QUANTUM OSCILLATIONS IN SUPERCONDUCTORS

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Quantum oscillations of the energy gap, current density, and surface impedance in superconductors are investigated. The distinctive features of the oscillations with respect to the magnetic field are elucidated, and the period and temperature dependence of the amplitude are found. The experimental feasibility of establishing the dispersion law $\xi(p)$ and the gap $\Delta(p)$ from the period and temperature dependence of the amplitude of the oscillations is proved. The conditions on Maxwell's equations at the interface between metal and vacuum are discussed. The rule for determining the phase γ in the conditions for quasiclassical quantization is found in general form.

1. INTRODUCTION

SINCE Pincus[1] pointed out the existence of surface levels in superconductors, a large number of articles have appeared which are devoted to this question. The possibility of a quasiclassical consideration of such levels was first mentioned in article [2] and the Bohr rules of quantization for the excitations in superconductors are derived. In [3] the quasiclassical treatment was used in order to study vortex lines in a pure superconductor. The asymptotic form of the exact wave functions of the excitations is found in [4] in the quasiclassical domain, and with their aid the values of the phase in the quantization roles are obtained. In [5] it is shown that the number of levels increases rapidly with increasing magnetic field and decreases abruptly upon an increase of the angle of reflection of the excitation from the surface of the superconductor. An analysis of the nature of the classical trajectories and of the quantized spectrum is carried out in [6]; there it was shown that the strong dependence of the number of discrete levels on the magnetic field leads to resonances and to quantum oscillations of the various physical quantities with respect to the magnetic field.[1]

The present article is devoted to the calculation of the quantum oscillations of the energy gap, current density, and surface impedance in a superconductor occupying the half-space $y \geq 0$ in the presence of a magnetic field $\mathbf{H} \parallel \mathbf{z}$. Since the quantum oscillations are essentially connected with the quasiclassical nature of the corresponding levels, we shall discuss this question in somewhat more detail. In the classical case the energy of the excitations in a superconductor in the presence of a magnetic field is changed by the amount $-(e/c)\mathbf{A}\cdot\mathbf{v}$ (see [2,6]):

$$\varepsilon(\mathbf{p}) = \pm\sqrt{\xi_p{}^2 + \Delta^2} - \frac{e}{c}\mathbf{A}\mathbf{v} = \pm\,\varepsilon_0(\mathbf{p}) - \frac{e}{c}\mathbf{A}\mathbf{v}, \quad (1.1)$$

$$\xi_p = \varepsilon_e(\mathbf{p}) - \mu, \quad \mathbf{v} = \partial\xi_p/\partial\mathbf{p}, \quad (1.2)$$

where μ is the chemical potential, Δ is the energy gap, and $\mathbf{A}$ is the vector potential of the magnetic field $\mathbf{B}$.

In the geometry of interest to us

$$A_x = A_x(y) = \int_y^\infty B(y')dy', \quad A_y = A_z = 0. \quad (1.3)$$

According to Eqs. (1.1) and (1.3) the phase trajectories $p_y = p_y(y)$ associated with the quadratic dispersion law $\epsilon_e = p^2/2m$ are determined by the equation

$$p_y = \pm p_\pm = [2m\{\mu_\perp \pm \sqrt{(\varepsilon+\rho)^2 - \Delta^2}\}]^{1/2}, \quad (1.4)$$

$$\rho = \rho(y) = eA_x v_x / c,$$

$$\mu_\perp = \varepsilon_F - (P_x{}^2 + P_z{}^2)/2m = \varepsilon_F - p_\parallel{}^2/2m. \quad (1.5)$$

The form of the phase trajectory essentially depends on the parameters P_x, P_z, and ϵ, where in general there are four branches $p_y(y)$ corresponding to each set P_x, P_z, ϵ in accordance with the four possibilities for the choice of the plus and minus signs in Eq. (1.4). In order to write down the condition for the validity of the quasiclassical treatment, let us turn to the characteristic trajectory depicted in Fig. 1. One can talk about classical motion in that case when a quantum transition from one orbit to the other is excluded. It is clear that the uncertainty of the coordinate is characterized by the attenuation depth δ of the field (or by the dimension ξ_0 of a pair; in our case δ will play the fundamental role).
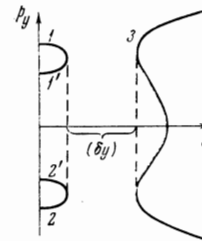

FIG. 1

Therefore, the obvious and easily satisfied condition for the absence of transitions from trajectory 1 to trajectory 3 is $(\delta y) \gg \delta$ (see Fig. 1), where the transitions are absent with exponential accuracy in $[-(\delta y)/\delta]$. On the other hand, the uncertainty in the momentum p_y is given by $(\delta p_y) \sim \hbar/\delta$ and the absence of transitions (also with exponential accuracy) from trajectory 1 to trajectory 2 is guaranteed by the condition

$$|p_y| \gg \hbar/\delta. \quad (1.6)$$

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[1]The construction of a phenomenological theory of superconductivity was also discussed in [6]. In order to avoid misunderstandings, we emphasize that in this connection we had in mind the derivation of the well-known equations of the microscopic theory of superconductivity from intuitive semi-phenomenological considerations.

The fulfilment of condition (1.6) is equivalent to the "usual" criterion for the quasiclassical nature of the motion (see, for example, [7]) and corresponds to a "division" of the motion along upper and lower orbits[2] (see Fig. 1); it is certainly satisfied for $p_\| \sim p_F$. Precisely this simplification has been used in a large number of articles (see, for example, [8]) and precisely this criterion, in essence, is cited in [4]. For the validity of the "true" quasiclassical treatment, however, it is also necessary to exclude quantum transitions from orbit 1 to 1' (or from 2 to 2'). The appropriate condition obviously has the form

$$(\delta p_y^{(i)}) = (p_+ - p_-) \gg \hbar/\delta. \tag{1.7}$$

The fulfilment of condition (1.7), as already mentioned, essentially depends on the values of P_x, P_z, and ϵ; thus, for example, for $p_\| = p_F$, $\epsilon \approx \Delta$, and $\rho \approx \Delta$ from Eq. (1.4) it follows that

$$p_y(0) \sim (m^2\Delta\rho)^{1/4} \sim \sqrt{m\Delta} \sim \sqrt{\hbar p_F/\xi_0} \sim \sqrt{\hbar^2/a\xi_0} \gg \hbar/\xi_0,\ \hbar/\delta \tag{1.8}$$

(see Fig. 2, where in order to be specific ϵ is taken to be less than Δ).

A detailed analysis of the situations when the quasiclassical treatment is valid is carried out in [6]. Of course, the question of whether to solve this or a different physical problem quasiclassically requires a determination of what values of P_x, P_z, and ϵ are important in the problem, and whether they satisfy inequalities (1.6) and (1.7).

Now let us go on to the description of the quantum picture of the energy spectrum in a magnetic field,[6] which is essential for what follows. In the absence of a magnetic field, for a given value of P_x the allowed energy values $\epsilon(P_x) = \epsilon_n(P_x)$ are indicated by the shaded region shown on Fig. 3 (as long as $P_x < p_F$ values of p_y and P_z can always be found such that $\epsilon_{min} = \Delta$; for $P_x > p_F$ one has

$$\varepsilon_{min} = [\,[(P_x^2 - p_F^2)/2m]^2 - \Delta^2\,]^{1/2}. \tag{1.9}$$

If $H < H_1$ then one discrete level (curve 1 on Fig. 3) is split off from the continuous spectrum ($\epsilon > 0$, $P_x > 0$) for each value of P_x, where the distance of this level from the "old" cutoff of the spectrum has a maximum value for $P_x = p_F$ and its order of magnitude is given by

$$\Delta - \varepsilon \sim \Delta \frac{H^4}{H_2 H_1^3},\quad H_1 \sim \frac{\Phi_0 a^2\xi_0}{\delta^5} \sim 10^{-5} - 10^{-2}\text{Oe},\ H_2 \sim \frac{\Phi_0}{\delta\xi_0} \sim 10^2\text{Oe}. \tag{1.10}$$

The n-th discrete level (for a fixed value of P_x) is split off from the continuous spectrum in a field $H > H_1$. For $H \gg H_1$ the picture depicted in Fig. 4 is obtained; the minimum distance of the n-th level from the gap is of the order of

$$\Delta - \varepsilon_n \sim \Delta(H/H_2)n/n_{max} \tag{1.11}$$

(measurement of the levels is conducted from below upwards). In the London case $H_2 \sim H_c$, and in the Pippard case $H_2 \sim H_c \kappa^{1/3} \ll H_c$ with $\kappa = \delta_L/\xi_0$.

For $H > H_2$ the levels cross the Fermi boundary $\epsilon = 0$; new levels appear in the magnetic field $H = H_n$, where $H_n \sim H_4 n^2$ and $H_4 \sim H_c a/\delta_L$. Their general form
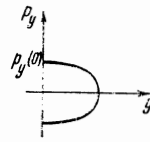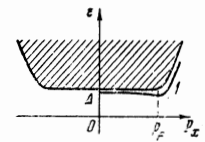

FIG. 2   FIG. 3
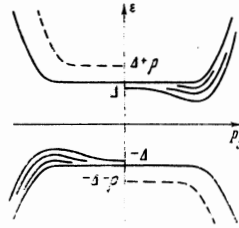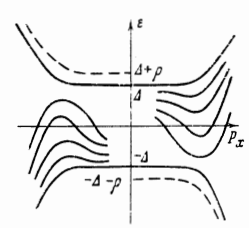
FIG. 4   FIG. 5

is shown in Fig. 5 (for convenience both $\epsilon > 0$ and $\epsilon < 0$ are shown in Figs. 4 and 5); the scaling in the figures is not maintained. The distance between the levels is of the order of $(\delta\epsilon) \sim \rho(0)/n \sim \Delta(HH_4/H_2^2)^{1/2}$ and is appreciably larger than, for example, the quantity $\hbar\Omega = \hbar eH/mc$. This is quite natural: The question is about levels resulting from "sliding" orbits with very small periods and large frequencies, which are essentially imbedded in a layer of thickness δ (Fig. 6).

Quassiclassical quantization is determined by the usual Bohr formula $S = \oint p_y\,dy = 2\pi n\hbar$ (see [7,2]). The more exact notation for the quantization condition has the form

$$S = 2\pi\hbar(n + \gamma), \tag{1.12}$$

where the phase γ depends on the nature of the trajectory. Its form in the general case is derived in the Appendix (where, in particular, a mistake which was made during the calculation of one of the values of γ in [4] is corrected). It is found that one can formulate the following simple rule for the determination of γ: if the number of turning points is even then $\gamma = -\frac{1}{2}$, or what amounts to the same thing, $\gamma = \frac{1}{2}$; if the number of turning points is odd, then $\gamma = -\frac{1}{4}$.[3]

FIG. 6

Now, when the structure of the spectrum is clear, one can pass to a physical picture of the quantized oscillations in superconductors.

2. PHYSICAL PICTURE OF THE QUANTIZED OSCILLATIONS

The periodic splitting-off of the discrete levels from the continuous band spectrum would appear as if it might lead to quantized oscillations of the different physical quantities. Let us consider, however, the situation which arises in more detail. At low temperatures the levels which are split-off from above ($\epsilon \geq \Delta$, see Fig. 4) are occupied to an exponentially small extent (to the extent $\exp\{-\Delta/T\}$); at high temperatures ($T \sim \Delta$) the thermal "broadening" of a given level (i.e., the possibility of

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[2]In the Gor'kov equations this corresponds to a change from two equations of the second order to two equations of the first order.

[3]This rule is valid both for the excitations in a superconductor as well as for "ordinary" particles and quasiparticles.

transitions to other levels and from other levels) leads to an exponentially small amplitude of the oscillations (small with respect to $T/(\delta\epsilon)$, where $(\delta\epsilon)$ is the distance between the discrete levels). The levels for $\epsilon < 0$ are, of course, occupied. However, high temperatures ($\sim \Delta$) according to the same reasons as above do not lead to oscillations. At low temperatures the total number of states N for $\epsilon < 0$ is obviously conserved with exponential accuracy (in Δ/T) and since all thermodynamic quantities can be expressed in terms of the number of states since

$$\Omega(\mu) = -\int_0^\mu N(\mu')\,d\mu' \,,$$

the amplitude of the corresponding oscillations again turns out to be exponentially small. Therefore the observable oscillations appear when the discrete levels cross the Fermi boundary $\epsilon = 0$, i.e., for $H > H_2$; thereby the gap in the excitation spectrum vanishes and the oscillating picture develops in principle just like in a normal metal. In normal metals the oscillations are a bulk property; in the interior of a metal the discrete spectrum only depends on a single continuous parameter (P_Z) while degeneracy exists with respect to the position of the center of the orbit, which is determined by P_X. In superconductors the spectrum is discrete only near the surface, where the quantizing magnetic field is not small, and it depends on both parameters (P_X, P_Z). Therefore the amplitude of the oscillations contains the first power of the parameter $(\delta\epsilon)/\epsilon_F \sim \rho/\epsilon_F n_{max}$, which determines the fraction of the participating energies. (In a normal metal the corresponding parameter $\hbar\Omega/\epsilon_F$ enters raised to the power $\frac{1}{2}$.)

Since with an increase of the magnetic field the superconductor changes into a normal metal, specifically the "superconducting" oscillations, incipient for $H > H_2$, continue only up to $H_{c2} \sim \kappa H_2 \gg H_2$ in the London case and up to $H_c \sim \kappa^{-1/3} H_2 \gg H_2$ in the Pippard case. It is only necessary to mention that, beginning with $H_{c1} \sim \kappa^{-1} H_2 \ll H_2$, in the case of a London superconductor the picture is complicated because of the appearance of vortex lines, which leads to a two-dimensional problem. However, since the distance between the vortices in fields $H \gtrsim H_2$ is much smaller than the size of the orbit $(r\delta)^{1/2}$ (r is the Larmor radius), then in the x direction the dependence of A on x is averaged and in the basic approximation one can restrict oneself to the one-dimensional case (with Δ, of course, depending on y). All of the stated information can be directly transferred to the case of an arbitrary dispersion law and arbitrary dependence of $\Delta(\mathbf{p})$ in Eqs. (1.1) and (1.2). It is clear that the specific form of the oscillations depends on $\xi(\mathbf{p})$ and on $\Delta(\mathbf{p})$; it is important, however, that the basic formula (1.12) of quasiclassical quantization is retained. Therefore, the period of the oscillations will always be determined by that change of the field ΔH for which the number of levels under the Fermi boundary changes by one:

$$\Delta S_{extr}(\varepsilon = 0, H) = 2\pi\hbar. \quad (2.1)$$

The extremum of $S(P_X, P_Z)$ is taken with respect to P_X and P_Z, which guarantees the maximum density of oscillation of the determined states. It is curious that the extremal nature of the number of levels as a function of P_X and P_Z indicates periodicity of the motion of the excitations along the appropriate orbits in coordinate space. In fact, the displacement during the period T is given by

$$(\Delta x)_T = \oint v_x \frac{dy}{v_y} = -\frac{\partial}{\partial P_x}\oint p_y\,dy = -\frac{\partial S}{\partial P_x}, \quad (\Delta z)_T = -\frac{\partial S}{\partial P_z} \quad (2.2)$$

and for $S = S_{extr}$ we have $(\Delta x)_T = (\Delta z)_T = 0$. The shape of these orbits is shown in Fig. 7. We note that analogous cross sections are also distinguished in cyclotron resonance, in the de Haas–van Alphen effect, etc.
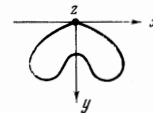

FIG. 7

As usual, the temperature dependence of the amplitude of the oscillations is determined by the distance between the levels:

$$(\delta\varepsilon) = \hbar\omega_{extr}^{cl} = 2\pi\hbar \Big/ \frac{\partial S_{extr}}{\partial \varepsilon}\Big|_{\varepsilon=0}. \quad (2.3)$$

Experimental investigation of the oscillations makes it possible, in principle, to determine two functions of H (namely, $\partial S/\partial\epsilon$ and $\partial S/\partial H$), where in contrast to the case of normal metals the dependence on the direction and magnitude of the field is not a universal function but depends on the specific form of ξ_p and $\Delta(\mathbf{p})$. Thus, experimentally one can obtain two functions of three variables whereas all of the quantities ($\xi(\mathbf{p})$, $\Delta(\mathbf{p})$, and $\partial\xi/\partial\mathbf{p}$) entering into formulas (2.1) and (2.2) are evaluated only on the Fermi surface and therefore only depend on two variables. This implies the possibility in principle of experimentally establishing all of the functions which enter into the theory.[4]

In conclusion, before going on immediately to the calculations, let us turn our attention to the nature of the conditions on Maxwell's equations at the interface between the metal and vacuum. Since the following discussions pertain to both superconductors and to normal metals, we shall demonstrate them for the example of normal metals. As is well known, the boundary conditions on Maxwell's equations are considered to be continuity of the tangential components, E_t and H_t, of the fields, and also continuity of the normal components B_n and D_n. These conditions are obtained after a suitable averaging of the microscopic equations of the field. The problem of specific interest to us about the quantum oscillations is the presence of several "microscopic" distances of different orders of magnitude, over which the macroscopic quantities are formed. These distances are: the intercharge distance $a \sim \hbar/p_F$, the Larmor radius r, and the mean free path length l. Upon averaging over distances $\Lambda \gg a, r, l$ the usual macroscopic

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[4]The point is that knowledge of a function of three variables is equivalent to knowledge of an infinite number of functions of two variables, for example

$$f(x, y, z) = \sum_{k=0}^\infty \frac{z^k f_k(x, y)}{k!}, \quad f_k = \frac{\partial^k f}{\partial z^k}\Big|_{z=0}.$$

equations and boundary conditions are naturally obtained. However, as a rule we are interested in averaging over distances $a \ll \Lambda \ll r, l$. In this connection in the equation connected with the magnetic field,

$$\operatorname{rot} \mathbf{B} = \frac{4\pi}{c} \mathbf{j}, \qquad (2.4)$$

both terms which are determined by distances of the order of a (the spin paramagnetism $\mathbf{M_S}$, the constant Landau diamagnetism $\mathbf{M_L}$, and the lattice magnetism $\mathbf{M}_l$) as well as the oscillating diamagnetic moment,[5] which changes over distances of the order of r, enter into **j**. Therefore, in connection with the well-known derivation of the boundary conditions, the integral of the oscillating term on the right hand side of Eq. (2.4) turns out to be small (to the extent that Λ/r is small) and the quantity $(\mathbf{B} - 4\pi\mathbf{M_S} - 4\pi\mathbf{M_L} - 4\pi\mathbf{M}_l)_t$ turns out to be continuous. Since we are not interested in small monotonic terms, in what follows we shall write the boundary condition simply as the continuity of the vector **B**.

Let us describe the plan of the calculation of the quantum oscillations to be given below. As already mentioned the oscillations are always associated with quasiclassical states; therefore Sec. 3 is devoted to a description of the quasiclassical wave functions and Green's functions. The necessity for a description of both the wave functions and the Green's functions is connected with the following. In order to calculate the energy gap and the current density it is necessary to carry out a summation over all of the states. In weak fields ($H < H_2 \lesssim H_C$) in the presence of an essential "non-quasiclassical nature" in the energy spectrum— that is, a gap whose magnitude is appreciably larger than the distance between the levels,[6] it is more convenient to use the Green's functions since the quasiclassical Poisson summation in this case turns out to be extremely complicated. Namely, in Sec. 4 the exponential smallness of the amplitude of the quantum oscillations is proved for $H < H_2$ with the aid of the Green's functions. In strong fields ($H > H_2$) the gap in the spectrum of the states of interest to us is absent, and the calculations turn out to be simpler in terms of the wave functions (Sec. 4). Finally, Sec. 5 is devoted to the determination of the oscillations of an easily measured physical quantity—the surface impedance.

3. QUASICLASSICAL WAVE FUNCTIONS AND GREEN'S FUNCTIONS

The form of the wave functions of the excitations in a superconductor in the quasiclassical region was derived in [6]. In [4] the asymptotic form of the wave functions is written down for two limiting cases in the re-

gion sufficiently close to the turning points, and the value of the phase γ in the Bohr quantization rules is found. Let us carry out an investigation of the general case in a symmetric notation. Strictly discrete levels correspond to finite phase trajectories. For $p_\parallel \sim p_F$ the corresponding trajectories, as follows from Eq. (1.4), have the form shown in Fig. 8a (the dotted line indicates the motion which would formally occur in the absence of the boundary at y = 0); with the reduction of $(p_F - p_\parallel)$ the trajectory takes the form shown in Fig. 8b; finally, for $(p_F - p_\parallel) \leq 0$ the trajectory is shown on Fig. 8c.
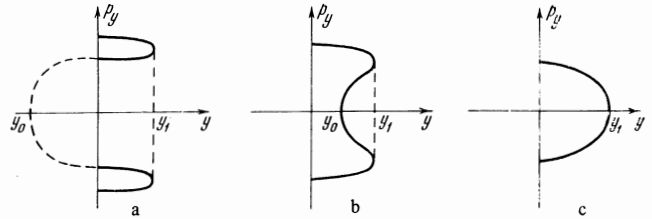

FIG. 8

Now let us find the wave functions in the quasiclassical regions, i.e., sufficiently far away from the turning points y_0, y_1. The general equation for the wave functions Ψ is given by

$$\begin{pmatrix} \hat{\xi} - \hat{\rho}, & -\Delta \\ -\Delta, & -\hat{\xi} - \hat{\rho} \end{pmatrix} \Psi = \varepsilon\Psi, \quad \Psi = \begin{pmatrix} u \\ v \end{pmatrix}. \qquad (3.1)$$

In (3.1) the small terms $(eAv_x/mc)^2$ are at once discarded. We shall seek the solution in the form

$$\Psi = (\mathbf{F_0} + \hbar\mathbf{F_1})\exp\left\{\frac{i}{\hbar}\int^y (p_0 + \hbar p_1)dy' + \frac{iP_xx}{\hbar} + \frac{iP_zz}{\hbar}\right\}. \qquad (3.2)$$

Substitution of (3.2) into (3.1) leads, in the zero-order approximation in $\hbar$, to the following homogeneous equation for $\mathbf{F_0}$:

$$(\hat{H}_0 - \varepsilon)\mathbf{F_0} = \begin{pmatrix} \dfrac{p_0^2}{2m} - \dfrac{p_F^2 - p_\parallel^2}{2m} - \rho - \varepsilon, & -\Delta \\ -\Delta, & -\dfrac{p_0^2}{2m} + \dfrac{p_F^2 - p_\parallel^2}{2m} - \rho - \varepsilon \end{pmatrix} \mathbf{F_0} = 0, \quad (3.3)$$

which contains p_0 as a parameter. The condition for the solvability of the homogeneous equation (3.3) determines p_0, coinciding with the "classical" $p_y(y)$ in Eq. (1.4); and the solution of (3.3) associated with this p_y gives the "classical" solutions for u and v. In the next approximation in $\hbar$ an inhomogeneous equation is obtained for $\mathbf{F_1}$

$$(\hat{H}_0 - \varepsilon)\mathbf{F_1} = \frac{p_0}{m}\hat{\sigma}_z\left\{i\frac{d\mathbf{F_0}}{dy} + \frac{i}{2}\mathbf{F_0}\frac{d}{dy}(\ln p_0) - p_1\mathbf{F_0}\right\},$$
$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (3.4)$$

Since the homogeneous equation (3.3) corresponding to (3.4) has a nontrivial solution $\mathbf{F_0}$, for (3.4) to have a solution the right-hand side (which depends on p_0 as on a parameter) must be orthogonal to the solution of the homogeneous equation which is the adjoint of (3.3), that is, to $\mathbf{F_0}$:

$$i\left\{\mathbf{F_0}\hat{\sigma}_z\frac{d\mathbf{F_0}}{dy} + \frac{1}{2}\mathbf{F_0}\hat{\sigma}_z\mathbf{F_0}\frac{d}{dy}(\ln p_0)\right\} = p_1\mathbf{F_0}\hat{\sigma}_z\mathbf{F_0}. \qquad (3.4a)$$

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[5] The separation of the electronic diamagnetic moment (more precisely, the moment associated with the diamagnetic orbital quantization) into a monotonic part (determined by the interval of the energies ϵ_F) and an oscillating part (connected with an interval of energies of the order of $\hbar\Omega$) is, of course, nonunique; however, it is nonunique only to the same extent as the inaccuracy (of order $\Lambda/r \sim a/r$) of the boundary condition itself.

[6] Of course, the "non-quasiclassical nature" in the sense of a summation does not in any way contradict the quasiclassical nature of the motion of the excitations.

From Eq. (3.4a) it follows that

$$p_1 = \frac{i}{2} \frac{d}{dy} \ln[\rho_0(F_0 \hat{\sigma}_z F_0)]. \qquad (3.5)$$

Substituting (3.5) into (3.2), in the basic approximation (in order to write down this approximation, knowledge of F_1 is not required) we obtain the natural quasiclassical formula

$$\Psi = \frac{C}{\gamma v_y} F_0 \exp\left\{\frac{i}{\hbar} \int^y p_y dy'\right\}, \quad F_0 = \begin{pmatrix} [{}^{1/2}(1 + \xi/(\varepsilon + \rho))]^{{}^{1/2}} \\ -[{}^{1/2}(1 - \xi/(\varepsilon + \rho))]^{{}^{1/2}} \end{pmatrix}. \quad (3.6)$$

In connection with the fact that the phase trajectory has several branches, finally one can write down the following answer for Ψ:

$$\Psi = C \sum_\alpha \frac{\exp\{i\varphi_\alpha\}}{\gamma v_y{}^\alpha} F_0{}^\alpha \exp\left\{\frac{i}{\hbar} \int^y p_y{}^\alpha dy'\right\}. \qquad (3.7)$$

For $y < y_1$ the summation in formula (3.7) goes over all values $p_y(y)$ which are classically possible, and for $y > y_1$ it goes over those $p_y(y)$ for which Im $p_y(y) > 0$, which guarantees attenuation of the wave function at infinity. In order to determine the phase factors $\exp(i\varphi_\alpha)$, it is necessary, as usual, to match the quasiclassical solutions to the right and to the left of the turning points y_0 and y_1, near which the quasiclassical solution is not admissible, and it is also necessary to take the boundary condition $\Psi|_{y=0} = 0$ at the surface of the metal into account. The constant C is found from the normalization conditions:

$$C^2 \sum_{\alpha, \beta} \int_0^\infty \frac{\exp\{i(\varphi_\alpha - \varphi_\beta)\}}{\gamma v_y{}^\alpha v_y{}^\beta} F_0{}^\alpha F_0{}^\beta \exp\left\{\frac{i}{\hbar} \int^y (p_y{}^\alpha - p_y{}^\beta) dy'\right\} dy = 1. \quad (3.8)$$

It is obvious that all of the integrals over y appearing in Eq. (3.8) are small for $\alpha \neq \beta$ (to the extent that the quasiclassical parameter $1/n$ is small, where n denotes the number of the level); with the same accuracy one can assume that the integration in (3.8) only goes over the region $0 \leq y \leq y_1$. As a result for C and Ψ_y we obtain

$$C^2 \sum_\alpha \int_0^{y_1} \frac{dy}{v_y{}^\alpha} = C^2 \oint \frac{dy}{v_y} = C^2 \frac{\partial S}{\partial \varepsilon}, \qquad (3.9)$$

$$\Psi = \frac{1}{\gamma |\partial S/\partial \varepsilon|} \sum_\alpha \frac{e^{i\varphi_\alpha}}{\gamma v_y{}^\alpha} F_0{}^\alpha \exp\left\{\frac{i}{\hbar} \int^y p_y{}^\alpha dy'\right\}. \qquad (3.10)$$

From the last discussion it is also seen that to the accuracy we are interested in the phase factors $\exp(i\varphi_\alpha)$ only affect the phase γ in the conditions for quasiclassical quantization (see Appendix 1).

Now let us find the quasiclassical temperature-dependent Green's function

$$G(\omega_s, P_x, P_z, y, y') = \begin{pmatrix} G(\omega_s) \\ F^+(\omega_s) \end{pmatrix}. \qquad (3.11)$$

Here $\omega_s = (2s+1)\pi T$ (s is a natural number). For this purpose let us utilize the following property of the Gor'kov equations, namely: $G(\omega_s) = g_1(i\omega_s)$ where $g_1(\omega)$ is the solution of the corresponding equations for the time-dependent Green's functions obtained under the condition that $\omega \neq \epsilon_n(P_x, P_z)$, where the ϵ_n denote the energy levels of the excitations in the superconductor. The equation for the determination of $g_1(\omega)$ has the same form as (3.1); it is only necessary to replace ϵ by ω, and for $y = 0$ and $y = y'$ it is necessary to estab-

lish the following boundary conditions: $g_1(0, y') = 0$, $g_1(y' - 0, y') = g_1(y' + 0, y')$,

$$\frac{ag_1}{dy}\bigg|_{y=y'+0} - \frac{dg_1}{dy}\bigg|_{y=y'-0} = \frac{2m}{\hbar^2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Solving Eq. (3.1) for g_1 is basically the same as for the determination of the wave functions; taking into consideration that $\omega \neq \epsilon_n$ one can easily obtain the following expression for the part of $g_1(\omega)$ of interest to us, the part connected with the quasiclassical discrete levels:

$$g_1(\omega) = F(\omega, y, y') / \sin(S/2\hbar - \pi\gamma); \qquad (3.12)$$

$F(\omega, y, y')$ is related to the wave functions in a simple manner and for $y = y'$ it depends on ω according to a power law; $S = S(\epsilon_n)|_{\epsilon_n=\omega}$ and $\gamma = \gamma(\epsilon_n)|_{\epsilon_n=\omega}$. Having utilized the relation (3.15), one can finally write down that

$$G(\omega_s) = F(i\omega_p, y, y') / \sin[S(i\omega_s)/2\hbar - \pi\gamma]. \qquad (3.13)$$

Since an approximate calculation of the thermodynamic characteristics in the presence of a gap in the energy spectrum is a rather delicate matter, it is helpful to have an idealized model, which admits an exact solution, as a check on the correctness of the results. Such a model is provided by the potential introduced by Pincus:[1]

$$A(y) = \begin{cases} H\delta, & 0 < y < \delta, \\ 0, & \delta < y < \infty. \end{cases} \qquad (3.14)$$

With the vector potential given by (3.14), the Gor'kov equation is, in essence, a system of two second-order equations with constant coefficients. Their solution, corresponding to G and F^+ being equal to zero on the boundary of the metal, is given by

$$G_\omega(y, y') = \begin{cases} \sum_r A_r \sin \alpha_r y, & 0 < y < y' < \delta \\ \\ \sum_r B_r e^{i\beta_r y}, & \delta < y' < y < \infty. \end{cases} \qquad (3.15)$$

Here the following notation has been introduced: $r = \pm 1$,

$$\alpha_r = \frac{1}{\hbar}[2m(\mu_\perp - r\overline{(i\omega + \rho)^2 - \Delta^2})]^{{}^{1/2}},$$

$$\beta_r = \frac{1}{\hbar}[2m(\mu_\perp + ir\overline{\omega^2 + \Delta^2})]^{{}^{1/2}}, \quad \text{Im } \beta_r \geq 0. \qquad (3.16)$$

The coefficients A_r and B_r are easily determined from the boundary conditions. In a sufficiently weak magnetic field, when not only in Eq. (3.11) but also for u and v one can restrict one's attention to the linear approximation in H, it is reasonable to use a somewhat more realistic (in the sense of analytic properties) model:

$$A(y) = \begin{cases} H(\delta - y), & 0 < y < \delta \\ 0, & \delta < y < \infty. \end{cases} \qquad (3.17)$$

4. QUANTUM OSCILLATIONS OF THE GAP AND OF THE CURRENT DENSITY

The gap and the current density in the superconductor are related in the well-known way to the Green's functions F^+ and G.[9] However, as will be evident from what follows, for calculations of the quantum oscillations of

these quantities it is convenient to express them in terms of the wave functions of the excitations.

The corresponding formulas have the form

$$\Delta = |\lambda| \sum_{states} u_n{}^* v_n n_F(\varepsilon_n),$$

$$j_x = j = \frac{e}{m} \sum_{states} \left(P_x - \frac{e}{c} A_x \right) |u_n|^2 n_F(\varepsilon_n). \quad (4.1)$$

In each of the relations (4.1) there is an integration over the continuum and a summation over the discrete spectrum. As already discussed, the oscillating term of interest to us is contained only in the summations over the discrete states. For their computation we shall use Poisson's formula,[10] and the wave functions of the excitations and the energy spectrum we take in the quasiclassical approximation. As a result, for the oscillating correction to the current density we have

$$j_1 = \frac{e}{2\pi^2 m \hbar^2} \operatorname{Re} \sum_{k=1}^{N_{max}} \int dP_x dP_z \int_{N_{min}}^{} P_x |u_n|^2 n_F(\varepsilon_n) e^{i 2\pi k n} dn, \quad (4.2)$$

N_{min} and N_{max} are the minimum and maximum values for the number of levels associated with fixed values of P_X and P_Z. Now in the integral over n let us change the variable according to the formula $n = S(\epsilon, P_X, P_Z)/2\pi\hbar$, after which one can easily perform the integration over the energy under the following assumptions: $|\epsilon_{min,max}| \gg T$, sign $\epsilon_{min} = -$ sign $\epsilon_{max},$[7] where ϵ_{min} and ϵ_{max} denote the upper and lower boundaries of the discrete spectrum for fixed values of P_X and P_Z. As a result we find

$$j_1 = \operatorname{Re} \sum_{k=1}^{\infty} \frac{1}{4\pi^3 \hbar^2 k} \int_0^{\infty} dP_x \int_{-\infty}^{\infty} dP_z \frac{e}{m} P_x |\Psi|^2_{\varepsilon=0} \theta(k\lambda) \exp \left\{ i 2\pi k n_{\varepsilon=0} - \frac{i\pi}{2} \right\},$$

$$\theta(x) = \frac{x}{\operatorname{sh} x}, \qquad \lambda = 2\pi^2 T \frac{\partial n}{\partial \varepsilon} \Big|_{\varepsilon=0}. \quad (4.3)$$

The integration over P_X and P_Z, which appears in formula (4.3), is elementary to perform using the method of steepest descents, after which we obtain the following answer:

$$j_1 = \sum_{k=1}^{\infty} \frac{e P_x{}^{extr}}{m} \frac{|\Psi|^2_{extr}}{8\pi^3 \hbar^2 k^2} \theta(k\lambda_{extr})$$

$$\times \left(\left| \frac{\partial^2 n}{\partial P_x^2} \right|_{extr} \left| \frac{\partial^2 n}{\partial P_z^2} \right|_{extr} \right)^{-1/2} \cos(2\pi k n_{extr} + \varphi), \quad (4.4)$$

where P_X^{extr} and P_Z^{extr} are determined from the equations $\partial n/\partial P_X = 0$ and $\partial n/\partial P_Z = 0$; $\varphi = 0$ if $\partial^2 n/\partial P_X^2$ and $\partial^2 n/\partial P_Z^2$ have the same sign, and $\varphi \cong -\pi/2$ if they have different signs. Similar calculations for the gap Δ_1 lead to the following answer:

$$\bar{\Delta}_1 = \sum_{k=1}^{\infty} \frac{|\lambda|(uv)_{extr}}{4\pi^3 \hbar^2 k^2} \left(\left| \frac{\partial^2 n}{\partial P_x^2} \right|_{extr} \left| \frac{\partial^2 n}{\partial P_z^2} \right|_{extr} \right)^{-1/2} \cos(2\pi k n_{extr} + \varphi). \quad (4.5)$$

As already referred to in (2.2), the extremal cross sections correspond to excitations with average velocity equal to zero.

The absence of a gap in the spectrum was very essential in all of the calculations which have been car-

ried out. In the presence of a gap (i.e., in a magnetic field $H < H_2$) it is more convenient to write the energy gap and the current density in terms of the Green's function:[9]

$$j_x = \frac{T}{2\pi^2 \hbar^2} \int dP_x dP_z \sum_{\omega} \left(\frac{e P_x}{m} - \frac{e^2 A_x}{cm} \right) G(\omega),$$

$$\Delta = |\lambda| \sum_{\omega} \int dP_x dP_z \frac{T}{4\pi^2 \hbar^2} F^+(\omega). \quad (4.6)$$

From formulas (4.6) and (3.13) it follows that for $\rho < \Delta$ the series in (4.6) are represented by a sum of exponentially small terms, i.e., the series are exponentially small. In fact, as is clear from (3.17), if $\rho < \Delta$, then even for T = 0 one finds Im S $\neq$ 0 and G $\sim$ exp $\{-$ Im S$\}$.

Now let us demonstrate this same result for the exactly solvable model of Pincus. With the aid of (3.15) j_X and Δ are easily calculated according to formulas (4.6). One can represent G_ω as the sum of two terms: G_ω^{sm} —the smooth part and G_ω^{osc} —the oscillating part. One can verify that the current j_X, calculated by using G_ω^{sm}, is equal to the expression cited in [9] if the vector potential is taken as (3.14) plus a small smooth correction, connected with the effect of the surface. The function G^{osc} is proportional to exp$\{i\alpha_r\delta\}$, and for $|\rho| < \Delta$ (see (3.16)) $G_\omega^{osc} \sim$ exp$\{-\delta/(a\xi_0)^{1/2}\}$ —which is an exponentially small quantity. Thus, the oscillations exist in that region of magnetic field where $|\rho| > \Delta$. Similarly, for the temperature dependence of the amplitude of the oscillations ($|\rho| > \Delta$) we obtain

$$G_\omega^{osc} \sim \exp \left\{ -\frac{2\pi T \rho}{(\hbar^2/2m\delta^2)^{1/2}(\rho^2 - \Delta^2)^{1/2}} \right\}.$$

5. QUANTUM OSCILLATIONS OF THE SURFACE IMPEDANCE AND OF THE GAP

As is well known, the surface impedance $Z(\omega)$ is given by

$$Z(\omega) = \frac{4\pi i \omega}{c^2} \frac{A_\omega(0)}{B_\omega(0)} = \frac{4\pi i \omega}{c^2} \int_0^{\infty} B_\omega dy' / B_\omega(0). \quad (5.1)$$

Here $A_\omega(0)$ and $B_\omega(0)$ are the variable components, respectively, of the vector potential and of the magnetic induction on the boundary of the superconductor. For not too large frequencies, the j$\{B\}$ in Maxwell's equations for the determination of the field can be taken in the form j$\{B\} = \hat{Q}A + j\{B\}$, where $\hat{Q}$ is the operator connecting j and A in the "usual" case when no oscillations are present,[9] and $\tilde{j}$ is calculated according to formula (4.5). Let $B_\omega(0)$ be smaller than the period ΔH of the oscillations and, of course, smaller than the external constant magnetic field H. In this case the equation

$$\operatorname{rot} B + \operatorname{rot} B_\omega = \frac{4\pi}{c} \{\hat{Q}A + \hat{Q}A_\omega + \tilde{j}\} \quad (5.2)$$

can be easily solved in the linear approximation with respect to B_ω. From (5.2) we have the following equations for the determination of B and B_ω:

$$\operatorname{rot} \tilde{B} = \frac{4\pi}{c} \hat{Q}A + \frac{4\pi}{c} \tilde{j}\{B\}, \quad (5.3)$$

$$\operatorname{rot} B_\omega = \frac{4\pi}{c} \hat{Q}A_\omega + \frac{4\pi}{c} [\tilde{j}\{\tilde{B} + B_\omega\} - \tilde{j}\{B\}] = \frac{4\pi}{c} \hat{Q}A_\omega + \frac{4\pi}{c} (\delta\tilde{j}). \quad (5.4)$$

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[7] That is, the minimum energy of excitation in a superconductor is equal to zero.

In the last term of Eq. (5.4) it is necessary to separate out the part which is linear in B_ω. It is clear that in this connection it is necessary to keep the dependence on B_ω only in the argument of the cosine in (4.5). Let us consider the expression for n_{extr} in more detail: $2\pi \hbar n_{extr} = \oint p_y \{B - B_\omega\}\, dy$. For the determination of $p_y(y)$ we have the equation

$$\rho(y) + \rho_\omega(y) = \sqrt{\xi_p^2 + \Delta^2} = \varepsilon_0(p), \qquad (5.5)$$

where $\rho_\omega = (e/c) A_\omega (\partial \xi_p / \partial P_x)$. We shall seek p_y in the form $p_y = p_y^{(0)} + p_y^{(\omega)}$, where $p_y^{(0)}$ satisfies the relation $\rho(y) = \varepsilon_0(p)$; then from (5.5) it follows that $p_y^{(\omega)} = \rho_\omega / v_y$. Having substituted this expression into n_{extr}, we find

$$n_{extr}\{B + B_\omega\} = \frac{1}{2\pi\hbar} \oint_{extr} p_y^{(0)}\, dy + \frac{1}{2\pi\hbar} \oint_{extr} \frac{\rho_\omega}{v_y}\, dy. \qquad (5.6)$$

Having made use of formula (5.5) for the correction to the current density which is linear in B_ω and of interest to us, we obtain

$$(\delta \tilde{j}) = -\sum_{n=1}^{\infty} \frac{e P_x^{extr}}{m} \frac{|\Psi|_{extr}^2}{8\pi^3 \hbar^3 k} \theta(k\lambda_{extr})$$

$$\times \left(\left| \frac{\partial^2 n}{\partial P_x^2} \right|_{extr} \left| \frac{\partial^2 n}{\partial P_x^2} \right|_{extr} \right)^{-1/2} \oint \frac{\rho_\omega}{v_y}\, dy \sin(2\pi k n_{extr} + \varphi). \qquad (5.7)$$

We note that in formula (5.7) for ρ_ω and B one can use the basic approximation, that is, one can use the solution of the equations

$$\text{rot } B = \frac{4\pi}{c} \hat{Q} A, \qquad \text{rot } B_\omega = \frac{4\pi}{c} \hat{Q} B_\omega. \qquad (5.8)$$

From (5.8) it is clear that ρ_ω and ρ decrease inside the metal in the same way and differ only in their amplitudes, that is,

$$\rho_\omega = \frac{B_\omega(0)}{B(0)} \rho, \quad \text{and} \quad (\delta \tilde{j}) = B_\omega(0) \frac{\partial \tilde{j}}{\partial H}. \qquad (5.9)$$

In Eq. (5.4) we change to Fourier components, having continued the field B_ω onto the semi-axis $y < 0$ in an even manner. We recall that this corresponds to making the assumption about the spectral nature of the reflection of the excitations at the boundary of the metal. After this the equation is easily solved, and one obtains the following answer for the surface impedance:

$$Z = \frac{8i\omega}{c^2} \int_0^\infty \left[1 + \frac{4\pi^2}{c} \left(\frac{\partial \tilde{j}}{\partial H} \right)_q \right] (q^2 + Q(q))^{-1/2}\, dq, \qquad (5.10)$$

$$\left(\frac{\partial \tilde{j}}{\partial H} \right)_q = \frac{1}{\pi} \int_0^\infty \cos qy\, \frac{\partial \tilde{j}}{\partial H}\, dy.$$

We note that the attenuation depth of a static field (the solution of Eq. (5.3)) is given by

$$\delta_{st} = \frac{2}{\pi} \int_0^\infty \left[1 + \frac{4\pi^2}{Bc} \tilde{j}_q \right] (q^2 + Q(q))^{-1/2}\, dq. \qquad (5.11)$$

Now let us again return to the quantum oscillations of the gap Δ, which it is natural to represent in the following form:

$$\Delta = \Delta_0(y) + \tilde{\Delta}(y), \qquad (5.12)$$

where $\tilde{\Delta}(y)$ denotes the corresponding oscillating part. For the calculation of both Δ_0 and $\tilde{\Delta}$ one must use the self-consistency condition:[9]

$$\Delta = |\lambda| F_{+\to+0}^+(y, y). \qquad (5.13)$$

The Green's function $F^+\{\Delta(y)\}$ in turn is the sum of the contribution $F_0^+\{\Delta(y)\}$ from the continuous part of the spectrum and the contribution $\tilde{F}^+\{\Delta(y)\}$ from the discrete part of the spectrum. During the calculation of $F^+\{\Delta(y)\}$ one can neglect the small correction $\tilde{\Delta}(y)$; it is precisely this quantity $\tilde{\Delta}_1(y) = \tilde{F}^+\{\Delta_0\}$ which was calculated in Sec. 4. As a result, the following equation is obtained for the determination of the gap:

$$\Delta(y) + \tilde{\Delta}(y) = F_0^+\{\Delta_0(y) + \tilde{\Delta}(y)\} + \tilde{\Delta}_1(y). \qquad (5.14)$$

The solution of (5.14) in a magnetic field of arbitrary magnitude is an extremely complicated problem. However, if it is assumed that $H_2 < H < H_c$, then it is found that in the basic approximation one can use the "usual" expression for F_0^+;[9] after this it is not difficult to obtain the result that $\tilde{\Delta} = \tilde{\Delta}_1 \ln(\Theta_D / \Delta_0)$;[8] having substituted here the value of $\tilde{\Delta}_1$ from (4.5), we finally obtain

$$\tilde{\Delta}(y) =$$
$$= \sum_{k=1}^{\infty} \frac{u_{extr} v_{extr} \hbar}{2mp_F k^2} \left(\left| \frac{\partial^2 n}{\partial P_x^2} \right|_{extr} \left| \frac{\partial^2 n}{\partial P_z^2} \right|_{extr} \right)^{-1/2} \theta(k\lambda_{extr}) \cos(2\pi k n_{extr} + \varphi). \qquad (5.15)$$

6. DISCUSSION OF THE RESULTS

First of all let us consider the temperature dependence of the amplitude of the oscillations. As is clear from formula (5.10), the temperature dependence is determined by the function $\theta(k\lambda_{extr})$. If the temperature $T > (\delta\varepsilon)_{extr}/2\pi^2$, then the amplitude of the oscillations is proportional to

$$[4\pi^2 T / (\delta\varepsilon)_{extr}] \exp\{-2\pi^2 T / (\delta\varepsilon)_{extr}\},$$

i.e., it is exponentially small. For low temperatures

$$(T < (\delta\varepsilon)_{extr}/2\pi^2), \qquad \theta(k\lambda_{extr}) = 1.$$

Now let us estimate the relative amplitude, for example, of the oscillations of the surface impedance. It is most convenient to do this by assuming that the dispersion law is quadratic, and the field decreases deep inside the metal according to the exponential law $\exp\{-y/\delta\}$. Then $p_y = \pm p_\pm$ (see Eq. (1.4)), and all of the quantities appearing in (4.4) are easily calculated in explicit form. As a result we obtain

$$n_{extr} = \frac{\sqrt{2m\rho_0}\delta}{\pi\hbar}, \quad \left(\frac{\partial^2 n}{\partial P_z^2} \right)_{extr} = -\frac{n_{extr}}{2p_F^2}, \quad \left(\frac{\partial^2 n}{\partial P_x^2} \right)_{extr} = -\frac{9n_{extr}}{p_F^2} \frac{\varepsilon_F^2}{\rho^{4/3}\Delta^{2/3}},$$

$$\left(\frac{\partial n}{\partial H} \right)_{extr} = \frac{n_{extr}}{2H}, \quad \left(\frac{\partial n}{\partial \varepsilon} \right)_{extr} = \frac{n_{extr}}{\rho};$$

here it is assumed that $\rho(y)|_{y=0} = \rho_0 > \Delta$. With the aid of these formulas we obtain the following result for the relative amplitude of the oscillations of the surface amplitude, $\Delta Z/Z$:

$$\frac{\Delta Z}{Z} \sim 10^{-1} \frac{a}{\delta_L} \left(\frac{\xi_0}{\delta_L} \right)^{1/3} \theta \left(\frac{2\pi T}{(\rho_0 \hbar^2 / 2m\delta^2)^{1/2}} \right), \qquad (6.1)$$

where $a \sim \hbar/p_F$, $(1/\delta_L^2) = 4\pi Ne^2/mc^2$, and $\xi_0 = \hbar v_F/\pi\Delta$. From formula (6.1) it follows that $(\Delta \tilde{Z}/\tilde{Z}) \sim 10^{-3}$ to 10^{-5}, which is the same order of magnitude, obviously, as $\tilde{\Delta}/\Delta_0$.

Now let us consider an analogous estimate for the derivative of the surface impedance with respect to the

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[8]The similar local relation is, of course, an approximation since strictly speaking $\tilde{\Delta} = \tilde{\Delta} \{\tilde{\Delta}_1\}$. We further note that the smooth (in the sense of the dependence on given values of the gap and of the magnetic field) part of the current density also contains oscillations due to $\tilde{\Delta}$; however, this part is $(H/H_c)(\delta_L/\xi_0)^{2/3}$ times smaller than $\tilde{j}_1$.

magnetic field. At low temperatures $(T < T_c)$ the smooth part of the surface impedance depends on the magnetic field weakly due to the fact that the numerical coefficient of $(H/Z)(dZ/dH) \sim 10^{-2}$ to 10^{-4};[11] for the oscillating correction the corresponding quantity is of the order of

$$\frac{H_0}{Z}\frac{d\Delta Z}{dH_0} \sim 10^{-1}n_{extr}\frac{a}{\delta_L}\left(\frac{\xi_0}{\delta_L}\right)^{1/3} \sim 10^{-2} - 10^{-4}. \quad (6.2)$$

From here it is seen that the quantum oscillations of the derivative of the surface impedance with respect to the magnetic field may turn out to be dominant. If the dispersion law is cylindrical, then all of the relative amplitudes of the oscillations obviously turn out to be $(n_{extr})^{1/2}$ times larger.

Now let us return to the question of the possibility of establishing the dependences of the amplitude of the oscillations through the functions $(\partial n/\partial H)$ and $(\partial n/\partial \epsilon)$ (in what follows the subscript extr is omitted everywhere). Let us assume, for simplicity, that the dispersion law is known to us, and the field falls off inside the metal according to an exponential law. Then

$$\frac{\partial}{\partial H_0}\frac{\partial n}{\partial \epsilon} = \frac{\delta}{2\pi\hbar\rho_0}\frac{d}{dH_0}(p^{(1)}(0) - p^{(2)}(0)),$$

$$\frac{\partial n}{\partial H_0} = \frac{\delta}{2\pi\hbar H_0}(p^{(1)}(0) - p^{(2)}(0)), \quad (6.3)$$

where $p^{(1)}(0)$ and $p^{(2)}(0)$ are possible values of the momentum p_y of the excitations on the boundary of the superconductor. From the picture under consideration of the classical motion of the excitations, it is clear that the surface states are connected with those excitations whose velocity along the y axis is small. Making use of this property, it is not difficult to obtain the result that

$$p^{(1)}(0) - p^{(2)}(0) = 2\sqrt{2m^*(\mu_\perp + \sqrt{\rho_0^2 - \Delta^2})},$$

$$\mu_\perp = \xi_p(p_y^{(0)}, P_x, P_z), \quad m^* = \frac{\partial^2\xi_p}{\partial p_y^2}(p_y^{(0)}, P_x, P_z),$$

where $p_y^{(0)} = p_y^{(0)}(P_x, P_z)$ is the solution of the equation $(\partial\xi_p/\partial p_y) = 0$. From relations (6.3) it now follows that

$$\Delta(p) = \rho_0 m^*((\varphi g/m^*)^2 - 1)^{1/2}/\varphi g. \quad (6.4)$$

Here

$$\varphi = \frac{\pi H_0\hbar}{\delta}\frac{\partial}{\partial H_0}\left(\frac{\partial n}{\partial \epsilon}\right), \quad g = \frac{\pi H_0\hbar}{\delta}\frac{\partial n}{\partial H_0}$$

are directly measurable quantities. Formula (6.4) solves the problem which has been posed, provided the extremal point (P_x, P_z) is known. We note that with an accuracy of the order of $(\Delta/\rho(H_c))^2 \sim 5$ to 20%, we have the following equations for the determination of the values P_x and P_z of interest to us:

$$\frac{\partial\xi_p}{\partial p_v} = 0, \quad \frac{\partial^2\xi_p}{\partial p_v^2} = \varphi(H_c)g(H_c), \quad \frac{\partial\xi_p}{\partial P_x} = \frac{g(H_c)}{\varphi(H_c)}\frac{c}{2e\delta H_c}. \quad (6.5)$$

7. CONCLUSIONS

In this article the quantum oscillations of the energy gap, current density, and surface impedance in superconductors have been investigated, and the corresponding expressions (5.15), (4.4), and (5.10) have been derived for these oscillations.

It has been ascertained that the oscillations in superconductors have an essential singularity with respect to the magnetic field—said singularities being present only for $H > H_2$, that is, when the gap in the energy spectrum vanishes. It is shown (see expression (6.1)) that the relative amplitude of the oscillations of the gap and of the current density is determined by the quantity

$$\frac{1}{n}\left(\frac{a}{\delta_L}\right)\left(\frac{\xi_0}{\delta_L}\right)^{1/3}\exp\left\{-\frac{2\pi^2 T}{(\delta\epsilon)}\right\},$$

and for the surface impedance the relative amplitude is determined by

$$\left(\frac{a}{\delta_L}\right)\left(\frac{\xi_0}{\delta_L}\right)^{1/3}\exp\left\{-\frac{2\pi^2 T}{(\delta\epsilon)}\right\},$$

provided that the Fermi surface is spherical; the relative amplitude is $n^{1/2}$ times larger if the Fermi surface is cylindrical; here n denotes the number of discrete levels.

It is shown (see Sec. 2) that from the temperature dependence of $\partial n/\partial \epsilon$ and from the period $\partial n/\partial H$ of the oscillations one can, in principle, establish ξ_p and $\Delta(p)$ as functions of p. In Sec. 6, under the simplest assumptions, formula (6.4) is derived for the determination of $\Delta(p)$ from $(\partial n/\partial \epsilon)$ and $(\partial n/\partial H)$. The result is obtained that the condition on Maxwell's equations at the interface between the metal and vacuum is the continuity of B.

A general rule is found for determining the phase γ in the Bohr quantization conditions (see the Appendix): $\gamma = \frac{1}{2}$ if the number of turning points is even and $\gamma = -\frac{1}{4}$ if the number of turning points is odd.

APPENDIX

Let the parameters ϵ, P_x, and P_z be such that the classical trajectory corresponding to the motion of the excitation has the form (see Figs. 8a and 8b) where the quasiclassical approximation is applicable in the region $\{0, y_0\} < y < y_1$, but the point y_0 lies close to the surface of the metal, $y = 0$. Here $\{a, b\} = \max(a, b)$. Let us find the value of the phase γ in the Bohr quantization rules for such trajectories. With quasiclassical accuracy the equation for the component v of the wave function $\Psi = \binom{u}{v}$ has the form

$$\left\{\left(\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \mu_\perp\right)^2 + \Delta^2 - (\epsilon + \rho)^2\right\}v = 0. \quad (A.1)$$

First of all let us write the quasiclassical solution in the interval $\{0, y_0\} < y < y_1$. It has the form

$$v = A_1\exp\left\{\frac{i}{\hbar}\int_0^y p_+dy'\right\} - A_2\exp\left\{-\frac{i}{\hbar}\int_0^y p_+dy'\right\} +$$

$$+ B_1\exp\left\{\frac{i}{\hbar}\int_{y_0}^y p_-dy'\right\} - B_2\exp\left\{-\frac{i}{\hbar}\int_{y_0}^y p_-dy'\right\}. \quad (A.2)$$

In order to determine the coefficients A_i and B_i, we match this solution with the asymptotic exact solution of Eq. (A.1) to the left of the point y_1. Near this point, to the accuracy we require Eq. (A.1) has the form

$$v^{IV} + 2v''k_\perp^2 + k_\perp^4 + \beta(y - y_1)v = 0.$$

$$k_\perp^2 = 2m\mu_\perp/\hbar^2,$$

$$\beta = -8\Delta\rho'(y_0)m^2/\hbar^4 > 0.$$

Solving this equation by Laplace's method, we obtain

$$v = C \int_L \exp\left\{\Delta y t + \frac{1}{\beta}\left(\frac{t^5}{5} + \frac{2}{3}k_\perp^2 t^3 + k_\perp^4 t\right)\right\}. \qquad (A.3)$$

We choose the contour L so that the solution (A.3) decreases at infinity. In order to achieve this, it is clear that it is sufficient to position the ends of the integration contour in the region where Re $t^5 > 0$. The asymptotic form of (A.3) to the left of the point y_1 is obtained by the method of steepest descents. In this connection, the saddle points are determined from the equation $\beta\Delta y + (t^2 + k_\perp^2)^2 = 0$, from which it follows that

$$t_{1,2,3,4} = \pm i(k_\perp^2 \pm \sqrt{-\beta\Delta y})^{1/2}.$$

The choice of contours and the saddle points are shown in Fig. 9. After simple calculations we obtain asymptotic expressions for the two linearly independent solutions of Eq. (A.1) in the region $y < y_1$:

$$v \approx C\left[\exp\left\{i\frac{\pi}{4} + \frac{i}{\hbar}\int_{y_1}^y p_+ dy'\right\} + \exp\left\{i\frac{3\pi}{4} + \frac{i}{\hbar}\int_{y_1}^y p_- dy'\right\}\right],$$
$$\qquad\qquad (A.4)$$
$$v \approx C'\left[\exp\left\{i\frac{3\pi}{4} - \frac{i}{\hbar}\int_{y_1}^y p_+ dy'\right\} + \exp\left\{i\frac{\pi}{4} - \frac{i}{\hbar}\int_{y_1}^y p_- dy'\right\}\right].$$

By comparing formulas (A.4) with the quasiclassical solution (A.2) we obtain two relations for the coefficients A_i and B_i.
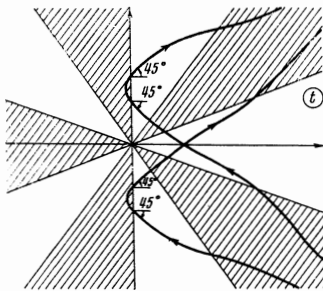

FIG. 9

Equations which are not sufficient for the determination of A_i and B_i follow from the boundary conditions on the surface of the metal. Having written the solution of Eq. (A.1) near y_0 in the form

$$v(y) = A\exp\left\{\frac{i}{\hbar}\int_0^y p_+ dy'\right\} - A\exp\left\{-\frac{i}{\hbar}\int_y^y p_+ dy'\right\} + v_1,$$

where v_1 satisfies the equation

$$v'' + \alpha(y - y_0)v = 0 \qquad (A.5)$$

and vanishes at the boundary of the sample, we also guarantee, with the necessary degree of accuracy, fulfilment of the boundary condition $u(0) = 0$, as is easy to verify. Here we have introduced the notation

$$\alpha = -\frac{2m\rho'(y_0)}{\hbar^2}\frac{\sqrt{\mu^2 + \Delta^2}}{\mu_\perp} > 0.$$

The solution of Eq. (A.5) of interest to us is obviously a linear combination of Airy functions. Matching the asymptotic form of the thus obtained exact solution with the wave function (A.2), we obtain two more relations between A_i and B_i. The system of equations for the determination of A_i and B_i has a nontrivial solution provided

$$S = 2\pi\hbar(n - 1/2 + \varphi/\pi), \qquad (A.6)$$

where

$$S = 2\left\{\int_0^{y_1} p_+ dy - \int_{L_0}^{y_1} p_- dy\right\}, \quad \text{tg}\,\varphi = \frac{\sin(\pi/12) - \lambda(y_0)\cos(\pi/12)}{\cos(\pi/12) - \lambda(y_0)\sin(\pi/12)},$$
$$\lambda(y_0) = -\frac{J_{-1/3}[^2/_3\sqrt{\alpha}\,y_0^{3/2}]}{J_{1/3}[^2/_3\sqrt{\alpha}\,y_0^{3/2}]}.$$

Let us consider a few special cases. Let y_0 be positive and large; then by using the asymptotic form of the Bessel functions one can easily obtain the result $\varphi = \pi/4$, from which it follows that for trajectories of type b on Fig. 8 we have $S = 2\pi(n - \frac{1}{4})$ (compare with [4]).[9] Similarly for trajectories of type a shown in Fig. 8, that is, if $y_0 < 0$ and large in absolute value, we find that $S = 2\pi(n - \frac{1}{2})$. We note that for an "ordinary" particle in the case of two turning points on the classical phase trajectory $\gamma = \frac{1}{2}$ (for a single turning point $\gamma = -\frac{1}{4}$).[7] Thus we see that in the Bohr quantization rules the value of the phase is determined by the evenness (or oddness) of the number of turning points both for the excitations in a superconductor as well as for "ordinary" particles and quasiparticles. For an arbitrary number of turning points the same statement is easily proved by induction.

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[9] An error was noted in [4], which was made in [2] in connection with writing down the general form of the transition matrix relating the coefficients in the wave function to the right and to the left of the turning points. This remark is based on a misunderstanding: the author of article [4] ignored the fact that the matrices in [2] and [4] differ only by an interchange of the rows, i.e., the components of the corresponding vector are relabeled.

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