NONLINEAR WAVES IN UNCOMPENSATED ELECTRON BEAMS

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Submitted December 24, 1970

Zh. Eksp. Teor. Fiz. 61, 198–207 (July, 1971)

Nonlinear stationary waves propagating along the axis of a ring electron beam similar to the E-layer[7] is investigated. Dispersion relations for nonlinear periodic waves are obtained, the maximum amplitudes of the electric field in them are determined, and conditions for the existence of solitary waves are considered.

1. In stationary electron configurations, such as the Budker self-stabilized beam and the Veksler relativistic electron rings[2], a reduction in the Coulomb repulsion of the electrons is achieved with the aid of the Lorentz forces of the self-pinched relativistic currents. Thus, in a self-stabilized electron beam, the condition for equilibrium of the electrons can be written in the form[3,4]

$$-eE_r+\frac{e}{c}v_0H_\varphi+F^{ext}=0.$$

Here, v_0 is the longitudinal velocity of the electrons of the beam, F^{ext} is the force exerted on the electrons by the external fields, and E_r and H_φ are the electric and magnetic fields in the electron beam, defined by the equations

$$\frac{1}{r}\frac{d}{dr}(rE_r)=4\pi e(n_i-n_e),\qquad \frac{1}{r}\frac{d}{dr}(rH_\varphi)=-\frac{4\pi e}{c}v_0n_e.$$

It follows from these equations that the Coulomb force of repulsion of the electrons in the beam, $-eE_r$, is weakened by a factor of $\gamma_0^2=(1-v_0^2/c^2)^{-1}$, owing to the Lorentz force ev_0H_φ/c, and equilibrium is attained either as a result of partial compensation of the electron charge by ions of density $n_i=n_e/\gamma_0^2$, or with the aid of external fields. A similar reduction in the Coulomb repulsion of the electrons, owing to the intrinsic magnetic field of the electron current, occurs in the relativistic ring[2]. As is well known, the stationary electron configurations considered in[1,2] can very effectively be employed for the realization of a collective method of acceleration.

Also of interest is the collective method of acceleration in which the merits of the method of acceleration by electron rings and of acceleration by running waves of charge density are combined. According to[6], such a method of acceleration can be realized by exciting charge-density waves that propagate along the axis of the E-layer[7]. In that case, the wave causes the layer to split up into electron rings, the Coulomb repulsion in which is weakened by the orbital motion of the electrons, which should lead to a substantial increase in the highest possible intensity of the field of the wave and, consequently, to a considerable rise in the effectiveness of the acceleration.

In the present paper we investigate, in connection with the method of acceleration proposed in[6], nonlinear waves propagating along the axis of an electron beam, the particles of which revolve in the azimuthal direction. In such a beam the charge-density wave leads to oscillations in the particle current $j'_\varphi=-ev_0n'(z-v_{ph}t)$ (v_0 is the azimuthal velocity of the electrons and v_{ph}—the phase velocity of the wave) and, consequently, to the appearance of a magnetic field of the wave $H_r(z-v_{ph}t)$. The corresponding pinching force $F^H=ev_0H_r/c$ of the bunches into which the wave breaks up the beam is, as in the stationary case, out of phase with the Coulomb force $F^E=-eE_z$ and balances it when $v_0\approx\sqrt{c^2-v_{ph}^2}$. Such a compensation of the forces acting on the electrons should lead to a considerable increase in the largest possible amplitudes of the field, since the breaking of the wave front at sufficiently large values of the amplitude of the wave is usually connected with Coulomb repulsion of the electrons in the field of the wave (see[8,9]).

In the second section of the paper we investigate nonlinear periodic waves in an uncompensated electron beam, consider the dispersion relations for these waves and determine the maximum amplitudes of the field.

In the third section we show that solutions of the type of solitary waves, in which the energy of the field is localized within a distance of the order of a wavelength, are also possible in such a beam.

2. In the present paper we investigate, for simplicity, the propagation of waves in a rectangular geometry. As a stationary state we choose a plane uncompensated—with respect to charge—electron beam bounded by the x-axis and moving along the y-axis with velocity v_0. The self-magnetic field of the current $H_z(x)$ weakens the Coulomb repulsion of the electrons by a factor of γ_0^2 (see Sec. 1).

Considering the stationary wave propagating along the axis of the electron beam and assuming that the transverse gradients of the wave quantities f are negligibly small in comparison with the longitudinal:

$$\left|\frac{\partial\ln f}{\partial\xi}\right|\Big/\left|\frac{\partial\ln f}{\partial x}\right|\sim ka\gg1$$

($\xi=z-v_{ph}t$, k is the wave number and a is the transverse dimension of the beam), we obtain the following system of equations, which describes the propagation of the wave in the beam:

$$(v_z-v_{ph})\frac{dp_z}{d\xi}=e\frac{d\varphi}{d\xi}-\frac{e}{c}\left(v_y\frac{dA_y}{d\xi}+v_x\frac{dA_x}{d\xi}\right),\qquad(1)$$

$$(v_z-v_{ph})\frac{dp_y}{d\xi}=-\frac{e}{c}(v_z-v_{ph})\frac{dA_y}{d\xi}+\frac{e}{c}v_xH_z,\qquad(2)$$

$$(v_z-v_{ph})\frac{dp_x}{d\xi}=\frac{e}{c}(v_z-v_{ph})\frac{dA_x}{d\xi}-\frac{e}{c}(v_y-v_0)H_z,\qquad(3)$$

$$\frac{d}{d\xi}[n(v_z - v_{ph})] = 0, \tag{4}$$

$$d^2\varphi / d\xi^2 = 4\pi e(n - n_0), \tag{5}$$

$$\left(1 - \frac{v_{ph}^2}{c^2}\right)\frac{d^2A_y}{d\xi^2} = \frac{4\pi e}{c}(nv_y - n_0v_0), \tag{6}$$

$$\left(1 - \frac{v_{ph}^2}{c^2}\right)\frac{d^2A_x}{d\xi^2} = \frac{4\pi e}{c}nv_x. \tag{7}$$

In these equations $p = \mathscr{E}v/c^2$ is the momentum of the beam particles, $\mathscr{E} = \sqrt{m^2c^4 + c^2p^2}$ is their energy, and $\varphi(\xi)$ and $A(\xi)$ are the scalar and vector potentials of the wave. Linearizing Eqs. (1)–(7) with respect to the amplitude of the wave and substituting in them the wave quantities in the form $f \sim \exp[i(k\xi - \omega t)]$, we reduce them to the following dispersion equation:

$$\left(c^2k^2 - \omega^2 - \frac{\omega^2\omega_0^2}{\omega_H^2/\gamma_0^2 - \omega^2}\right)\left(c^2k^2 - \omega^2 + \frac{\omega_0^2k^2v_0^2}{\omega^2 - \omega_0^2} - \frac{\omega^2\omega_0^2}{\gamma_0^2(\omega_H^2/\gamma_0^2 - \omega^2)}\right) - \frac{\omega_0^4}{\gamma_0^4}\frac{\omega_H^2\omega^2}{(\omega_H^2/\gamma_0^2 - \omega^2)^2} = 0, \tag{8}$$

where $\omega_0 = (4\pi e^2n_0/m\gamma_0)^{1/2}$ is the Langmuir frequency of the beam, $\omega_H = eH_z/mc\gamma_0$ is the cyclotron frequency in the self-magnetic field and $\gamma_0 = \mathscr{E}_0/mc^2$. We investigate the case when the self-magnetic field "magnetizes" the transverse wave motion in the beam. The conditions for "magnetization" have the form

$$\omega_H \gg \omega\gamma_0, \quad \max(\omega, ck) \gg \omega_0^2/\omega_H.$$

When these conditions are met, Eq. (8) breaks up into two independent equations:

$$\omega^2 = c^2k^2 / (1 + \omega_0^2\gamma_0^2/\omega_H^2), \tag{9}$$

$$\omega^4 - \omega^2(\omega_0^2 + c^2k^2) - \omega_0^2k^2(v_0^2 - c^2) = 0. \tag{10}$$

The last of these equations describes two branches of the oscillations in an uncompensated beam—a fast wave with $v_{ph} > c$ and frequency $\omega > \omega_0$ and a slow wave whose phase velocity varies within the limits $0 < v_{ph} < c/\gamma_0$, and frequency $\omega \leq \omega_0/\gamma_0$ (see Fig. 1). Our analysis will, in the main, pertain to the slow branch.

Substituting the characteristic values $k \sim \omega_0/c$, $\omega \sim \omega_0/\gamma_0$ and $\omega_H = eH_z/mc\gamma_0 \approx \omega_0^2a/c$ in the "one-dimensionality" condition $ka \gg 1$ and in the "magnetization" condition, we find that these conditions are fulfilled for sufficiently "wide" beams for which

$$\delta = \omega_0 a / c \gg 1. \tag{11}$$

Allowance for the "finiteness" of the self-magnetic field of the beam for the wave under consideration becomes important for sufficiently long wavelengths $kc \ll \omega_0$, $\omega \ll \omega_0/\gamma_0$. When these conditions are fulfilled the dispersion equation (8) becomes simplified and takes the form

$$(\omega^2 - k^2c^2)\left(\omega^2 - \frac{k^2c^2}{\gamma_0^2} + \omega^2\frac{k^2v_0^2}{\omega_0^2}\right) - \frac{\omega_0^4}{\omega_H^2}\omega^2 = 0. \tag{12}$$

It follows from this equation that a dependence of the wave frequency ω on ω_H arises when $k \sim \omega_0^{3/2}/c\omega_H^{1/2}(1 - \beta_{ph}^2)^{1/2}$. For such values of k the dispersion equation for the wave under consideration may be written in the form

$$\omega^2 = \frac{k^2c^2}{\gamma_0^2}\left(1 - \frac{k^2v_0^2}{\omega_0^2}\right) - \frac{c^2}{\gamma_0^2}\frac{\omega_0^4}{\omega_H^2v_0^2}. \tag{13}$$

The maximum value of the phase velocity for the periodic wave is attained at $k = (\omega_0/v_0)(\omega_0/\omega_H)^{1/2}$ and is equal to $v_{ph}^{max} = (c/\gamma_0)\sqrt{1 - 2\omega_0/\omega_H}$. For small k, the phase velocity of the wave decreases and when $k \ll \omega_0^2/c\omega_H$ (i.e., when $\beta_{ph}\gamma_0 \ll 1$) the branch of the oscillations under consideration goes over into the helicons $\omega = k^2c^2\omega_H/\omega_0^2\gamma_0$[1)].

In this section we shall consider with the aid of Eqs. (1)–(7) nonlinear periodic waves under the assumption of "magnetization" of the transverse wave motion and of the one-dimensionality of the wave. The finiteness of the self-magnetic field and deviation from the one-dimensionality condition for such waves are taken into account in the following section of the paper.

We note that the nonlinear theory of the high-frequency longitudinal and transverse oscillations of a relativistic plasma in the absence of orderly motion of the electrons has been considered in[10,11]. In the present paper (see also[12]) we investigate nonlinear oscillations for which the decisive role is played by the ordered motion of the electrons which leads to the appearance of an additional focusing force and to an increase in the maximum intensity of the field of the wave. This focusing proves to be substantial in the case of $\beta_{ph}\gamma_0 \approx 1$ which we shall, in the main, consider. The case of $\beta_{ph}\gamma_0 \ll 1$ and of long wavelengths corresponds to low-frequency electromagnetic oscillations of the helicon type, the nonlinear theory of which was constructed in[13].

The equations of motion of the beam particles (1)–(3) have the energy integral

$$\mathscr{E} - v_{ph}p_z - v_0\left(p_y - \frac{e}{c}A_y\right) = \mathscr{E}_0 - v_0p_0 + e\varphi. \tag{14}$$

Here, $\mathscr{E}_0 = \sqrt{m^2c^4 + c^2p_0^2}$ is the value of the energy of the electrons at the point $p_y = p_0$, $p_z = 0$ and $\varphi = A_y = 0$. Under the condition of "magnetization" of the transverse wave motion in the beam, we have

$$v_x = p_x = 0, \quad v_y = v_0,$$

$$p_y = \frac{v_0\mathscr{E}}{c^2} = p_0\left[1 + \frac{e(\varphi - v_0A_y/c) + p_zv_{ph}}{\mathscr{E}_0 - p_0v_0}\right]. \tag{15}$$

In that case, we have from Eqs. (5)–(7) the following relations for the vector potential of the wave:

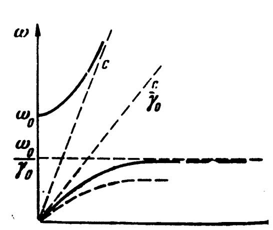

FIG. 1

[1)]Our analysis is applicable in the long wavelength region $k \lesssim \omega_0^2/c\omega_H$ only when an external magnetic field of intensity substantially exceeding the intensity of the self-magnetic field of the beam is present in the beam.

$$A_x = 0, \quad A_y = \frac{v_0}{c}\frac{\varphi}{1 - v_{ph}^2/c^2}. \tag{16}$$

Determining the longitudinal momentum of the particles from the energy integral

$$p_z = \frac{w\beta_{ph}\gamma_0 - \sqrt{w^2 - m^2c^4(1-\beta_{ph}^2\gamma_0^2)}}{c(1-\beta_\Phi^2\gamma_0^2)}, \tag{17}$$

where

$$w = mc^2 + e\varphi\frac{1-\beta_{ph}^2\gamma_0^2}{(1+\beta_{ph}^2)\gamma_0}, \quad \beta_{ph} = \frac{v_{ph}}{c},$$

and the density of the particles from the continuity equation

$$n = -n_0\frac{v_{ph}}{v_z - v_{ph}}, \quad v_z = \frac{p_z c^2}{\mathscr{E}}, \tag{18}$$

we obtain from (5) the following equation for the scalar potential of the wave $\varphi(\xi)$:

$$\frac{d^2\varphi}{d\xi^2} = \frac{4\pi e n_0}{1-\beta_{ph}^2\gamma_0^2}\left[\frac{w\beta_{ph}\gamma_0}{\sqrt{w^2 - m^2c^4(1-\beta_{ph}^2\gamma_0^2)}} - 1\right]. \tag{19}$$

The first integral of this equation can be obtained in the usual manner. It has the form

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{8\pi n_0}{1-\beta_{ph}^2\gamma_0^2}\left\{\frac{(1-\beta_{ph}^2)\beta_{ph}\gamma_0^2}{1-\beta_{ph}^2\gamma_0^2}\sqrt{w^2 - m^2c^4(1-\beta_{ph}^2\gamma_0^2)} - e\varphi - \mathscr{E}_0\frac{(1-\beta_{ph}^2)\beta_{ph}^2\gamma_0^2}{1-\beta_{ph}^2\gamma_0^2}\right\} + E_{max}^2. \tag{20}$$

In this equation E_{max} is the amplitude of the longitudinal electric field of the wave $E_z = -d\varphi/d\xi$, i.e., the value of E_z at the point $\varphi = 0$. Using (20), we can without difficulty express the solution of the equation for $\varphi(\xi)$ in terms of elliptic functions. To do that, it is convenient to change to the dimensionless variables

$$\eta = \frac{\omega_0}{c}\sqrt{\frac{1-\beta_{ph}^2\gamma_0^2}{1-\beta_{ph}^2}}\,\xi, \quad 2\vartheta = \arccos\frac{t + \sqrt{t^2-1} - \lambda\alpha}{\alpha\sqrt{\lambda^2-1}},$$

where t is related to the potential φ through the relation

$$t = \frac{1}{\sqrt{1-\beta_{ph}^2\gamma_0^2}}\left(1 + \frac{e\varphi}{\mathscr{E}_0}\frac{1-\beta_{ph}^2\gamma_0^2}{1-\beta_{ph}^2}\right),$$

$$\lambda = 1 + \frac{E_{max}}{8\pi n_0\mathscr{E}_0}\frac{1-\beta_{ph}^2\gamma_0^2}{1-\beta_{ph}^2}, \quad \alpha = \sqrt{\frac{1+\beta_{ph}\gamma_0}{1-\beta_{ph}\gamma_0}}.$$

Equation (19) may be written in terms of these variables in the following form (see [11]):

$$\eta = 2\beta_{ph}\gamma_0(\lambda + \sqrt{\lambda^2-1})^{1/2}E(\varkappa, \vartheta) + \frac{(1-\beta_{ph}\gamma_0)\sqrt{\lambda^2-1}}{(\lambda+\sqrt{\lambda^2-1})^{1/2}}\frac{\sin 2\vartheta}{\sqrt{1-\varkappa^2\sin^2\vartheta}} \tag{21}$$

Here, $E(\kappa, \vartheta)$ is an elliptic integral of the second kind,

$$\varkappa^2 = 2\sqrt{\lambda^2-1}\,/\,(\lambda + \sqrt{\lambda^2-1}).$$

The potential φ of the wave determined by the relation (21) is a periodic function of ξ. For the wave frequency $\omega = \pi v_{ph}/L$ (L is the period of the function $\varphi(\xi)$) we have from (21) the following equation:

$$\omega^2 = \frac{\omega_0^2}{\gamma_0^2}\frac{1-\beta_{ph}^2\gamma_0^2}{1-\beta_{ph}^2}\left[\frac{\pi}{2E(\varkappa)}\right]^2\frac{1}{\lambda+\sqrt{\lambda^2-1}}. \tag{22}$$

$E(\kappa)$ is a complete elliptic integral of the second kind.

For small amplitudes of the wave ($\lambda \to 1$, $\kappa \to 0$) this equation coincides, as is not difficult to see, with the dispersion equation of the linear theory (10). For larger amplitudes ($\lambda \to 1$) two waves—a fast ($v_{ph} > c$) and a slow wave for which $0 < v_{ph} < c/\gamma_0$—exist in the beam as before. The dispersion dependence for the slow wave for the largest possible amplitude of the field is shown in Fig. 1 by the dashed curve. The growth of the amplitude of the wave only leads to some decrease in its frequency for a given k.

The slow wave possesses substantial components of both the longitudinal $E^l = -d\varphi/d\xi$ and the transverse $E^t = \beta_{ph}dA_y/d\xi$ electric fields. For the coupling between these components of the field we have from (16) the following relation:

$$E^t \approx \frac{v_0}{c}\frac{\beta_{ph}}{1-\beta_{ph}^2}E^l. \tag{23}$$

The magnetic force, acting on the electrons in such a wave

$$F_z^H = \frac{e}{c}v_0H_x = -\frac{e}{c}v_0\frac{dA_y}{d\xi} = -e\frac{v_0^2}{c^2-v_{ph}^2}\frac{d\varphi}{d\xi},$$

balances the electric force $F_z^E = ed\varphi/d\xi$ when $v_{ph} \approx \sqrt{c^2 - v_0^2}$. Under these conditions, we must expect a substantial increase in the maximum amplitude of the field of the wave. Indeed, substituting in Eq. (20) the minimum of the possible values

$$w_{min} = mc^2\sqrt{1-\beta_{ph}^2\gamma_0^2},$$

which corresponds to

$$e\varphi_{min} = -\mathscr{E}_0\frac{1-\beta_{ph}^2}{1-\beta_{ph}^2\gamma_0^2}(1-\sqrt{1-\beta_{ph}^2\gamma_0^2}),$$

we obtain the following relation for the maximum value of the amplitude of the longitudinal electric field in the slow wave, on exceeding which a breaking of the wave front occurs.

$$\frac{E_{max\,max}^2}{8\pi} = n_0\mathscr{E}_0(1-\beta_{ph}^2)\frac{1-\sqrt{1-\beta_{ph}^2\gamma_0^2}}{\sqrt{(1-\beta_{ph}^2\gamma_0^2)^3}} \quad (0 < \beta_{ph} < 1/\gamma_0). \tag{24}$$

When $\omega \approx \omega_0/\gamma_0$ ($\beta_{ph}\gamma_0 \ll 1$) the amplitude $E_{max\,max}$ is sufficiently small:

$$E_{max\,max}^2/8\pi \approx n_0 m v_{ph}^2\gamma_0^3 \ll n_0\mathscr{E}_0. \tag{25}$$

On approach of v_{ph} to c/γ_0, the amplitude of the field $E_{max\,max}$ significantly increases. The limitation of $E_{max\,max}$ in this case is connected with the fact that owing to the "one-dimensionality" and "magnetization" conditions on the wave, the "detuning" $c/\gamma_0 - v_{ph}$ cannot be made as small as we wish.

Thus, for example, setting

$$k_{min} \sim k_0 = \omega_0^{3/2}/c\omega_H^{1/2}(1-\beta_{ph}^2)^{1/2} \tag{25'}$$

and determining the corresponding "detuning" $\Delta v = c/\gamma_0 - v_{ph}$ from the dispersion equation (22) for $E_{max} = E_{max\,max}$:

$$\Delta v = \frac{2}{\pi^{4/3}}\frac{c}{\gamma_0}\left(\frac{kv_0}{\omega_0}\right)^{4/3} \text{ for } \Delta v \ll \frac{c}{\gamma_0},$$

we obtain from (24) the following condition on $E_{max\,max}$, on the fulfillment of which the analysis carried out in this section becomes applicable:

$$\frac{E_{max\,max}^2}{8\pi} \lesssim n_0\mathscr{E}_0\frac{\omega_H}{\omega_0}(1-\beta_{ph}^2), \tag{26}$$

i.e., the energy of the wave propagating in the uncompensated beam can considerably exceed the energy of the particles of the beam.

3. At small k ($kc \ll \omega_0$) allowance for the finiteness of the self-magnetic field of the beam becomes important for the slow branch of the oscillations. In this case we should retain the terms $\sim 1/H_z$ in the equations of motion of the beam particles in the directions perpendicular to the magnetic field (2) and (3). As a result, we obtain for the transverse components of the velocity the following relations:

$$v_y = v_0 - \frac{c}{eH_z}(v_z - v_{ph})\frac{d}{d\xi}\left(p_x - \frac{e}{c}A_x\right) \approx v_0 + \frac{v_z - v_{ph}}{H_z}\frac{dA_x}{d\xi} \quad (27)$$

$$v_x = \frac{c}{eH_z}(v_z - v_{ph})\frac{d}{d\xi}\left(p_y - \frac{e}{c}A_y\right) \approx -\frac{v_z - v_{ph}}{H_z}\frac{dA_y}{d\xi}. \quad (28)$$

We have neglected on the right hand sides of these relations addends $\sim dp_x/d\xi$ and $dp_y/d\xi$ which are small in respect of $k^2c^2/\omega_0^2 \ll 1$. Substituting (27) and (28) into the equations for the components of the vector potential of the wave A_x and A_y, we arrive at these equations

$$\frac{dA_x}{d\xi} = \frac{4\pi e n_0 \beta_{ph}}{H_z}\frac{A_y}{1-\beta_{ph}^2}, \quad (29)$$

$$(1-\beta_{ph}^2)\frac{d^2A_y}{d\xi^2} + \frac{16\pi^2 e^2 n_0^2 \beta_{ph}^2}{H_z^2(1-\beta_{ph}^2)}A_y = \frac{4\pi e}{c}v_0(n-n_0). \quad (30)$$

To obtain (29) we integrated with respect to ξ with the condition $dA_x/d\xi = 0$ at the point $A_y = 0$ and used (29) in writing down Eq. (30).

In the leading order with respect to the parameter $\sim 1/H_z$, the perturbation of the particle density in the wave is determined by the relations (17) and (18). In the following order in this small parameter we obtain with the aid of the continuity equation

$$n = n^{(0)} - \frac{n^{(0)}}{v_z^{(0)} - v_{ph}}\frac{m^2c^6}{\mathscr{E}^{(0)3}}(p_z - p_z^{(0)}). \quad (31)$$

Here, $p_z - p_z^{(0)}$ is the perturbation of the longitudinal momentum of the particles which may be found from the energy integral (14):

$$p_z - p_z^{(0)} = -c^2p_x^2 / \mathscr{E}^{(0)}(v_z^{(0)} - v_{ph}). \quad (31')$$

The index 0 in the formulas (31) and (31′) pertains to the zeroth approximation with respect to $1/H_z$. With the aid of (31), (31′) and (28), we obtain finally the following equation for $\varphi(\xi)$:

$$\frac{d^2\varphi}{d\xi^2} = -4\pi e(n-n_0) = 4\pi e n_0\Bigg\{\frac{1}{1-\beta_{ph}^2\gamma_0^2}\cdot\left(\frac{\beta_{ph}\gamma_0\bar{w}}{\sqrt{\bar{w}^2-1+\beta_{ph}^2\gamma_0^2}} - 1\right) + \frac{1}{\gamma_0^2H_z^2}\left(\frac{dA_y}{d\xi}\frac{1+\beta_{ph}^2\gamma_0^2}{\bar{w}-\beta_{ph}\gamma_0\sqrt{\bar{w}^2-1+\beta_{ph}^2\gamma_0^2}}\right)^2\Bigg\},$$

$$\bar{w} = 1 + \frac{e(\varphi - \beta_0 A_y)}{mc^2}\gamma_0, \quad \beta_0 = \frac{v_0}{c}. \quad (32)$$

The amplitude of the wave propagating in the beam is limited by the condition

$$\frac{\beta_0}{2\pi e n_0}\frac{d^2A_y}{d\xi^2} \leqslant 1$$

(see below).

In such amplitudes the addend $\sim 1/H_z^2$ on the right hand side of (32) is small compared with $d^2\varphi/d\xi^2$ in the ratio $c^2/a^2\omega_0^2$ and may be neglected. Then we have from (32) for $kc \ll \omega_0$ and $\beta_{ph}\gamma_0 \approx 1$,

$$\varphi \approx \beta_0 A_y + \frac{mc^2}{e\gamma_0}\left\{\left[1 + \frac{\beta_0}{2\pi e n_0}\frac{d^2A_y}{d\xi^2}\right]^{-1/2} - 1\right\}. \quad (33)$$

Substituting this relation into the right hand side of Eq. (30), we obtain the following equation of the fourth order for A_y:

$$\frac{(1-\beta_{ph}^2\gamma_0^2)}{\gamma_0^2}\frac{d^2A_y}{d\xi^2} + \frac{16\pi^2e^2n_0^2\beta_{ph}^2}{H_z^2(1-\beta_{ph}^2)}A_y = \frac{mc^2}{e\gamma_0}\beta_0\frac{d^2}{d\xi^2}\left[1 + \frac{\beta_0}{2\pi e n_0}\frac{d^2A_y}{d\xi^2}\right]^{-1/2}. \quad (34)$$

For $H_z \to \infty$ this equation is equivalent to Eq. (19) in the region $kc \ll \omega_0$. Allowance for the finiteness of the self-magnetic field in Eq. (34) becomes essential at $k \sim k_0^2$).

Besides periodic solutions, Eq. (34) also has the solitary-wave type of solutions for which $A_y \to 0$ as $\xi \to \pm\infty$. The possibility of an effective use of such waves for acceleration in a plasma was pointed out in[14]. However, solitary waves are possible in a plasma only in the case of low-frequency oscillations in which charges of both signs participate, for instance, ion-acoustic waves[15] and magnetic sound[16]. In an uncompensated electron beam, contraction of the electrons in a wave, due to the action of the magnetic force, leads to the possibility of the appearance of a solitary wave even in the absence of ions.

Linearizing Eq. (34) and substituting $A_y \sim \exp(ik\xi)$, we obtain the following equation for k^2, which coincides, as is not difficult to see, with the dispersion relation of the linear theory (12):

$$k^4 - \omega_0^2\frac{k^2}{c^2}\frac{1-\beta_{ph}^2\gamma_0^2}{1-\beta_{ph}^2} + \frac{\omega_0^6}{c^4\omega_H^2}\frac{1}{(1-\beta_{ph}^2)^2} = 0. \quad (35)$$

The existence for this equation of roots with $k_i = \operatorname{Im} k \neq 0$ indicates the possibility of construction of solutions with the asymptotic forms $A_y \sim \exp(-|k_i|\xi)$ for $\xi \to \infty$ and $A_y \sim \exp(|k_i|\xi)$ for $\xi \to -\infty$, which correspond to solitary waves. However, it is necessary for the construction of such a solution, which is continuous for all values of ξ, to turn to the nonlinear equation (34) or to (19) for $H_z \to \infty$. If $H_z \to \infty$, then $k_i \neq 0$ when $\beta_{ph}\gamma_0 > 1$, but it follows from Eq. (19) that it is impossible in this case to construct a solitary wave type of solution which is everywhere continuous up to the second derivative. Indeed, integrating Eq. (19) under the condition that $\varphi \to 0$, $d\varphi/d\xi \to 0$ as $\xi \to \pm\infty$, we obtain

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{8\pi n_0}{1-\beta_{ph}^2\gamma_0^2}\left[\frac{(1-\beta_{ph}^2)\beta_{ph}\gamma_0^2}{1-\beta_{ph}^2\gamma_0^2} \times \sqrt{w^2 - m^2c^4(1-\beta_{ph}^2\gamma_0^2)} - e\varphi\right] - 8\pi n_0\mathscr{E}_0\frac{(1-\beta_{ph}^2)\beta_{ph}^2\gamma_0^2}{(1-\beta_{ph}^2\gamma_0^2)^2}. \quad (36)$$

It follows from this relation that at the maximum point of the potential, $d\varphi/d\xi = 0$, only the value $\varphi = 0$ is

2)The wave propagating in the beam may here, as before, be assumed to be one-dimensional. Allowance for a "multi-dimensionality" of the wave $\partial A/\partial x \neq 0$ leads in Eq. (34) to negligibly small corrections:

$$\frac{1-\beta_{ph}^2\gamma_0^2}{(1-\beta_{ph}^2)\gamma_0^2}\frac{\partial^2 A_y}{\partial x^2} \approx \frac{c}{\omega_0 a}\frac{1-\beta_{ph}^2\gamma_0^2}{\gamma_0^2}\frac{\partial^2 A_y}{\partial\xi^2},$$

$$\frac{4\pi e n_0}{H_z(1-\beta_{ph}^2)}\frac{\partial}{\partial x}(\varphi - \beta_0 A_y) \approx \frac{c}{\omega_0 a}\frac{1}{\beta_0}\frac{1-\beta_{ph}^2\gamma_0^2}{\gamma_0^2}\frac{\partial^2 A_y}{\partial\xi^2}.$$

possible, i.e., a solitary wave is impossible in this case. The obtained result is connected with the fact that, besides the regions of contraction of the electrons, there should exist in a solitary wave regions in which the Coulomb repulsion predominates. Such a situation may occur at sufficiently small k when allowance for the finiteness of the self-magnetic field becomes essential. In that case Eq. (35) also has roots with $k_i \neq 0$ when

$$1 - 2\frac{\omega_0}{\omega_H} < \beta_{ph}^2\gamma_0^2 < 1. \quad (37)$$

In the present paper we confine ourselves to the analysis of the case

$$1 - \beta_{ph}^2\gamma_0^2 = 2\frac{\omega_0}{\omega_H}(1-\varepsilon), \quad \varepsilon \ll 1,$$

when $k_i \ll k_r$ and we may in constructing the solution of Eq. (34) limit ourselves to the small nonlinearity approximation. In this case we have from Eq. (35)

$$k_r = k_0(1 - \varepsilon/4), \quad k_i = k_0\sqrt{\varepsilon/2} \quad (38)$$

(the quantity k_0 is given by formula (25′)).

It is sufficient to limit ourselves in Eq. (34) to terms up to the cubic in the amplitude of the wave. As a result we obtain from (34) the following equation for $A_y(\xi)$:

$$\frac{c^2}{\omega_0^2}\frac{\beta_0^2}{\gamma_0^2}\frac{d^4A_y}{d\xi^4} + \frac{1-\beta_{ph}^2\gamma_0^2}{\gamma_0^2}\frac{d^2A_y}{d\xi^2} + \frac{16\pi^2e^2n_0^2\beta_{ph}^2}{H_z^2(1-\beta_{ph}^2)}A_y = -\frac{3}{2}\frac{e\beta_0^3c^2}{m\gamma_0^3\omega_0^4}\frac{d^2}{d\xi^2}\left(\frac{d^2A_y}{d\xi^2}\right)^2 + \frac{5}{2}\frac{e^2\beta_0^4c^2}{m^2\gamma_0^3\omega_0^6}\frac{d^2}{d\xi^2}\left(\frac{d^2A_y}{d\xi^2}\right)^3. \quad (39)$$

We seek the solution of this equation in the form

$$A_y(\xi) = a(\xi)\cos(k_r\xi + \theta(\xi)) + b(\xi)\cos[2(k_r\xi + \theta(\xi))], \quad (40)$$

where $a(\xi)$ and $\theta(\xi)$ are the slowly varying—with respect to ξ—amplitude and phase of the solution. For the amplitude of the second harmonic we have from (39)

$$b = -\frac{1}{3}\frac{ea^2}{mc^2\gamma_0}\frac{\omega_0}{\omega_H}\frac{1}{(1-\beta_{ph}^2)^{1/2}}. \quad (41)$$

Substituting the solution in the form (40) into Eq. (39) and collecting the terms $\backsim \sin(k_r\xi + \theta)$, and $\cos(k_r\xi + \theta)$, we obtain by the usual methods the following equations for the amplitude and phase:

$$\frac{d^2a}{d\xi^2} = \frac{k_0^2a}{2}\left[\varepsilon - \frac{1}{16}\left(\frac{ea}{mc^2\gamma_0}\right)^2\left(\frac{\omega_0}{\omega_H}\right)^2\frac{1}{1-\beta_{ph}^2}\right], \quad (42)$$
$$d\theta/d\xi = 0.$$

The first integral of the equation for $a(\xi)$, which satisfies the condition $da/d\xi \to 0$ as $a \to 0$, has the form

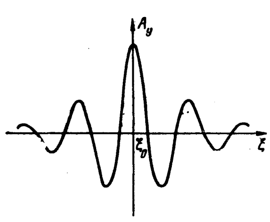

FIG. 2

$$\left(\frac{da}{d\xi}\right)^2 = \frac{k_0^2a^2}{2}\left[\varepsilon - \frac{1}{32}\left(\frac{ea}{mc^2\gamma_0}\right)^2\left(\frac{\omega_0}{\omega_H}\right)^2\frac{1}{1-\beta_{ph}^2}\right]. \quad (43)$$

The formulas (40) and (43) describe a solitary-wave type of solution, called in the theory of nonlinear waves a soliton envelope (an E-soliton)[17] in the case when the derivative $da/d\xi$ changes sign at some point $\xi = \xi_0$ (see Fig. 2). At this point the amplitude of the wave is a maximum: $a = a_{max}$. The obtained solution will be continuous at $\xi = \xi_0$ up to and including the fourth derivative if we set $\theta = 0$, $da/d\xi\,|_{\xi=\xi_0} = 0$. From the last condition we obtain for the wave under consideration, after making use of (43), a dispersion equation relating the maximum value of the amplitude of the wave to the phase velocity:

$$1 - \beta_{ph}^2\gamma_0^2 = 2\frac{\omega_0}{\omega_H}\left[1 - \frac{1}{32}\left(\frac{ea_{max}}{\mathscr{E}_0}\right)^2\left(\frac{\omega_0}{\omega_H}\right)^2\frac{1}{1-\beta_{ph}^2}\right]. \quad (44)$$

Characteristic values of the maximum amplitude of such a wave (for $\epsilon \sim 1$) are determined from the relation $ea_{max} \sim \mathscr{E}_0(\omega_H/\omega_0)(1-\beta_{ph}^2)^{1/2}$ (here, $d^2A/d\xi^2 \sim k_0^2a \sim 4\pi e n_0/\beta_0$). The corresponding value of the energy of the longitudinal electric field in the wave coincides, by order of magnitude, with (26) and substantially exceeds the energy of the beam:

$$\frac{E_{z\,max}^2}{8\pi} \sim n_0\mathscr{E}_0\frac{\omega_H}{\omega_0}(1-\beta_{ph}^2). \quad (45)$$

The increase in the maximum amplitude of the field is also connected with the compensation of the electric and magnetic forces when $\beta_{ph}\gamma_0 \approx 1$.

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Translated by A. K. Agyei
21