We now consider the case when a transport current flows in the plate in the y-direction. This means that the field at the surfaces of the plate is given as

$$H(\pm d/2) = H_0 \mp H_I,$$

where H_I is the field produced by the transport current at the surfaces of the plate. It follows at once from the results of the preceding section that also in this case there will be no vortex density gradient. There will simply occur an equilibrium shift of the whole vortex structure by an amount Δ in the direction of the Lorentz force.

The restoring force, or the pinning force, is easily determined from (26), if we bear in mind that $\Delta \ll 1$:

$$f_p = -\frac{\partial\mathcal{G}}{\partial\Delta} = \left(\frac{64\pi^2}{3\varkappa^2 a^4} - \frac{16\pi H_0}{\sqrt{3}\varkappa a^2}\right)\Delta. \tag{27}$$

This force must be balanced by the Lorentz force acting upon the vortices due to the transport current.

By virtue of the linearity of the problem in the range of fields under consideration we find at once the transport current distribution from the London equation:

$$j_T = H_I \operatorname{ch} x / \operatorname{sh}(d/2).$$

The Lorentz force acting on one vortex in the l-th row is equal to $4\pi\kappa^{-1} j_T(x_l)$ and the total Lorentz force acting on all vortices of the plate in a unit strip along the y-axis is thus equal to

$$f_L = \frac{4\pi}{\varkappa a}\frac{H_I}{\operatorname{sh}(d/2)}\sum_l \operatorname{ch} x_l.$$

Substituting here $x_l = bl$, summing and using the fact that $b = \sqrt{3}\, a/2$, we get

$$f_L = 16\pi H_I / \sqrt{3}\varkappa a^2. \tag{28}$$

Equating (27) and (28) we find the equilibrium displacement Δ of the vortex lattice for a given H_I:

$$\Delta = H_I/(H_0 - B), \quad B = 4\pi/\sqrt{3}\varkappa a^2. \tag{29}$$

Here B is the average magnetic field (induction) in the interior of the plate.

As in Sec. 2 we define the critical current as the one which leads to a displacement $\Delta = b/2$. Then

$$H_{I_c} = (H_0 - B)a\sqrt{3}/4.$$

We now change to absolute Gaussian units:

$$H_{I_c} = -3^{1/4}\sqrt{2}\pi\frac{M}{\delta_0}\left(\frac{\Phi_0}{B}\right)^{1/2}, \qquad M = \frac{B - H_0}{4\pi}.$$

Here M is the reversible diamagnetic moment of a bulk superconductor in the mixed state, $\Phi_0 = hc/2e$ is a flux quantum.

The average critical current density is determined from the formula $\bar{j}_c = cH_{I_c}/2\pi d$,

$$\bar{j}_c = -\frac{3^{1/4}}{2^{1/2}}\frac{c\sqrt{\Phi_0}}{\delta_0}\frac{M(H_0)}{d\sqrt{B}}. \quad (30)$$

A formula very similar to this one was obtained earlier by Campbell, Evetts, and Dew-Hughes.[6]

5. DISCUSSION OF THE RESULTS

We discuss first the result obtained for a film.

1. The critical current density (20) of a film is caused by the pinning of the vortex structure at the surfaces of the film. The physics of this process is completely clear. It follows from Eq. (17) that the restoring—or pinning—force is proportional to the external field H_0. This field produces in the films Meissner currents that have opposite directions in the two halves of the film (with $x > 0$ and $x < 0$). These currents interact with the vortex structure in such a way that only its symmetric arrangement in the film corresponds (in the absence of transport currents) to an equilibrium state. Any displacement of the whole structure in either the positive or the negative x-direction at once leads to the appearance of a resultant interaction force between the Meissner currents and the vortex structure, i.e., essentially to the same Lorentz force which tends to restore the whole structure to the symmetric position. The fact that the critical current density depends on the film thickness as $j_c \sim d^{-1}$ expresses the same fact: pinning of the vortex structure at the film surfaces. This is in agreement with the well-known experimental fact that the critical current density determined by pinning is proportional to the area of the dividing boundary between the superconductor and the non-superconductor, per unit volume of the superconductor. Indeed, the quantity d^{-1} is directly proportional to the surface area of the film per unit volume of the film.

Our result is in this respect essentially different from the result of the GL theory where the critical current of a film, caused by the mechanism of disrupting Cooper pairs, was evaluated. According to the GL theory $j_c(GL) \sim cH_{cm}/\delta_0$ is independent of d. This result is valid for films with $d \ll (\delta_0, \xi(T))$ as in that case the dimensions of the film are too small for the formation of vortices and the only mechanism leading to an instability of the superconducting state when there is a transport current flowing is the breaking up of Cooper pairs. In our case, however, $d > \xi(T)$ and another mechanism for instability arises: vortex instability. The critical current density j_c evaluated by us for this case of instability turns therefore out to be less than $j_c(GL)$:

$$j_c / j_c(GL) \sim \xi(T) / d,$$

since $\kappa = \delta_0(T)/\xi(T)$. It follows from this that for thick films $(d > \xi(T))$ in the range of external fields $H_{c1}(d) \ll H_0 \ll H_{c2}$ the destruction of superconductivity by a current will occur through the appearance of vortex instability.

It is relevant to note here that Eq. (20) near T_c gives a temperature-dependence of the critical current of the form $j_c \sim T_c - T$, while according to the GL theory $j_c(GL) \sim (T_c - T)^{3/2}$.

2. We now discuss the dependence of the critical current on the external magnetic field H_0. According to (20) j_c is independent of H_0. This result, which seems at first sight to be paradoxical, has an explanation. According to (17) the pinning of the vortices in the film increases in proportion to the field H_0 but the Lorentz force density (18) acting upon the vortices also increases in proportion to H_0. This occurs because the Lorentz force acting on one vortex is independent of H_0 but the vortex density turns out to be proportional to H_0. As a result the critical current turns out to be independent of H_0.

What can we now say altogether about the H_0-dependence of the critical current? For small fields ($H_0 < H_{c1}(d)$) the critical current decreases linearly.[1] If the film surface is not perfect, one must expect even near the field $H_0 \sim H_{c1}(d)$ the appearance of the mixed state in the film. An increase of j_c with increasing H_0 must then start at $H_0 \sim H_{c1}(d)$.[1] Furthermore, in a wide range of fields, $H_{c1}(d) \ll H_0 \ll H_{c2}$, there must occur a plateau in the function $j_c(H_0)$. Finally, when $H_0 \sim H_{c2}$ the critical current must fall steeply since here only a surface current remains which vanishes at H_{c3}. We did not consider that range of fields.

3. We estimate the order of magnitude of the critical current obtained from Eq. (20). If we take $H_{cm} \sim 10^3$ Oe, $d \sim 10^{-5}$ cm, and $\kappa \sim 100$, we get $j_c \sim 3 \times 10^6$ A/cm^2.

4. We note finally what the relation is between the results obtained and real heterogeneous type II superconductors, i.e., hard superconductors.

Nowadays there is widespread use of alloys which in the high-temperature range form a uniform solid solution. They retain this uniform state (albeit metastably) under fast cooling (quenching). At low temperatures such a metastable solid solution turns out to be superconducting. When manufacturing wires or strips from such a solid solution (by drawing or rolling) the grains of the metal are stretched into fibers and the wire or strip acquires a fiber-filament microstructure. If one submits a wire or strip manufactured in this way to annealing by aging, the uniform metastable solid solution starts to disintegrate with a precipitation of lumps of a finely dispersed non-superconducting phase. Under certain conditions of annealing such a disintegration occurs initially mainly at the boundaries of the fibers. As a result the fibers remain separated from one another by normal phase particles.

The film considered in the present paper is an idealized case of one such fiber. It is inessential that the fiber is in contact with normal metal while the film considered is in contact with the vacuum: the influence of the normal metal on the superconductor takes place over distances $\sim\xi(T)$ into the interior of the superconductor, and we have assumed that $d \gg \xi(T)$. We must still note that for many superconducting alloys there is a well-pronounced plateau in the function $j_c(H_0)$ or even an increase in the critical current with magnetic field.

It is necessary to note in that connection that in[7], where the critical current of rolled specimens of the eutectic Nb-Th alloy was studied such a plateau was observed in a large range of magnetic fields.

We now turn to a discussion of the result for a bulk plate. Here again the interaction of the vortices with the surface of the plate leads to the stability of the mixed state with respect to a transverse transport current. However, here the critical current depends on the external field H_0.

We can apply the result obtained to estimate the critical current in eutectic alloys when the superconducting phase is split in the form of plates of thickness $d > \delta_0$ or, the other way round, it is separated by non-superconducting parts, the distance between which $d > \delta_0$. In that case we must replace $d^{-1} \to S_{V\perp} = S_V/3$, where $S_{V\perp}$ is the average area of the surface separating the superconducting from the normal phase per unit volume and situated at right angles to the action of the Lorentz force; S_V is the total area of the phase separation boundary per unit volume of the alloy. The critical current density in the eutectic alloy will then be

$$j_c \approx 0.3c\sqrt{\Phi_0}S_v M(H_0) / \delta_0\sqrt{B}.$$

Apart from the coefficient this is also the formula given by Campbell, Evetts, and Dew-Hughes.[6] For the Pb-Bi alloy $\delta_0 = 1.5 \times 10^{-5}$ cm[6] so that

$$j_c = 9cS_v M / \sqrt{B}.$$

This result agrees with the empirical formula proposed and thoroughly checked in[6]

$$j_c = 3.3cS_v M / \sqrt{B}.$$

We have thus constructed a theory for the critical current in uniform films and plates in the mixed state. The pinning of the vortices occurs as the result of a purely electrodynamic interaction of the vortices with the surface of the superconductor. It is clear that if the microstructure of a bulk heterogeneous superconductor is such that thin superconducting films or plates occur in it separated by layers of the non-superconducting phase this theory can also be applied to such heterogeneous superconductors.

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APPENDIX

1. Energy of the Vortices in a Superconductor

We prove in this section of the Appendix some very general statements. We consider a type II superconductor with $\kappa \gg 1$ in the form of an arbitrary cylinder in an external field H_0 parallel to the generatrix of the cylinder. Let there be in the cylinder a system of vortex lines with an arbitrary configuration. The distances between them are much larger than κ^{-1}. We can write the free energy $\mathcal{F}$ in the form

$$\mathcal{F} = \int_{V_s} (\mathbf{H}^2 + (\operatorname{rot}\mathbf{H})^2)\, dV_s, \tag{A.1}$$

where V is the volume of the superconductor. One can check that

$$\mathcal{F} = \mathcal{F}_0 + \frac{2\pi}{\kappa}\int_{\mathcal{L}} \mathbf{H}_v d\mathbf{l}, \tag{A.2}$$

where $\mathcal{F}_0$ is the value $\mathcal{F}$ would have if all vortices would be removed from superconductor, i.e., the energy purely connected with the Meissner currents. The integral in (A.2) is taken along the cores of all vortices and $\mathbf{H}_V$ is the field produced by the vortex system alone.

We now prove this. Let the cores of all vortices be given parametrically in space: $\mathbf{r}_0 = \mathbf{r}_0(t)$. The field inside the superconductor then satisfies the equation

$$\mathbf{H} + \operatorname{rot}\operatorname{rot}\mathbf{H} = 2\pi\kappa^{-1}\mathbf{e}(\mathbf{r}_0)\,\delta_2(\mathbf{r} - \mathbf{r}_0),$$

$\delta_2(\mathbf{r} - \mathbf{r}_0)$ is a two-dimensional delta-function defined in the plane through $\mathbf{r}_0$ at right angles to $\mathbf{e}(\mathbf{r}_0)$, where $\mathbf{e}(\mathbf{r}_0)$ is a unit vector directed along the vortex at $\mathbf{r}_0$. We write the solution for $\mathbf{H}$ in the form $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_V$, where $\mathbf{H}_1$ is the field produced in the superconductor by the external field alone, i.e., the field connected only with the Meissner currents; $\mathbf{H}_V$ is the field produced by the vortex system alone. These fields satisfy clearly the following formulae

$$\mathbf{H}_1 + \operatorname{rot}\operatorname{rot}\mathbf{H}_1 = 0, \quad \mathbf{H}_1|_S = \mathbf{H}_0, \tag{A.3}$$

$$\mathbf{H}_v + \operatorname{rot}\operatorname{rot}\mathbf{H}_v = 2\pi\kappa^{-1}\mathbf{e}(\mathbf{r}_0)\,\delta_2(\mathbf{r} - \mathbf{r}_0), \quad \mathbf{H}_v|_S = 0. \tag{A.4}$$

Substituting $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_V$ into (A.1) we have

$$\mathcal{F} = \int_{V_s} [\mathbf{H}_1^2 + (\operatorname{rot}\mathbf{H}_1)^2]\, dV_s + \int_{V_s} [\mathbf{H}_v^2 + (\operatorname{rot}\mathbf{H}_v)^2]\, dV_s + 2\int_{V_s} (\mathbf{H}_1\mathbf{H}_v + \operatorname{rot}\mathbf{H}_1 \operatorname{rot}\mathbf{H}_v)\, dV_s. \tag{A.5}$$

The last integral vanishes. Indeed,*

$$\int_{V_s} \operatorname{rot}\mathbf{H}_1 \operatorname{rot}\mathbf{H}_v\, dV_s = \int_{V_s} \mathbf{H}_v \operatorname{rot}\operatorname{rot}\mathbf{H}_1\, dV_s + \oint_S [\mathbf{H}_v, \operatorname{rot}\mathbf{H}_1]\, d\mathbf{S}.$$

The surface integral vanishes as $\mathbf{H}_V|_S = 0$, and hence, using (A.3), we have

$$\int_{V_s} (\mathbf{H}_1\mathbf{H}_v + \operatorname{rot}\mathbf{H}_1 \operatorname{rot}\mathbf{H}_v)\, dV_s = \int_{V_s} \mathbf{H}_v(\mathbf{H}_1 + \operatorname{rot}\operatorname{rot}\mathbf{H}_1)\, dV_s = 0.$$

Using (A.4) we can transform the second integral in (A.5) to the form

$$\int_{V_S} [\mathbf{H}_v^2 + (\operatorname{rot}\mathbf{H}_v)^2]\, dV_S = \int_{V_S} \mathbf{H}_v(\mathbf{H}_v + \operatorname{rot}\operatorname{rot}\mathbf{H}_v)\, dV_S = \frac{2\pi}{\kappa}\int_{\mathcal{L}} \mathbf{H}_v d\mathbf{l}.$$

If we now denote the first integral in (A.5) by $\mathcal{F}_0$ we obtain (A.2).

However, strictly speaking, we have obtained a meaningless result since the field in the center of a vortex is infinite in the London approximation. There are several methods of obtaining a valid result for the field in the center of a vortex[2,8]. Since the field in a superconductor changes appreciably over distances of the order of the penetration depth (~1) while the normal core of a vortex (a concept is completely foreign to the London approximation) has a size $\sim\kappa^{-1} \ll 1$, we must understand by the field at the center of a vortex the field which one obtains in the London approximation at a distance κ^{-1} from the center. In the particular case of an infinite superconductor Abrikosov[4] obtained Eq. (A.2) earlier.

2. Evaluation of the Field H_{Vl}

We consider one infinite vortex row in our film with $d \ll 1$ which lies in the plane $x = x_{l'}$. The vortices are

*$[\mathbf{H}_V, \operatorname{rot}\mathbf{H}_1] \equiv \mathbf{H}_V \times \operatorname{curl}\mathbf{H}_1$.

parallel to one another and to the z-axis. They lie at distances a along the y-axis from one another, i.e., the coordinates of the vortices are $(x_{l'}, ma)$ where m takes on integer values from $-\infty$ to $+\infty$. The vortex with m = 0 lies on the x-axis.

We find the field $h_v^{l'}(x, 0)$ produced by this row at the point (x, 0) if $x > x_{l'}$. The field produced by one vortex in the film was found in[1]. Using the solution found there, we have for $h_v^{l'}$

$$h_v^{l'}(x,0)=\frac{1}{\varkappa}\int_{-\infty}^{\infty}dk\sum_{m=-\infty}^{\infty}\frac{e^{-ikma}}{u}\frac{\operatorname{sh}[u(^1/_2d-x)]\operatorname{sh}[u(^1/_2d+x_{l'})]}{\operatorname{sh}ud},$$
$$u=(k^2+1)^{1/2},\quad x>x_{l'}.$$

Using the relation

$$\sum_{m=-\infty}^{\infty}e^{ikma}=2\pi\sum_{n=-\infty}^{\infty}\delta(ka-2\pi n),\tag{A.6}$$

we get after simple calculations the final expression

$$h_v^{l'}(x,0)=\frac{\pi d}{2\varkappa a}-\frac{2\pi}{\varkappa ad}xx_{l'}-\frac{\pi}{\varkappa a}|x-x_{l'}| \tag{A.7}$$
$$-\frac{1}{\varkappa}\ln\left(1-\exp\left\{-\frac{2\pi}{a}|x-x_{l'}|\right\}\right).$$

This formula is valid both for $x > x_{l'}$ and for $x < x_{l'}$. In deriving this formula we used the inequalities $a \ll 1$ and $d \gg a$.

To evaluate the field H_{vl} produced at the center of a vortex in the l-th row by the whole vortex system we must find the contribution of all other rows and add to that result the field produced by the vortex considered itself as well as by all other vortices in the same row. We obtain the latter by putting $l' = l$ in (A.7) and taking $x \to x_l$. If we put $x = x_l$ the last term in (A.7) diverges, as it gives the eigen-field of the vortex. As the core of the vortex has dimensions $\sim\kappa^{-1}$ we must put in the last term $|x - x_{l'}| = \kappa^{-1}$. Bearing in mind that the lattice is triangular we sum separately the contributions from all even and of all odd rows reckoned from the vortex considered. As a result we obtain finally Eq. (3).

3. Field of a Linear Chain of Vortices

In this section of the Appendix we find the field produced by a linear chain of vortices in an infinite superconductor which are parallel to one another.

We consider two cases.

1. Let the coordinates of the vortices be equal to $(0, \pm ma)$, where m = 0, 1, 2, ... We find the field H produced by such a chain at the point (x, 0). The boundary of the superconductor is removed to infinity.

The field H is determined by the equation[4]

$$H-\Delta H=\frac{2\pi}{\varkappa}\sum_m\delta(\mathbf{r}-\mathbf{r}_m).\tag{A.8}$$

Expanding the right-hand and the left-hand sides in two-fold Fourier integrals we easily find the **k**-th Fourier component $H_{\mathbf{k}}$:

$$H_{\mathbf{k}}=\frac{2\pi}{\varkappa}\sum_m\frac{e^{-ik_yma}}{1+k_x^2+k_y^2},$$

whence

$$H(x,0)=\frac{2\pi}{\varkappa}\sum_m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_x\,dk_y}{(2\pi)^2}\frac{\exp\{i(k_xx-k_yma)\}}{1+k_x^2+k_y^2}.$$

Summing over m and using Eq. (A.6) we find

$$H(x,0)=\frac{1}{\varkappa a}\sum_n\int_{-\infty}^{\infty}\frac{dk_x\exp\{ik_xx\}}{1+k_x^2+(2\pi n/a)^2}.$$

Evaluating the integral we get

$$H(x,0)=\frac{\pi}{\varkappa a}\sum_n\frac{\exp\{-x\sqrt{1+(2\pi n/a)^2}\}}{[1+(2\pi n/a)^2]^{1/2}}.$$

As, by assumption, $a \ll 1$, we have

$$H(x,0)=\frac{\pi}{\varkappa a}e^{-x}+\frac{2\pi}{\varkappa a}\sum_{n=1}^{\infty}\exp\left\{-\frac{2\pi nx}{a}\right\}\Big/\left(\frac{2\pi n}{a}\right).$$

In the last equation we split off the term with n = 0. Evaluating the sum in the last term we find finally[2)]

$$H=\frac{\pi}{\varkappa a}e^{-x}-\frac{1}{\varkappa}\ln(1-e^{-2\pi x/a}).\tag{A.9}$$

2. In the second case we find the field H produced at the point (x, 0) by a chain of vortices with coordinates $(0, \pm(2m + 1)a/2)$, m = 0, 1, 2, ... The required field is, clearly, equal to the difference between the field produced by a chain of vortices at the points $(0, \pm ma/2)$ and the field of vortices at the points $(0, \pm ma)$. Both fields follow easily from Eq. (A.9). As a result we find

$$H(x,0)=\frac{\pi}{\varkappa a}e^{-x}-\frac{1}{\varkappa}\ln(1+e^{-2\pi x/a}).\tag{A.10}$$

To evaluate the field at the center of a given vortex produced by the other rows of vortices we must bear in mind that $x > a$ so that the logarithmic terms in (A.9) and (A.10) are small compared to the exponential term. Dropping them we get

$$H=\pi e^{-x}/\varkappa a\tag{A.11}$$

for both positions of the chain of vortices with respect to the point of observation.

4. Magnetic Flux of One Vortex

We find the magnetic flux of one vortex in a superconducting cylinder of arbitrary (not necessarily circular) cross section. The vortex is parallel to the generatrix of the cylinder and situated at an arbitrary point $\mathbf{r}_0$ where $\mathbf{r}_0$ is a two-dimensional vector defined in a plane at right angles to the generatrix of the cylinder. If the vortex is far from the surface of the cylinder (as compared to δ_0) its magnetic flux is well known: it is equal to a flux quantum, i.e., $2\pi/\kappa$ (in our relative units). If the vortex lies on the surface of the superconductor its flux is equal to zero as the vortex in that case is annihilated by its own image. Our problem is to find the flux in the general case.

We consider a cylinder of unit height (along the generatrix) and denote its volume by V_S. We introduce a vector $\mathbf{\Phi}_V$ with an absolute value equal to the looked-for flux:

$$\mathbf{\Phi}_v=\int_{V_S}\mathbf{H}_v\,dV_S,$$

where $\mathbf{H}_V$ is the field produced by the vortex considered in the superconductor; $\mathbf{H}_V$ satisfies the equation

2)The idea of the derivation of this formula is due to A. I. Rusinov.

$$\mathbf{H}_v + \text{rot rot } \mathbf{H}_v = 2\pi\varkappa^{-1}\mathbf{e}\delta(\mathbf{r}-\mathbf{r}_0), \qquad (A.12)$$

where e is a unit vector in the direction of the vortex, and on the surface of the cylinder $H_V = 0$. Using the identity

$$\int \text{rot } \mathbf{a}\, dV = \oint [d\mathbf{S}\mathbf{a}]$$

and Eq. (A.12) we have

$$\Phi_v = \frac{2\pi}{\varkappa}\mathbf{e} - \oint [d\mathbf{S} \text{ rot } \mathbf{H}_v].$$

Multiplying this equation by e we get

$$\Phi_v = \frac{2\pi}{\varkappa} - \mathbf{e}\oint [d\mathbf{S} \text{ rot } \mathbf{H}_v]. \qquad (A.13)$$

We introduce into our considerations an auxiliary unit uniform magnetic field h parallel to the surface of the cylinder in the direction of e. It is clear that we then can rewrite (A.13) in the form

$$\Phi_v = \frac{2\pi}{\varkappa} - \oint \mathbf{h}[d\mathbf{S} \text{ rot } \mathbf{H}_v] = \frac{2\pi}{\varkappa} - \oint [\text{rot } \mathbf{H}_v \mathbf{h}]\, d\mathbf{S}. \qquad (A.14)$$

Inside the cylinder the field h will, however, no longer be a unit field and will satisfy the equation

$$\mathbf{h} + \text{rot rot } \mathbf{h} = 0, \quad h|_s = 1. \qquad (A.15)$$

Changing in (A.14) from a surface to a volume integral we get

$$\Phi_v = \frac{2\pi}{\varkappa} - \int_{V_s} \text{div}[\text{rot } \mathbf{H}_v \mathbf{h}]\, dV_s = \frac{2\pi}{\varkappa} - \int_{V_s} \mathbf{h} \text{ rot rot } \mathbf{H}_v\, dV_s + \int_{V_s} \text{rot } \mathbf{H}_v \text{ rot } \mathbf{h}\, dV_s.$$

Using (A.12) and (A.15) and changing back in the last integral to a surface integral and using the fact that at the surface $H_V = 0$ we get finally the following formula:[3)]

$$\Phi_v = 2\pi\varkappa^{-1}(1 - h(\mathbf{r}_0)). \qquad (A.16)$$

The factor $(1 - h(\mathbf{r}_0))$ gives the decrease in flux of the vortex caused by the nearness of the superconductor surface. Calculating the field h at the point $\mathbf{r}_0$ is, of course, much simpler than finding Φ_V by a direct calculation.

We now apply Eq. (A.16) to evaluate the quantity W in the case of a semi-infinite superconductor. From (22) we have

$$W = 2H_0 \int H\, dS.$$

The field h satisfies in this case the equation $h - h'' = 0$, $h(0) = 1$ so that the flux of the vortex at the point x_l will be according to (A.16)

$$\Phi_v(x_l) = 2\pi\varkappa^{-1}(1 - e^{-x_l}).$$

We get at once from this equation an expression for W:

$$W = \text{const} - \frac{4\pi H_0}{\varkappa a}\sum_l e^{-x_l}. \qquad (A.17)$$

The constant term contains the terms independent of x_l.

[3)] Equation (A.16) was derived by G. S. Mkrtchyan.

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