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BROADENING OF A HIGH AMPLITUDE MONOCHROMATIC CYCLOTRON WAVE IN A PLASMA

A. A. IVANOV and V. V. PARAIL

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The possibility of broadening of a high-amplitude, initially-monochromatic electron-cyclotron wave in a plasma is considered. It is shown that as a result of the decay of such a wave, low-frequency ion-acoustic and high-frequency waves (with a frequency $\omega_h \sim \omega_{He}$) are produced, which possesses shifted phases and a finite frequency range ($\Delta\omega \leq \omega_{pi}$, where ω_{pi} is the ion plasma frequency). As a result of reverse coalescence of the waves, cyclotron waves with shifted phases are produced in the same frequency range $\Delta\omega \lesssim \omega_{pi}$. Hence, after a certain time τ the spectrum of the initially monochromatic cyclotron wave may be broadened to such an extent that quasilinear equations can be employed for the solution of the self-consistent problem of plasma-electron heating by a cyclotron wave.

ONE of the main questions in the determination of the effectiveness of the heating of plasma electrons by a large-amplitude wave is that of the stochasticity of the heating. There are two points of view. Some assume that the phase of a monochromatic cyclotron wave is significantly altered during the time of its passage of the electron heating,[1] while others[2] assume that the cyclotron wave has a fixed phase and construct, on this basis, a theory of wave absorption at the resonance point ($\Omega \sim \omega_{He}$), assuming that the distribution function is specified. In both cases, however, the problem is solved in a non-selfconsistent manner—either the electron distribution function is specified and is used to construct the theory of wave absorption, or conversely, the amplitude of the cyclotron wave is assumed given and is used to determine the equilibrium distribution function. The self-consistent problem can be solved by using a quasilinear theory.[3] To this end, however, it is necessary that the packet of cyclotron waves have a sufficient width, namely the phase-velocity scatter $\Delta(\omega/k)$ should exceed the velocity of the captured particles relative to the wave. An electron having a velocity v_z close to the phase velocity of the wave is acted upon by a Lorentz force $e\mathbf{v}_\perp \times \mathbf{H}/c$, and consequently the particle oscillation velocity is $v_z \sim \sqrt{2ev_\perp \tilde{H}/mk_z c}$ ($\tilde{H}$ is the alternating magnetic field of the cyclotron wave).

Using the dispersion law for cyclotron waves

$$\omega = \omega_{He}(1 - \omega_{pe}^2/k^2c^2), \tag{1}$$

we obtain ultimately

$$\Delta\omega \gtrsim \omega_{He}\sqrt{\frac{2\tilde{H}v_\perp\omega_{pe}}{H_0 c\,\omega_{He}}}\sqrt{\frac{\omega_{He}-\omega}{\omega_{He}}}. \tag{2}$$

In typical experiments on cyclotron heating of a plasma $\omega_{pe} \sim \omega_{He} \sim 10^{10}$ sec^{-1}, $T_e \sim 10$ eV, $E \sim 0.1$–0.3 cgs esu, and $k \sim 3$ cm^{-1}, and to satisfy the condition for the stochasticity of the heating it is necessary that the cyclotron wave have a frequency spectrum satisfying the condition

$$\Delta\omega/\omega \sim 3\cdot 10^{-3}, \tag{3}$$

i.e., $\Delta\omega \lesssim \omega_{pi}$. This condition is frequently not satisfied in the experiments. Nevertheless, after the lapse of a certain time $\tau \sim 10^{-7}$, the processes in the plasma begin to obey the stochasticity condition.[4–6] This can be explained as follows. Assume that a monochromatic electron-cyclotron wave is incident on the plasma. It is known that such a wave, with sufficiently large amplitude, can decay into two potential waves—hybrid with frequency $\omega_h \sim \omega_{He}$ and low-frequency ion sound with frequency $\omega_s \ll \omega_{He}$. The first to investigate the decays of monochromatic waves were Sagdeev and Oraevskiĭ.[7]

As will be shown below, as a result of such a decay the hybrid and ion-acoustic waves have a nonzero frequency scatter, and oscillations with shifted phases are produced, to which ordinary weak-nonlinearity theory is applicable. The reverse process, the coalescence of these waves, produces a cyclotron wave having a shifted phase and a finite frequency scatter (the magnitude of this scatter will be estimated below). Thus, after a certain time the frequency spectrum of the cyclotron wave can broaden to such an extent, that the condition (3) is satisfied and further heating of the plasma proceeds in the manner predicted by the quasilinear theory.

Let us consider now the foregoing processes in greater detail. Assume that an electron cyclotron wave of sufficiently large amplitude, satisfying Eq. (1), propagates along the z axis in the plasma. Such a wave can decay into a sum of two waves—hybrid and ion-acoustic. The dispersion laws for these waves have respectively the form

$$\omega_h \approx \omega_{He}\left(1 - \frac{\omega_{pe}^2}{2|\omega_{pe}^2 - \omega_{He}^2|}\frac{k_{\perp h}^2}{k_{г}^2}\right), \quad \omega_s = k_s c_s, \tag{4}$$

where $k_{\perp h}^2 = k_{xh}^2 + k_{yh}^2$ and $c_s = \sqrt{T_e/M}$; in the derivation of (4) we took into account the inequality $k_{\perp h}/k_h \ll 1$.

We shall assume henceforth that the amplitude of the fields in the cyclotron wave is much larger than the amplitudes of the fields in the hybrid and in the ion-acoustic waves, and we shall therefore derive only equations for the amplitude changes of the latter two waves, subject to the condition $W_{k_c}^c = \text{const}$ ($W_{k_c}^c$ is the spectral energy density of the cyclotron wave).

To obtain the sought equations we must have expressions for the particle-density perturbation in the oscillations, up to terms of second order in the field amplitudes, inclusive. These expressions can be easily obtained by using the system of magnetohydrodynamics equations together with Maxwell's equations (we can retain in the nonlinear terms, from the very outset, only the terms containing the fields of the supplementary oscillations, for example $n_s^{(2)} \sim E^c_{k_c}\varphi^h_{k_h}$; for all the remaining terms, the decay conditions will not be satisfied):

$$n_{\mathbf{k}}^{(e)} = n_0 \frac{e}{mv_{Te}^2}\varphi_{\mathbf{k}_s}^s - \frac{e^2 n_0}{2m^2 v_{Te}^2 k_{zs}} \int \varphi_{\mathbf{k}_h}^h E_{x\mathbf{k}_c}^c (k_{yh} - ik_{xh}) \frac{k_{zh}\, d\lambda_1}{(\omega_{He} - \omega_c)\,\omega_h} \quad (5)$$

$$d\lambda_1 = d\mathbf{k}_h d\mathbf{k}_c d\omega_h d\omega_c \delta(\mathbf{k}_s - \mathbf{k}_h - \mathbf{k}_c)\,\delta(\omega_s - \omega_h - \omega_c) \quad (6)$$

(we took into account in the derivation of (5) that in the cyclotron wave $E_y^c = iE_x^c$);

$$n_{\mathbf{k}_h}^{(e)} = \frac{en_0\varphi_{\mathbf{k}_h}^h}{m\omega_h}\left(\frac{\omega_h k_{\perp h}^2}{\omega_{He}^2 - \omega_h^2} - \frac{k_{zh}^2}{\omega_h}\right) + \frac{e^2 n_0}{2m^2\omega_h}\int E_{x\mathbf{k}_c}^c \varphi_{\mathbf{k}_s}^s (k_{ys} - ik_{xs}) \cdot$$
$$\times\left[-\frac{k_{zh}}{\omega_h}\frac{\omega_s}{k_{zs}v_{Te}^2(\omega_{He} - \omega_c)} - \frac{k_{\perp h}^2}{(\omega_{He} - \omega_c)(\omega_{He}^2 - \omega_r^2)} - \frac{k_c\omega_s}{(\omega_{He} - \omega_h)(\omega_{He} - \omega_c)k_{zs}v_{Te}^2}\right] d\lambda_2, \quad (7)$$
$$d\lambda_2 = d\mathbf{k}_s d\mathbf{k}_c d\omega_s d\omega_c \delta(\mathbf{k}_s - \mathbf{k}_h - \mathbf{k}_c)\,\delta(\omega_s - \omega_h - \omega_c);$$
$$n_{\mathbf{k}_s}^{(i)} = n_0 \frac{ek_s^2}{M\omega_s^2}\varphi_{\mathbf{k}_s}^s. \quad (8)$$

The only nonvanishing term in (8) is that of first order of smallness, since the ions do not move in either the cyclotron or the hybrid wave.

The obtained expressions (5)–(8) must be inserted in the Poisson equation, after which they take the form

$$\varphi_{\mathbf{k}_s}^s(\omega_s^2 - k_s^2c_s^2) = \frac{e}{2m}\frac{\omega_s^2}{k_{z_s}}\int \varphi_{\mathbf{k}_h}^h E_{x\mathbf{k}_c}^c (k_{yh} - ik_{xh})\frac{k_{zh}}{\omega_h(\omega_{He} - \omega_c)}\, d\lambda_1, \quad (9)$$

$$\varphi_{\mathbf{k}_h}^h\left(\omega_h^2 + \frac{\omega_{pe}^2\omega_h^2}{(\omega_{He}^2 - \omega_h^2)}\frac{k_{\perp h}^2}{k_h^2} - \omega_{pe}^2\frac{k_{zh}^2}{k_h^2}\right)$$
$$= -\frac{\omega_{pe}^2\omega_h}{k_h^2}\frac{e}{2m}\int E_{x\mathbf{k}_c}^c \varphi_{\mathbf{k}_s}^s (k_{ys} - ik_{xs})\left[-\frac{k_{\perp h}^2}{(\omega_{He} - \omega_c)(\omega_{He}^2 - \omega_h^2)} - \frac{k_{zh}\omega_s}{\omega_h k_{zs}v_{Te}^2(\omega_{He} - \omega_c)} - \frac{k_c\omega_s}{(\omega_{He} - \omega_h)(\omega_{He} - \omega_c)k_{zs}v_{Te}^2}\right] d\lambda_2. \quad (10)$$

Equations (9) and (10) coincide in structure with the equations described in [8]. Transforming them in accord with [8], we can obtain equations for the time variation of the field amplitudes in the hybrid and ion-acoustic waves:

$$\frac{\partial\varphi_{\mathbf{k}_s}^{*s}}{\partial t} = i\frac{e}{4m}\frac{\omega_s}{k_{zs}}\int \varphi_{\mathbf{k}_h}^h E_{x\mathbf{k}_c}^c (k_{yh} + ik_{xh}) \cdot \times \frac{k_{zh}e^{i\Delta\omega t}}{\omega_h(\omega_{He} - \omega_c)}\, d\mathbf{k}_c\, d\mathbf{k}_h \delta(\mathbf{k}_s + \mathbf{k}_h - \mathbf{k}_c), \quad (11)$$

$$\frac{\partial\varphi_{\mathbf{k}_h}^h}{\partial t} = i\frac{e}{4m}\frac{(\omega_{He}^2 - \omega_h^2)^2}{\omega_{He}^2 k_{\perp h}^2}\int E_{x\mathbf{k}_c}^c \varphi_{\mathbf{k}_s}^{s*} e^{-i\Delta\omega t}\, d\mathbf{k}_c\, d\mathbf{k}_s$$
$$\times\,\delta(\mathbf{k}_h + \mathbf{k}_s - \mathbf{k}_c)(k_{ys} - ik_{xs})\left[-\frac{k_{\perp h}^2}{(\omega_{He} - \omega_c)(\omega_{He}^2 - \omega_h^2)} + \frac{k_{zh}\omega_s}{\omega_h k_{zs}v_{Te}^2(\omega_{He} - \omega_c)} - \frac{k_c\omega_s}{(\omega_{He} - \omega_h)(\omega_{He} - \omega_c)k_{zs}v_{Te}^2}\right], \quad (12)$$

where $\Delta\omega = \omega_{\mathbf{k}_s} + \omega_{\mathbf{k}_h} - \omega_{\mathbf{k}_c}$.

Since we assume the cyclotron wave to be monochromatic, we can write $E^c_{x\mathbf{k}_c} = k_0 E^c_{\mathbf{k}_0}\,\delta(\mathbf{k}_c - \mathbf{k}_0)$ (here k_0 is the wave vector of the cyclotron wave). Thus the ingration in (11) and (12) can be carried through to conclusion. Expressing them with the aid of (11) $\varphi^h_{\mathbf{k}_h}$ in terms of $\varphi^{s*}_{\mathbf{k}_s}$ and substituting the result in (12), we get

$$\frac{\partial^2\varphi_{\mathbf{k}_s}^{s*}}{\partial t^2} - i\Delta\omega\frac{\partial\varphi_{\mathbf{k}_s}^{s*}}{\partial t} = \frac{1}{8}\frac{k_0^2W_{\mathbf{k}_0}^c}{n_0T_e}k_h c_s\omega_{He}\frac{k_{\perp h}^2}{k_h^2}$$
$$\times\left[\frac{2k_{\perp h}^2 v_{Te}^2}{\omega_{He}^2} + \frac{k_h c_s}{\omega_{He}} + 2\frac{k_0c_s}{\omega_{He}}\right]\varphi_{\mathbf{k}_s}^{s*} \equiv \gamma_{\text{nonl}}^2\varphi_{\mathbf{k}_s}^{s*}. \quad (13)$$

Maximizing the right-hand side of (13) with respect to ($k_h^2v_{Te}^2 \ll \omega_{He}^2$) and using the condition $\Delta\omega \approx 0$ ($\Delta\omega \ll \gamma_{\text{nonl}}$), we obtain

$$\frac{\partial^2\varphi_{\mathbf{k}_s}^{s*}}{\partial t^2} - i\Delta\omega\frac{\partial\varphi_{\mathbf{k}_s}^{s*}}{\partial t} = \frac{1}{4}\frac{k_0^2W_{\mathbf{k}_0}^c}{n_0T_e}\omega_s\omega_{He}\frac{\omega_{pe}^4}{k_0^4c^4}\varphi_{\mathbf{k}_s}^{s*}. \quad (14)$$

It should be noted that in the derivation (14) it is essential to satisfy the condition $\gamma_{\text{nonl}} < \omega_s$, and therefore (14) is valid only if

$$\frac{k_0^2W_{\mathbf{k}_0}^c}{n_0T_e}\omega_s\omega_{He}\frac{\omega_{pe}^4}{k_0^4c^4} < \omega_s^2,$$

otherwise there arises one more limitation on $k_h^{\max}$, namely, $k_h^{\max}$ must be chosen in accord with the condition $\gamma_{\text{nonl}}(k_h^{\max}) < \omega_s$.

The solution of (14) is

$$\varphi_{k_s}^{s*} = \varphi_{k_{s_0}}^{s*}e^{\gamma t}, \quad \gamma = \gamma_{\text{nonl}}\left(1 - \frac{(\Delta\omega)^2}{8\gamma_{\text{nonl}}^2}\right). \quad (15)$$

The presence in (15) of a small correction to the increment, due to $\Delta\omega \neq 0$, means that such a decay results in the production of ion-acoustic waves with a finite width of the frequency spectrum $\Delta\omega_s \sim \Delta\omega \lesssim \omega_s$. The time of formation of such a packet, τ_1, coincides in order of magnitude with the reciprocal nonlinear decay increment $\tau_1 > 1/\gamma_{\text{nonl}}$. Obviously, formula (15) and all the deductions that follow from it are valid also for ax the hybrid wave.

We shall show now that the packets of ion-acoustic and hybrid waves produced as a result of the decay can be considered stochastically. The question of the limit of applicability of the statistical method for the description of the behavior of the discrete wave packets was considered in detail in Zaslavskiĭ's book.[9] It turns out that the system of wave packets, for which the decay laws are satisfied, can be treated stochastically under the condition

$$\frac{\partial\Delta\omega_k}{\partial N_k}\Delta N_k \gg \Omega_k, \quad (16)$$

where $\Delta\omega_k$ is the change of frequency resulting from the nonlinear resonant perturbation, ΔN_k is the maximum change of the number of waves produced by the resonance, and Ω_k is the characteristic distance between the harmonics in the spectrum.

The physical meaning of (16) is as follows. Let a wave of large amplitude and fixed frequency ω decay resonantly into a sum of waves with frequencies ω_1 and ω_2. Since $\omega_1 = \omega_1(N_{k_1})$ and $\omega_2 = \omega_2(N_{k_2})$, it follows that after some time N_{k_1} and N_{k_2} can increase to such an extent that the decay condition $\omega = \omega_1 + \omega_2$ is no longer satisfied. It may turn out, however, that the decay condition will then be satisfied for another pair of waves with frequencies ω_1' and ω_2'. In the case when condition (16) is satisfied, the wave with frequency ω rapidly goes out of resonance with any pair of waves, owing to the strong nonlinearity, but on the other hand is always in resonance with some other pair of waves, which can thus be treated stochastically.

We shall continue the derivation for ion-acoustic oscillations; it is easy to verify that the results that follow are valid also for hybrid waves.

The quantity $\Delta\omega_s(N_s)$ should be determined from an equation analogous to (9), in which one should retain the terms of third order in the amplitudes of the interacting waves. The corresponding calculations give for $\Delta\omega$ the following value

$$\Delta\omega_s \sim \omega_s e^2 \varphi_s^2 / m^2 v_{Te}^4. \tag{17}$$

The quantity $\Delta\varphi_s$ can be easily found from (11) by assuming that φ_h and E_x are independent of the time. Assuming the inequality $N_s\Omega_s \gg \omega_s$ to be satisfied (N_s is the number of excited oscillations), we obtain from (11), in analogy with [9],

$$\Delta\varphi_s \sim \frac{e}{m}\omega_s\varphi_h E_x \frac{k_s}{\omega_{He}(\omega_{He}-\omega_c)}\frac{1}{\Omega_s}. \tag{18}$$

Replacing N_s by φ_s in (16), we obtain ultimately the stochasticity condition in the form

$$\frac{W_s\sqrt{W_c}}{(n_0 T)^{3/2}} \gg \frac{(2\pi)^2}{k_s k_{s\perp} L^2}, \tag{19}$$

L is the dimension of the system. (In (19) we substituted in place Ω_s the expression $\Omega_s = (\partial\omega_s/\partial k_s)\Delta k_s = c_s \cdot 2\pi/L$.) Since we are considering a system with strong nonlinearity ($W_c \lesssim n_0 T$), the inequality (19) is satisfied during the final stage of the decay instability ($W_c \sim W_s \sim W_h$) for all oscillations in the packet. Thus, the packets of ion-acoustic and hybrid waves produced as a result of the decay can be considered stochastically.

Before we proceed to a consideration of the reverse process of wave coalescence, we note that a similar result (the buildup of ion sound by a cyclotron wave of large amplitude) was obtained by Zyunder and Gradov [10] in an analysis of parametric excitation of ion sound by an external high-frequency electric field of frequency $\Omega \sim \omega_{He}$.

The dispersion relation for ion-acoustic oscillations, obtained in [10] and written out without allowance for the kinetic damping, is

$$\frac{1}{\delta\varepsilon_i}+\frac{1}{1+\delta\varepsilon_e} = -\frac{\delta^2}{4}\left[\frac{1}{\varepsilon(\Omega+\omega)}+\frac{1}{\varepsilon(\Omega-\omega)}\right], \tag{20}$$

where

$$\delta\varepsilon_i = -\frac{\omega_{pi}^2}{\omega^2}, \quad \delta\varepsilon_e = \frac{\omega_{pe}^2}{k^2 v_{Te}^2}, \quad \varepsilon(\omega) = 1+\frac{\omega_{pe}^2}{\omega_{He}^2-\omega^2}\frac{k_\perp^2}{k^2}-\frac{\omega_{pe}^2}{\omega_{He}^2}.$$

(In the derivation of $\epsilon(\omega)$ it was taken into account that $k_\perp/k \ll 1$.) Since $\omega \ll \Omega$, it follows that

$$\varepsilon(\Omega+\omega) \approx \varepsilon(\Omega)+\omega\frac{\partial\varepsilon}{\partial\Omega}, \quad \varepsilon(\Omega-\omega) \approx \varepsilon(\Omega)-\omega\frac{\partial\varepsilon}{\partial\Omega}$$

and then

$$\frac{1}{\delta\varepsilon_i}+\frac{1}{1+\delta\varepsilon_e} = -\frac{\delta^2}{4}\frac{2\varepsilon(\Omega)}{\varepsilon^2(\Omega)-\omega^2(\partial\varepsilon/\partial\Omega)^2}. \tag{21}$$

Substituting in (21) the values of $\epsilon(\Omega)$ and $\partial\epsilon/\partial\Omega$, we obtain ultimately

$$-\frac{\omega^2}{\omega_{pi}^2}+\frac{k^2 v_{Te}^2}{\omega_{pe}^2} = -\frac{e^2 k^2 E_0^2}{4m^2\Omega^4}\frac{2\omega_{He}\Delta}{\Delta^2-\omega^2}, \tag{22}$$

where*

$$\Delta = (\omega_{He}-\Omega)\left[\frac{2|\omega_{pe}^2-\omega_{He}^2|(\omega_{He}-\Omega)}{\omega_{He}\omega_{pe}^2 k_\perp^2/k^2}-1\right].$$

If we now put in (22) $\Delta = \mathrm{Re}\,\omega$, then

$$\gamma^2 = \frac{W_c}{n_0 T_e}\omega_s\omega_{He}\left(\frac{\omega_{He}-\Omega}{\omega_{He}}\right)^2,$$

which coincides with the expression obtained by us for $\gamma_{\rm nonl}$ in the decay of an electron cyclotron wave.

It can thus be stated that parametric buildup of ion sound by an external current perpendicular to the field $\mathbf{H}_0$ coincides with the nonlinear decay of a cyclotron wave into an ion-acoustic wave and a hybrid wave.

Let us now consider in detail the reverse process—the coalescence of an ion-acoustic with a hybrid wave to form a cyclotron wave. These waves, as shown above, are stochastic. Therefore in the derivation of the equation for the change in the noise level of the cyclotron waves we can use the standard method of averaging over the phases. Using the system of equations of magnetohydrodynamics in conjunction with Maxwell's equations, we can obtain expressions for the electron current, which contains terms linear and quadratic in the fields of the decaying waves:

$$\begin{aligned} J^c_{xk_c} = &-\frac{e^2 n_0}{m}\frac{i\omega_c E^c_{xk_c}+\omega_{He}E^c_{yk_c}}{\omega_{He}^2-\omega_c^2} - \\ &-\frac{en_0}{2}i\omega_{He}\cdot\frac{\int(v_{k_h}k_s v_{yk_s}+v_{k_s}k_h v_{yk_r})\,d\lambda}{\omega_{He}^2-\omega_c^2} \\ &+\frac{en_0}{2}\omega_c\cdot\frac{\int(v_{k_h}k_s v_{xk_s}+v_{k_s}k_h v_{xk_h})\,d\lambda}{\omega_{He}^2-\omega_c^2}-\frac{e}{2}\int(n_{k_h}v_{xk_s}+n_{k_s}v_{xk_h})\,d\lambda, \end{aligned} \tag{23}$$

$$\begin{aligned} J^c_{yk_c} = &-\frac{e^2 n_0}{m}\frac{i\omega_c E^c_{yk_c}-\omega_{He}E^c_{xk_c}}{\omega_{He}^2-\omega_c^2}+\frac{en_0}{2}\omega_c\frac{\int(v_{k_h}k_s v_{yk_s}+v_{k_s}k_h v_{yk_h})\,d\lambda}{\omega_{He}^2-\omega_c^2} \\ &+\frac{en_0}{2}i\omega_{He}\frac{\int(v_{k_h}k_s v_{xk_s}+v_{k_s}k_h v_{xk_h})\,d\lambda}{\omega_{He}^2-\omega_c^2}-\frac{e}{2}\int(n_{k_h}v_{yk_s}+n_{k_s}v_{yk_r})\,d\lambda, \end{aligned} \tag{24}$$

where

$$d\lambda = d\mathbf{k}_h\,d\mathbf{k}_s\,d\omega_h\,d\omega_s\,\delta(\mathbf{k}_c-\mathbf{k}_h-\mathbf{k}_s)\,\delta(\omega_c-\omega_h-\omega_s),$$

$$\mathbf{v}_{k_s} = -i\frac{e}{m\omega_{He}}\frac{[\mathbf{k}_s\mathbf{H}_0]}{|H_0|}\varphi^s_{k_s}+\frac{e}{m}\frac{\omega_s\mathbf{H}_0/|H_0|}{k_{zs}v_{Te}^2}\varphi^s_{k_s}, \quad n_{k_s} = \frac{en_0}{mv_{Te}^2}\varphi^s_{k_s}$$

$$\mathbf{v}_{k_h} = \frac{e}{m}\frac{\omega_h\mathbf{k}_h-i\omega_{He}[\mathbf{k}_h\mathbf{H}_0]/|H_0|}{\omega_{He}^2-\omega_h^2}\varphi^h_{k_h}-\frac{e}{m\omega_h}\frac{\omega_{He}^2}{\omega_{He2}-\omega_h^2}\frac{[\mathbf{k}_h\mathbf{H}_0]}{|H_0|}\varphi^h_{k_h},$$

$$n_{k_h} = \frac{en_0}{m\omega_h}\left(\frac{\omega_h k_{\perp h}^2}{\omega_{He}^2-\omega_h^2}-\frac{k_{zh}^2}{\omega_h}\right)\varphi^h_{k_h}, \quad \varphi_k = \varphi_{\mathbf{k},\omega}, \quad E_k^c = E^c_{\mathbf{k},\omega}.$$

(Just as in the case of the decay of a monochromatic cyclotron wave, we have retained in (23) and (24) only terms with the fields of the supplementary oscillations, and the decay conditions are not satisfied for all the other terms.)

Substituting (23) and (24) in the equation curl curl E $= 4\pi i\omega c^{-2}\mathbf{J}$ expressed in terms of its components, and expressing the field $E^c_{yk_c}$ in terms of $E^c_{xk_c}$, we obtain

$$\left(k_c^2-\frac{\omega_{pe}^2}{c^2}\frac{\omega_c}{\omega_{He}-\omega_c}\right)E^c_{xk_c} = -\frac{\pi e^3 n_0}{m^2c^2}\omega_c\int d\lambda$$

$$\times\,\varphi^h_{k_h}\varphi^s_{k_s}(k_{yh}+ik_{xh})\left\{-\frac{1}{\omega_h\omega_{He}}\left(\frac{\omega_r k_{\perp h}^2}{\omega_{He}^2-\omega_h^2}-\frac{k_{zh}^2}{\omega_h}\right)\right.$$

*$[\mathbf{k}_s\mathbf{H}_0] \equiv \mathbf{k}_s \times \mathbf{H}_0$.

$$+\frac{1}{v_{Te}^2(\omega_{He}-\omega_r)}-\frac{1}{\omega_{He}-\omega_c}\left(\frac{k_{zh}k_{zs}}{\omega_h\omega_{He}}+\frac{k_{zr}\omega_s}{(\omega_{He}-\omega_h)k_{zs}v_{Te}^2}+\frac{k_{\perp h}^2}{\omega_{He}^2-\omega_h^2}\right)\Bigg\}. \tag{25}$$

We multiply (25) by $E^c_{xk'}$ and average over the phases of the interacting waves; when averaging the expressions $E^c_{xk'}$, $\varphi^h_{k_h}$, and $\varphi^s_{k_s}$, it is necessary to substitute for $E^c_{xk'}$ its expression from the equation conjugate to (25). After averaging we obtain

$$\left(k_c^2-\frac{\omega_{pe}^2}{c^2}\frac{\omega_c}{\omega_{He}-\omega_c}\right)|E^c_{xk_c}|^2=\left(\frac{\pi e^3n_0\omega_c}{m^2c^2}\right)^2\Big/\left(k^2-\frac{\omega_{pe}^2}{c^2}\frac{\omega_c}{\omega_{He}-\omega_c}\right)$$
$$\times\int|\varphi^h_{k_h}|^2|\varphi^s_{k_s}|^2k_{\perp h}^2\left|-\frac{1}{\omega_{He}\omega_h}\left(\frac{\omega_r k_{\perp h}^2}{\omega_{He}^2-\omega_h^2}-\frac{k_{zh}^2}{\omega_h}\right)+\frac{1}{v_{Te}^2(\omega_{He}-\omega_h)}-\frac{1}{\omega_{He}-\omega_c}\left(\frac{k_{zr}k_{zs}}{\omega_h\omega_{He}}+\frac{k_{zh}\omega_s}{k_{zs}v_{he}^2(\omega_{He}-\omega_h)}+\frac{k_{\perp h}^2}{\omega_{He}^2-\omega_h^2}\right)\right|^2d\lambda. \tag{26}$$

In the zeroth approximation it follows from (26) that $|E^c_{xk_c}|^2=|E^c_{xk_c}|^2\delta(\omega_c-\omega_{k_c})$, and from equations analogous to (26) we have

$$|\varphi^h_{k_h}|^2=|\varphi^h_{k_h}|^2\delta(\omega_h-\omega_{k_h}),$$
$$|\varphi^s_{k_s}|^2=|\varphi^s_{k_s}|^2\delta(\omega_s-\omega^h_{k_s})$$

We note, in addition, that

$$\mathrm{Im}\left(k^2-\frac{\omega_{pe}^2}{c^2}\frac{\omega}{\omega_{He}-\omega}\right)\approx\frac{\omega_{pe}^2}{c^2}\frac{\omega}{(\omega_{He}-\omega)^2}2\gamma=-\frac{\omega_{pe}^2}{c^2}\frac{\omega}{(\omega_{He}-\omega)^2}\frac{\partial}{\partial t},$$
$$\mathrm{Im}\left(k^2-\frac{\omega_{pe}^2}{c^2}\frac{\omega}{\omega_{He}-\omega}\right)^{*-1}=-\delta(\omega_c-\omega_{k_c})\left(\frac{\omega_{pe}^2}{c^2}\frac{\omega}{(\omega_{He}-\omega)^2}\right)^{-1}$$

We now take the imaginary part of (26) and integrate with respect to $d\omega_c$; taking all the foregoing into account, we obtain

$$\frac{\partial|E^c_{xk_c}|^2}{\partial t}=\frac{1}{8}\frac{e^2}{m^2}(\omega_{He}-\omega_c)^4\int d\mathbf{k}_r\delta\left(\omega_{He}\frac{\omega_{pe}^2}{k_c^2c^2}-\omega_{He}\frac{k_{\perp h}^2}{k_h^2}-\omega_s\right)$$
$$\times k_{\perp h}^2\left|\frac{1}{\omega_h\omega_{He}}\left(\frac{\omega_r k_{\perp h}^2}{\omega_{He}^2-\omega_h^2}-\frac{k_{zh}^2}{\omega_h}\right)-\frac{1}{v_{Te}^2(\omega_{He}-\omega_h)}+\frac{1}{\omega_{He}-\omega_c}\left(\frac{k_{zh}(k_{zc}-k_{zh})}{\omega_h\omega_{He}}-\frac{k_{zh}c_s}{v_{Te}^2(\omega_{He}-\omega_h)}-\frac{k_{\perp h}^2}{\omega_{He}^2-\omega_h^2}\right)\right|^2\cdot|\varphi^h_{k_h}|^2|\varphi^s_{k_s}|^2. \tag{27}$$

Since we are considering oscillations with $kv_{Te}\ll\omega_{He}$, it follows, as can be readily seen, that the principal term in the right-hand side of (27) under the absolute-value sign is $1/v_{Te}^2(\omega_{He}-\omega_h)$. Introducing also the spectral density of the oscillations, we obtain from (27)

$$\frac{\partial W^c_{k_c}}{\partial t}=\omega_{He}\int dk_hW^h_{k_h}\frac{W^s_{k_c-k_h}}{n_0T_e}\delta\left(\frac{\omega_{pe}^2}{k_c^2c^2}-\frac{k_{\perp h}^2}{k_h^2}-\frac{|\mathbf{k}_c-\mathbf{k}_h|c_s}{\omega_{He}}\right)\frac{k_{\perp h}^4}{k_h^4}. \tag{28}$$

In most cases of practical interest we have $\omega_{pe}^2/k_c^2c^2\gg\omega_s/\omega_{He}$. Indeed, since it is necessary to satisfy the inequality $\omega_{pe}^2/k_c^2c^2\gg k_cv_{Te}^2/\omega_{He}$, this inequality leads to the following limitation on k_c: $k_c^2\ll(\omega_{pe}^2\omega_{He}/c^2v_{Te}^2)\ll(\omega_{pe}^2\omega_{He}/c^2v_{Te})^{2/3}$. If we now substitute the obtained value of k_c^2 in $\omega_{pe}^2/k_c^2c^2$ and compare it with ω_s/ω_{He}, then we obtain

$$\frac{\omega_{pe}^2/k_c^2c^2}{\omega_s/\omega_{He}}\gg\frac{(\omega_{pe}^2\omega_{He})^{1/3}}{\omega_{pi}}\left(\frac{v_{Te}}{c}\right)^{2/3}.$$

Usually this latter quantity is much larger than unity, i.e., $\omega_{pe}^2/k_c^2c^2\gg\omega_s/\bar{\omega}_{He}$. It then follows from the δ-function in (28) that $k_{\perp h}^2/k_h^2\approx\omega_{pe}^2/k_c^2c^2$ and Eq. (28) can be finally represented in the form

$$\frac{\partial W^c_{k_c}}{\partial t}\approx\omega_{He}\frac{\omega_{pe}^2}{k_c^2c^2}W^s_{k_s}\frac{W^h}{n_0T_e}. \tag{29}$$

Thus, the reverse process of coalescence of an ion-acoustic wave with a hybrid wave into a stochastic packet of cyclotron waves occurs within a characteristic time

$$\tau_2\sim\left(\omega_{He}\frac{\omega_{pe}^2}{k_c^2c^2}\frac{W^s_{k_s}}{W^c_{k_c}}\frac{W^h}{n_0T_e}\right)^{-1}. \tag{30}$$

Since the decay instability together with its reverse process, the coalescence of the ion-acoustic and hybrid waves, leads to a Rayleigh-Jeans distribution, we can put in (3) $W^h\sim W^s\sim W^c$, and we can rewrite finally (3) in the form

$$\tau_2\sim(\omega_{He}W^c/n_0T_e)^{-1}. \tag{31}$$

We can thus state that after the lapse of a time $\tau\gtrsim\tau_1+\tau_2$ (we note that for typical experiments on cyclotron heating these times are of the same order of magnitude) there is produced in the plasma a sufficiently broad spectrum of cyclotron waves, to which the theory of quasilinear relaxation can be applied. We note that at such a spectrum width practically all the particles will interact with the wave, at the characteristic parameters of the installations, so that

$$\frac{\Delta v_{res}}{v_{res}}\sim\frac{\Delta\omega_c}{\omega_{He}-\omega_c}\sim1\left(v_{res}=\frac{\omega_c-\omega_{He}}{k_c}\right),$$

and the longitudinal velocity remains practically unchanged during the cyclotron heating.[3]

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