

FORMATION OF CURRENT SHEETS IN A PLASMA WITH A FROZEN-IN STRONG MAGNETIC FIELD

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The conditions for the appearance of current (neutral) sheets during slow movements of a plasma with a frozen-in strong magnetic field are investigated. It is shown that a necessary condition for the appearance of a current sheet in the two-dimensional case is the presence of a neutral line in the magnetic field. The structure of the field in the vicinity of the current sheet is investigated and a method is proposed for the determination of a field satisfying prescribed boundary conditions. It is shown that, in the general case, forward and backward currents exist in the sheet. Under special conditions, however, when the total current in the layer is matched in a specific manner to the external field, backward currents are absent and the structure of the field corresponds to that of the familiar self-similar solutions.

1. INTRODUCTION

THE special role played in cosmic plasma physics by magnetic neutral points and the current sheets that develop around them^[1-4], is now becoming increasingly clear. Under the conditions of the high conductivity and the immense spatial dimensions of the cosmic plasma, the magnetic neutral points are, apparently, the only place where a rapid dissipation of the magnetic field can occur. Furthermore, as a rule, this dissipation is accompanied, in an unbalanced fashion, by the excitation of nonthermal fluctuations in the plasma and nonthermal distributions of the energetic particles.

All this allows us to think that we are here getting to understand the mechanism underlying the effective transformation of the magneto-hydrodynamic energy of a plasma into the energy of accelerated particles—a process which is characteristic of cosmic physics and takes place, for example, in the Earth's magnetosphere during solar flares and during the generation of cosmic rays by distant sources. It is highly probable that a similar process is realized in some laboratory experiments with high-current discharges^[5].

A systematic theoretical analysis of the problem of the motion of a plasma in a nonhomogeneous magnetic field containing neutral lines meets with serious difficulties even in the simplest case of the magneto-hydrodynamic approximation. In this connection, such general questions as, for example, the conditions for the appearance of current sheets in a plasma, are still not completely clear. Analysis of the stability of equilibrium near the neutral lines^[2], the method of small perturbations^[6], and also the exact self-similar solutions^[7-9] permit us to conclude that, generally speaking, a current sheet is formed at the time of variation of the frozen-in magnetic field containing a neutral line. This deduction cannot, however, be considered as rigorously proven on account of the many limitations used.

Thus, in the small perturbations method, the current sheet development phase lies outside the limits of applicability of the method. As to the exact self-similar

solutions, they pertain to a limited class of boundary conditions, the possibility of realization of which is not quite clear. Moreover, these solutions are cut off at the moment of formation of a current sheet.

Finally, the properties of the current sheet, and the structure of the magnetic field associated with it, have not been sufficiently investigated under the conditions when the current sheet forms. Only the particular solution by Green^[10] is known for a current sheet in a plasma between two parallel currents.

In the present paper we propose, first, to elucidate the conditions for the appearance of current sheets in a strong magnetic field and their connection with the existence of neutral points. Secondly, we shall consider the properties of current sheets and the structure of the magnetic field in the neighborhood of the sheets.

We shall consider only plane two-dimensional magnetic fields and plasma motions. In this case the force-free field is simply a potential field, while the neutral points constitute neutral lines in space. Moreover, the plasma will be assumed to be a perfect conductor, and the freezing-in condition—to be exact.

2. THE STRONG MAGNETIC FIELD APPROXIMATION

We shall call the field strong if everywhere, with the possible exception of negligibly small neighborhoods of isolated points or lines, the condition

$$p / \rho V_A^2 \ll 1 \text{ and } V^2 / V_A^2 \ll 1 \quad (1)$$

is satisfied. Here, p and ρ are the pressure and density of the plasma, V and $V_A = H / \sqrt{4\pi\rho}$ are characteristic values for the plasma and Alfvén velocities. In this approximation the equations of magnetohydrodynamics for a planar two-dimensional motion of a perfectly conducting plasma reduce to the following system^[6,11] (we set $p = 0$, which is not important for what follows):

$$\frac{dA}{dt} \equiv \frac{\partial A}{\partial t} + v \nabla A = 0, \quad (2)$$

$$\Delta A = 0, \quad (3)$$

$$\left[\frac{dv}{dt} \nabla A \right] = 0, \quad (4)^*$$

$$d\rho/dt = -\rho \operatorname{div} v. \quad (5)$$

Here $A(x, y, t)$ is the **only nonvanishing** component of the vector potential and **all** the quantities are assumed to be independent of the coordinate z . The magnetic intensity vector \mathbf{H} is, by definition, equal to

$$\mathbf{H} = \left\{ \frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0 \right\}. \quad (6)$$

Note that for a fixed t , the lines $A(x, y, t) = \text{const}$ are the lines of force of the magnetic field.

At first glance, the solution to the system (2)–(5) is completely determined if these equations are supplemented by the value of $A(\mathbf{x}, y, t)|_S$ at the boundary S of some region in the (x, y) -plane, and the values of the density $\rho(x, y, t_0)$ and components of the velocity along the lines of force $\mathbf{v}_{\parallel} = \mathbf{v} - (\mathbf{v} \nabla A) \nabla A / |\nabla A|^2$ (the velocity component transverse to the lines of force is, for specified $A(x, y, t)$, found from Eq. (2)). Indeed, we may then determine for any moment of time $A(x, y, t)$ from Eq. (3), $\mathbf{v}(x, y, t)$ from Eqs. (2) and (4) and $\rho(x, y, t)$ from Eq. (5).

In reality, however, the solutions of the system (2)–(5) do not always exist. We are, of course, here concerned with solutions corresponding to a continuous deformation of the magnetic field and continuous translations of the plasma in which there is a one-to-one correspondence between the coordinates of each plasma particle at any two moments of time.

Before proceeding to the formulation of the corresponding conditions, we introduce, for future convenience, the complex potential $F(z)$. Precisely because the potential A , on account of Eq. (3), is a harmonic function of the coordinates, we may associate with it in the complex plane $z = x + iy$ a function $F(z)$ which is analytic in the considered region, by setting

$$F(z, t) = A(x, y, t) + iB(x, y, t). \quad (7)$$

Here, the time t is considered as a parameter, while the conjugate harmonic function $B(x, y, t)$ is determined in the usual way:

$$B(x, y, t) = \int \left(-\frac{\partial A}{\partial y} dx + \frac{\partial A}{\partial x} dy \right) + B_0(t) = -\int \mathbf{H} d\mathbf{l} + B_0(t), \quad (8)$$

where $B_0(t)$ is a quantity which does not depend on x and y , and the definition (6) has been used. We also have in accordance with (6) and (7)

$$dF/dz = -H_y - iH_x. \quad (9)$$

3. CONDITIONS FOR THE NONEXISTENCE OF CONTINUOUS SOLUTIONS

It is easy to obtain this condition directly from (2) and (3). Indeed, on account of Eq. (3), the potential A is uniquely determined by its boundary values, which, generally speaking, are assigned quite arbitrarily. These boundary values may be such that the corresponding potential A has magnetic neutral points (points at which $\mathbf{H} \equiv |\nabla A| = 0$) including neutral points at which the electric field proportional to the quantity $\partial A/\partial t$, is

different from zero. However, the existence of such neutral points is not compatible with the freezing-in equation (2), according to which the electric field $\mathbf{E} = -c^{-1} \partial A/\partial t$ should vanish at the neutral points of the magnetic field. Thus, we come to the following conclusion: the freezing-in condition allows a continuous deformation of the strong magnetic field and, corresponding to it, a continuous motion of the plasma everywhere except at the magnetic neutral points where the electric field is not equal to zero.

We make here the following observations. The first one pertains to the conditions (1) of applicability of the approximation employed here. It is easy to see that these conditions cannot be fulfilled at the neutral points since $V_A = 0$ at these points. Therefore, it would appear that the neutral points should, beforehand, be excluded from the analysis. However, not all the neutral points turn out to be "dangerous," but only those at which the field $\mathbf{E} \neq 0$; besides, the properties of the system (2)–(5) are of interest regardless of how this system was obtained. The main thing, however, is—and this constitutes the second observation—that the conclusion about the peculiar role of neutral points is based on the freezing-in equation only and is not, therefore, connected with the approximation used. Furthermore, this conclusion holds for arbitrary three-dimensional motions, as follows from the general freezing-in equation, which is taken in the form

$$\mathbf{E} + \frac{1}{c} [\mathbf{v} \mathbf{H}] = 0. \quad (10)$$

The significance of the employed approximation, (1)–(5), lies in the fact that it shows how the neutral points with $\mathbf{E} \neq 0$ may actually be realized in virtue of the boundary conditions.

In the absence of neutral points, the solution of the system (2)–(5) does not present any fundamental difficulties. As an example, we mention the problem of plasma motion in a varying magnetic dipole field considered in^[11,12]. If, however, the magnetic field defined by Eq. (3) and by the boundary conditions has neutral points in the region under consideration, then, in conformity with the condition obtained above, two fundamentally different cases are possible: either the electric field vanishes at the neutral points, or it has a finite nonvanishing value.

For the sake of brevity, we shall call the neutral points of the first kind, at which the field $\mathbf{E} = 0$, simple, while the neutral points with $\mathbf{E} \neq 0$ we shall call singular.

4. DEFORMATION OF THE MAGNETIC FIELD IN THE VICINITY OF A SIMPLE NEUTRAL POINT

Let $z = \zeta(t)$ be a neutral point of the magnetic field the position of which, generally speaking, depends on time. It is easy to obtain a general expression for the complex potential $F(z)$ in the vicinity of a neutral point if we take into account the fact that $F(z)$ is an analytical function and, according to (9), $dF/dz|_{z=\zeta(t)} = 0$. Then, retaining the first nonvanishing term in the expansion in powers of $z - \zeta(t)$, we have

$$dF/dz = a(t) [z - \zeta(t)]^m \quad (11)$$

and

$$F(z, t) = \frac{a(t)}{m+1} [z - \zeta(t)]^{m+1} - \beta(t). \quad (12)$$

* $[d\mathbf{v}/dt] \nabla A \equiv (d\mathbf{v}/dt) \times \nabla A$.

Here, m is an integral number, known as the order of the neutral point (in three-dimensional space-line), while the complex coefficients $\alpha(t)$, $\beta(t)$, and $\zeta(t)$ depend on time in so far as the magnetic field in the region under consideration depends on it through the boundary conditions. The sign in front of $\beta(t)$ has been chosen such that for $\partial\beta/\partial t > 0$ the electric field on the neutral line

$$E = -\frac{1}{c} \frac{\partial A}{\partial t} \Big|_{z=\zeta(t)} = \frac{1}{c} \frac{d\beta}{dt}$$

is directed along the positive direction of the z axis of a right-handed system of coordinates.

Let us now prove the following proposition pertaining to simple neutral points: the order of the neutral points of a frozen-in magnetic field does not change during a continuous deformation of the field.

Indeed, by taking the freezing-in equation in the form

$$dH/dt = (H\nabla)\mathbf{v} - H \operatorname{div} \mathbf{v}, \tag{13}$$

we find that the magnetic field is always zero at some point in the moving plasma if it vanished at least at one moment of time at this point. For the n -th order derivative of the field the assertion is easily proved by differentiating (13) n times with respect to the coordinates.

Thus, simple neutral points cannot disappear or appear; they only participate in the convective motion of the plasma. This conclusion applies, of course, to only simple neutral points, since the freezing-in condition does not apply at singular neutral points and a continuous deformation of the field is impossible.

In the case of simple neutral points we have, in virtue of the condition $\mathbf{E} = c^{-1}\partial\mathbf{A}/\partial t = 0$, $\operatorname{Re} \partial F/\partial t|_z = \zeta(t) = 0$ or $\operatorname{Re} \beta(t) = \operatorname{const}$. Inasmuch as $\beta(t)$ has been determined except for an imaginary part (see (8)) and the constant in the expression for the potential is insignificant, the complex potential in the vicinity of a simple neutral line may be taken in the form

$$F(z, t) = \frac{\alpha(t)}{m+1} [z - \zeta(t)]^{m+1} \tag{14}$$

Notice that by the proper choice of the system of coordinates (of the translational and rotational motion), we can at any fixed moment t_0 make $\zeta(t_0) = 0$ and $\operatorname{Im} \alpha(t_0) = 0$. Therefore, we may, without loss of generality, choose the potential at the initial moment of time $t_0 = 0$ in the form

$$F(z_0, 0) = \frac{\alpha_0}{m+1} z_0^{m+1}, \tag{15}$$

where α_0 is a real coefficient.

We shall consider the potential $A(x, y, t)$ as a function of the Lagrange coordinates x_0, y_0, t , setting $x = x(x_0, y_0, t)$ and $y = y(x_0, y_0, t)$ or

$$z = z(z_0, t), \tag{16}$$

where $z_0 = x_0 + iy_0$ is the initial position of the fluid particle. The freezing-in equation (2) establishes the connection between the coordinates x and y of a plasma particle at the moment t and its initial coordinates x_0 and y_0 at the moment $t_0 = 0$:

$$A(x, y, t) = A(x_0, y_0, 0), \tag{17}$$

For the complex potential (14) and (15), Eq. (17) may be written as:

$$F(z, t) = F(z_0, 0) + iY(z_0, t), \tag{18}$$

where $Y(z_0, t)$ is some real function of the variables x_0, y_0, t . Its appearance is connected with the fact that the function (16) is not necessarily an analytic function of the variable z_0 and therefore the first integral (17) of the system (2)–(5) alone is not sufficient for its determination. The law of motion $z = z(z_0, t)$ is determined by a simultaneous integration of Eqs. (2)–(4).

That the representation $z(z_0, t)$ is not, generally speaking, an analytic function may be verified with the aid of (4). Indeed, since

$$\mathbf{v} = v_x + iv_y = \dot{x} + i\dot{y} = \dot{z}, \tag{19}$$

Eq. (4) may be written as

$$\operatorname{Im} \left(\dot{z} \frac{dF}{dz} \right) = 0. \tag{20}$$

The only analytic function satisfying this equation in a region with neutral points is

$$\dot{z} = \mathbf{v} = \operatorname{const}. \tag{21}$$

The expression (21) is a trivial solution to the system (2)–(5), corresponding to a uniform translation of the frozen-in magnetic field.

The existence of nontrivial solutions in the vicinity of a simple neutral line may be verified on the following example which allows a simple exact solution. Let us consider a potential of the form

$$A(x, y, t) = \frac{1}{\xi^2(t)} (x^2 - y^2), \tag{22}$$

where $\xi(t)$ is an unknown function of the time ($\xi(0) = 1$). Then, in virtue of the freezing-in equation (2), the components of the velocity should be equal to

$$\begin{aligned} v_x &= \frac{\dot{\xi}}{\xi} x + \frac{f}{\xi^2} y, \\ v_y &= \frac{f}{\xi^2} x + \frac{\dot{\xi}}{\xi} y, \end{aligned} \tag{23}$$

where f is for the moment an arbitrary function. These expressions satisfy the ‘‘transversality’’ condition (4) if $f = \operatorname{const}$ and

$$\xi\dot{\xi} = \sqrt{f^2 + C\xi^2}, \tag{24}$$

where C is a constant of integration. Equation (24) has the general solution

$$\xi^2 = \frac{1}{C} [(\xi_0 + Ct)^2 - f^2], \tag{25}$$

where $C = \xi_0^2 - f^2 \neq 0$ and ξ_0 denotes the initial value of $\xi = d\xi/dt$. For $C = 0$ a particular solution to Eq. (24) is

$$\xi^2 = 1 + 2\xi_0 t. \tag{26}$$

The expressions (22) and (23) together with $\xi(t)$ from (25) or (26) completely determine the variation law of the potential, and the plasma velocity field. According to Eqs. (5) and (23) the plasma density in the Lagrange coordinates varies then according to the law

$$\rho = \rho_0 / \xi^2, \tag{27}$$

where ρ_0 is the initial density at the moving point of the plasma.

5. SINGULAR NEUTRAL POINTS OF THE MAGNETIC FIELD AND CURRENT SHEETS

Let the external sources (currents) produce in the region being considered a field corresponding to a singular neutral point and described by the potential (12) with $\beta(t) \neq \text{const}$. Since the imaginary part of the potential (12) has been determined to within an arbitrary function of the time (see (8)), we henceforth set $\text{Im } \beta(t) = 0$, i.e., we assume $\beta(t)$ to be a real function. The values of the coefficients $\alpha(t)$, $\beta(t)$ and $\zeta(t)$ are determined by the spatial distribution and time dependence of the external sources. The simplest examples of neutral points are neutral points arising between two parallel linear currents and also neutral points in the case of a magnetic dipole oriented along a homogeneous magnetic field. In the first case, the coefficients in the expression (12) are simply expressible in terms of the current strength $I(t)$ in the conductors, the distance between them and the dimensions of the closing conductors, and in the second case—in terms of the magnetic moment of the dipole and the intensity of the external field^[6,13]. Examples of neutral points which are of interest in cosmic plasma physics are also given in^[3].

Strictly speaking, the potential (12) produced by the external currents can only serve as a boundary condition for our problem. However, a function having the prescribed form of (12) at the boundary of a region and regular in virtue of Eq. (3) in the whole region, is uniquely determined by its analytic continuation. Therefore, the analyticity condition requires that (12) be a solution to Eq. (3) in the whole region under consideration including the neutral point $z = \zeta(t)$.

At the same time, as has already been pointed out, such a solution cannot satisfy Eq. (2) if the neutral point is singular, i.e., if $\beta(t) \neq \text{const}$. It follows from this that the potential $F(z, t)$ cannot remain analytic in the vicinity of a neutral line as the external sources vary.

Physically this means that finite currents arise in the vicinity of singular neutral points during any arbitrarily slow variation of the external fields. As a result, Eq. (3) becomes inapplicable at these points.

Thus, if the field of the external sources has in the volume being considered singular neutral points, then there exist regions of non-analyticity of the solution of the system (2)–(5). Let us consider the nature of these regions.

In virtue of the supposition (1) a region of nonanalyticity may consist of only isolated points (linear currents) and cuts (plane currents).

Let us first consider the isolated singular points which correspond to linear currents flowing along neutral lines. The appearance of such currents corresponds to the additional singular potential

$$F_s = \sum_k \left\{ -\frac{2}{c} I_k(t) \ln [z - \zeta_k(t)] - \beta_k(t) \right\} \quad (28)$$

being added to the external potential (12). Here, $I_k(t)$ is the strength of the k -th linear current ($I_k(t) > 0$ if the current flows in the positive direction of the z -axis) and $\beta_k(t)$ is a quantity depending on the nature of the closure of the current at infinity.

It is easy to verify that the addition of the potential (28) leads to the appearance of closed lines of force in

the vicinity of a neutral point. Such lines, however, cannot be obtained by a continuous deformation of the initial potential field under the freezing-in condition. In this way the isolated singular points of the system (2)–(5) would have, in consequence, whole regions in which no continuous solution of the system (2)–(5) exists, which contradicts the conditions (1).

Let us consider this situation in greater detail with a simple example. Let the potential of the external currents and of the current arising on a neutral line has the form

$$F(z, t) = \frac{\alpha(t)}{2} z^2 - \beta(t) - \frac{2}{c} I(t) \ln z, \quad (29)$$

where at the initial moment $I(0) = 0$. Then the field at the initial moment corresponds to the simplest neutral point of the potential field (neutral point of the type X) while at the moment t there appear two X-type neutral points and a region of closed lines of force between them (Fig. 1). A "bifurcation," as it were, of the neutral point takes place.

On account of the freezing-in condition, the plasma cannot get into the region of closed lines of force since it would have to cross the boundary lines of force which pass through new neutral points. Therefore, a solution to the system (2)–(5) cannot be constructed not only for the point $z = 0$ but, also, for the entire region of closed lines of force.

Thus, it is impossible to construct for the system (2)–(5), with the help of isolated singular points, a solution that is continuous in the rest of the space. As a result, only the solutions with cuts remain admissible and we come to the following assertion: if the external field contains singular neutral lines, then current sheets appear in the plasma in the region of these lines.

The arrangement of the cuts, which correspond in the complex plane to current sheets, should be such that everywhere, outside the cuts, the boundary-value problem for Eqs. (2)–(5) has a continuous solution. The following rule may be formulated for the determination of the location of a cut.

A cut should pass through the initial neutral point and all the neutral points that arise when a linear current, which varies from zero to some finite value, is introduced at the initial neutral point. The direction of the linear current should coincide with the direction of the electric field

$$E = \text{Re} \left(\frac{1}{c} \frac{d\beta}{dt} \right), \quad (30)$$

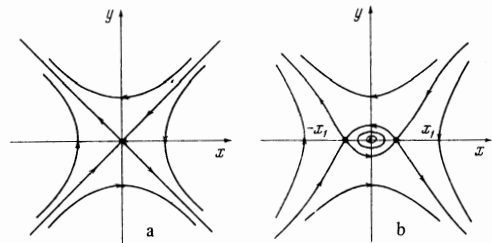


FIG. 1. On the appearance at a singular neutral point (a) of a linear current $I(t)$, the neutral point "bifurcates" (b). The resulting two new neutral points are, for $I(t)/\alpha(t) > 0$, situated on the x axis at the points $x = \pm [2I(t)/\alpha(t)]^{1/2}$ (see the expression (29)). However, this pattern of lines of force cannot be realized if the field is not frozen-in.

or, in the formula (29), the sign of $I(t)$ should coincide with the sign of $\beta(t)$ (it is assumed that $\beta(t)$ depends monotonically on time).

6. THE STRUCTURE OF THE FIELD IN THE VICINITY OF A SINGULAR NEUTRAL POINT

Let us consider a neutral point of order $m = n - 1$ produced by the external sources at some moment $t = t_1$. We may, without loss of generality, put $\xi(t_1) = 0$. Then, in accordance with (12), the potential of the external currents is equal to

$$F_e(z) = \frac{\alpha}{n} z^n - \beta. \tag{31}$$

The corresponding pattern of the lines of magnetic force for a neutral point of order four ($n = 5$) is represented schematically in Fig. 2 a.

If the neutral point is singular ($\beta(t) \neq \text{const}$), then the complex potential of the field in the plasma should include cuts which pass through the initial point $z = 0$ and the neutral points of the potential

$$F'(z) = \frac{\alpha}{n} z^{n-1} - \frac{2}{c} I(t) \ln z - \beta, \tag{32}$$

corresponding to a linear current in the region of the initial neutral line. The latter points are determined from the condition $dF'/dz = 0$ and, evidently, lie on the rays

$$\varphi = \varphi_k = \text{Arg} \left(\frac{2I}{c\alpha} \right)^{1/n} = \frac{\psi + 2\pi k}{n}, \quad |z| \leq |b|, \tag{33}$$

where $k = 0, 1, \dots, n - 1$ and $\psi = \arg(I/\alpha)$. On account of the symmetry, all the rays must have the same length; it is denoted by $|b|$.

Thus, the problem amounts to the determination of a complex potential $F(z, t)$ which is an analytic function of z in a plane from which a star of n equally spaced rays of length $|b|$ is excluded, and satisfies prescribed boundary conditions.

Let us consider these boundary conditions.

In the presence of the cuts (33) the function $F(z, t)$ is, generally speaking, not single-valued and its value is increased by $i\Gamma$ when it goes around the star, where, according to (8), the cyclic constant Γ is equal to

$$\Gamma = - \oint H dl = - \frac{4\pi}{c} I(t) \tag{34}$$

and $I(t)$ is the total current flowing through the star of cuts. The magnitude of this current depends on the external circuit (on the conditions under which the current

is closed through the plasma) and remains a free parameter, subject to the condition $I(0) = 0$. In the case of an open external circuit $I(t) = 0$.

At large distances from the cuts, when $z \gg b$, the field is determined by the external currents and the total current $I(t)$ in the region of the cuts. Consequently, the terms in the expansion of $F(z, t)$ in powers of b/z which do not vanish as $b/z \rightarrow 0$, should have the form

$$F(z, t) = \frac{\alpha}{n} z^n - \beta + \frac{\Gamma}{2\pi} \ln z - \frac{\Gamma}{4\pi} L_0, \tag{35}$$

where the constant L_0 is determined by the geometry of the current contour enclosing the star and the conductors that close it (the star).

Furthermore, as a boundary condition on the cuts of the star, we assume that the lines of magnetic force do not intersect the cuts, i.e.,

$$H_n|_z = \frac{\partial A}{\partial \tau} \Big|_z = 0 \quad \text{or} \quad A|_z = A(t). \tag{36}$$

Here $\partial/\partial\tau$ denotes differentiation along a cut and it is assumed that the value of the potential along the edges of the cuts is, generally speaking, a function of the time.

The condition (36) follows from the continuity of the lines of force, a property which is conserved, on account of the freezing-in condition, during the deformation of the lines. In this connection, we must, strictly speaking, assume that on the cut $A|_\Sigma = \text{const}$ and is time independent. Indeed, on account of Eq. (2) the variation of A at some point is equal to the magnetic flux transported through this point by the plasma. Therefore, a variation of A on the edges of a cut would mean that the magnetic field was transported into the current sheet but that its flux was not conserved. This would contradict the assumed freezing-in condition. Nevertheless, by allowing a slight deviation from the frozen-in state near the current sheets, we can expect the pattern of the field also to change insignificantly and we can apply the method developed here for the construction of the field. In this case the quantity

$$\Delta\Phi = A(t) - A(0) \tag{37}$$

determines the magnetic flux which disappears from each side of the current sheet.

The conditions (35) and (36) together with the analyticity condition for the potential completely determine the solution.

To obtain this solution, we consider the function

$$f(z, t) = F(z, t) - \frac{\alpha}{n} z^n - \frac{\Gamma}{2\pi} \text{Ln } z. \tag{38}$$

This function is single-valued, regular at infinity and assumes on the star-like cut Σ , the values

$$\text{Re } f(z, t) = \text{Re} \left\{ A(t) - \frac{\alpha}{n} z^n - \frac{\Gamma}{2\pi} \text{Ln } z \right\} \Big|_\Sigma. \tag{39}$$

Let us find this function. For this end, we map with the aid of the conformal transformation^[14]

$$w = \frac{1}{b} [z^{n/2} - \sqrt{z^n - b^n}]^{2/n}, \tag{40}$$

where $b = |b| \exp(i\varphi_0)$, $\varphi_0 = \psi/n$, the exterior of the star (33) unto the interior of the unit circle in such a way that to the successive angles of the star $z = |b| \exp(i\varphi)$, $0, |b| \exp [i(\varphi_0 + 2\pi/n)], 0, \dots$ correspond the points

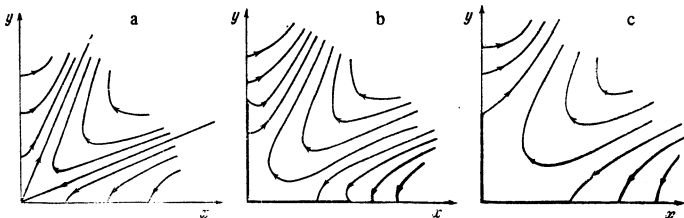


FIG. 2. a) Neutral point of the fourth order; only the portion of the lines of force occupying one quadrant is represented; b) the deformed field in the same region; the heavy lines represent the cuts; c) the same in the absence of backward currents (condition (47)).

$$w = 1, e^{in/n}, e^{i2n/n}, e^{i3n/n}, \dots$$

on the unit circle.

With the aid of the inverse transformation

$$z = \frac{b}{4^{1/n}} \frac{(1+w)^{2/n}}{w} \quad (41)$$

we find that the required function $f[z(w)] = \varphi(w)$ should satisfy on the circle $|w| = 1$ the condition:

$$\operatorname{Re} \varphi(w) \Big|_{|w|=1} = \operatorname{Re} \left\{ A(t) - \frac{\alpha b^n}{2n} (1+w^n) - \frac{\Gamma}{2\pi} \operatorname{Ln} \frac{b(1+w^n)^{2/n}}{4^{1/n}} \right\} \Big|_{|w|=1} \quad (42)$$

Here we have used the fact that on the circle $|w| = 1$,

$$w^{-1} = w^* \text{ and } \operatorname{Re} \{ \operatorname{Ln} w \} = 0.$$

We see from Eq. (42) that the function

$$\varphi(w) = A(t) - \frac{\alpha b^n}{2n} (1+w^n) - \frac{\Gamma}{2\pi} \operatorname{Ln} \frac{b(1+w^n)^{2/n}}{4^{1/n}} \quad (43)$$

is single-valued and analytic inside the unit circle and satisfies the necessary boundary conditions on the circle. Consequently, the function (43) is the required solution.

Returning with the aid of (38) and (40) to the function $F(z, t)$, we obtain finally

$$F(z, t) = \frac{\alpha}{n} z^{n/2} \sqrt{z^n - b^n} + \frac{\Gamma}{2\pi} \operatorname{Ln} \frac{[z^{n/2} + \sqrt{z^n - b^n}]}{b} + A(t). \quad (44)$$

Let us consider the properties of the function (44) which maps the z -plane from which the n -ray star (33) has been thrown unto the plane of the complex potential F (the time t is considered as a parameter). The points

$$z = |b| e^{i(\varphi_0 + 2\pi k/n)}, \quad k = 0, 1, \dots, n-1 \quad (45)$$

are branch points of the function (44). The introduction of cuts connecting these points and forming a star with center at the point $z = 0$, separates out the continuous and single-valued real part of this function, i.e., the potential $A(x, y, t)$ which is of interest to us. The imaginary part of $F(z, t)$ increases by the constant value $i\Gamma$ when it turns around the cut.

The derivative of the potential $F(z, t)$, which, according to (9), determines the components of the magnetic field, is equal to

$$\frac{dF}{dz} = \alpha z^{n/2-1} \frac{z^n - 1/2 b^n + \Gamma/2\pi\alpha}{\sqrt{z^n - b^n}}. \quad (46)$$

It has singularities at the ends of the cuts—at the points (45)—and, consequently, the intensity of the magnetic field at these points are unbounded. An exception is the special case

$$\Gamma = -\pi\alpha b^n, \quad (47)$$

which we shall consider somewhat later.

At the points

$$z = \left[\frac{|b|^n}{2} + \frac{1}{2\pi} \left| \frac{\Gamma}{\alpha} \right| \right]^{1/n} e^{i(\varphi_0 + 2\pi k/n)}, \quad k = 0, 1, \dots, n-1, \quad (48)$$

which, evidently, lie on the rays of the star (33), and also at the point $z = 0$ when $n > 2$, the derivative of (46) vanishes. These points are branch points of the inverse function $z = z(F)$ of (44) and correspond to the intersections of the equipotential lines $A = \text{const}$, i.e., of the lines of force of the magnetic field. For $|\Gamma| > \pi|\alpha b^n|$, these are simply neutral points of the first order lying

on the rays (33) but outside the cuts. Since new isolated neutral points cannot appear under the conditions when the field is frozen-in, it is necessary to assume

$$|\Gamma| \leq \pi|\alpha b^n|. \quad (49)$$

The pattern of the lines of magnetic force corresponding to the potential (44) is shown in Fig. 2 b.

A characteristic feature of the obtained solution is the presence of currents flowing in opposite directions on each ray and separated by the points (48). A current in the central section of the star up to the points (48), in accord with the above given rule, is in the direction of the external electric field. At the same time, currents at the borders of rays between the points (45) and (48) are in the opposite direction. Thus, the obtained solution points to the possibility of the appearance of backward currents and characteristic distortions of the magnetic field—the drawing off of the field by the edges of the current sheets.

In the particular case when the total current in a sheet $I = -c\Gamma/4\pi$ is so matched unto the external field that the condition (47) is fulfilled, the points (48) merge with the points (45) and the singularities in the derivative of (46) disappear. The last quantity becomes equal to

$$dF/dz = \alpha z^{n/2-1} \sqrt{z^n - b^n}. \quad (50)$$

In this case, the backward currents are absent and the intensity of the magnetic field H on the shores of the cuts are bounded. The corresponding pattern of the lines of magnetic force is shown in Fig. 2 c.

In Green's paper¹⁰¹ an expression was obtained for the magnetic field of two parallel linear currents with a plane current sheet between them. In the case when the width of the current sheet is considerably smaller than the distance between the currents, this expression reduces to formula (50) for $n = 2$.

7. DISCUSSION

We see that the neutral points of a strong magnetic field are those points in the plasma at which current sheets may develop under the action of an external electric field.

If the considered region of the plasma initially did not contain neutral points, then they may either appear at the boundary of the region and be brought into the region, or appear inside the region as a result of the finite conductivity and the corresponding violation of the freezing-in condition. Under cosmic conditions, the last process usually proceeds very slowly.

For a given strong external magnetic field (i.e., for given boundary conditions), the field in the plasma may be constructed with the aid of the following procedure.

It is necessary to position current sheets at the locations of singular neutral lines of the external field in such a way that they passed through all the secondary neutral lines which arise if a linear current, varying from zero to some finite value determined by the width of the current sheet, is placed along the initial neutral line. Then, the potential field (analytic function in the complex plane), satisfying the given boundary conditions at the boundary of the region and having a vanishing normal component of the magnetic field intensity on the

surface of a current sheet (on the cuts in the complex plane), is sought.

The procedure yields a solution to within the still undetermined length of the cuts (i.e., the width of the current sheets). Let us dwell on this question.

For $z \gg b$, the potential (44) is, to within terms of order $(b/z)^n$, equal to

$$F(z, t) = \frac{\alpha}{n} z^n - \frac{\alpha}{2n} b^n + \frac{\Gamma}{2\pi} \text{Ln} \frac{2z}{b} + A(t). \quad (51)$$

Comparing this expression with the condition (35), we obtain the equation

$$\frac{\alpha}{2n} b^n + \frac{\Gamma}{2\pi} \text{Ln} \frac{b}{2} - \beta - \frac{\Gamma}{4\pi} L_0 - A(t) = 0, \quad (52)$$

which relates the length of the cut rays $b = b(t)$ to the parameters of the external field $\alpha(t)$ and $\beta(t)$, to the total current through a cut $I = -c\Gamma/4\pi$, and to the quantity $A(t)$, which is equal to the magnetic field flux that disappears (is annihilated) in the current sheet.

In Eq. (52), besides $b(t)$, $A(t)$ and $\Gamma(t)$ are also known (the parameters $\alpha(t)$ and $\beta(t)$ are given since the external field is given). We do not see any way of determining these parameters without solving the more complete problem, including the stage when current sheets appear and taking into consideration the finite conductivity of the plasma. When the freezing-in condition is strictly fulfilled, we must assume $A(t) = 0$. However, this requirement is apparently too stringent since, in reality, a current sheet has a finite width and a finite conductivity and with its existence is inevitably connected a more or less rapid annihilation of the field^[15].

As regards the total current in a sheet or the coefficient $\Gamma(t)$, its magnitude depends on the detailed pattern of development of the current sheet in time (it can be shown that the current is transported to the initial neutral line by magnetosonic waves^[6]), as well as on the condition under which the current sheets are closed at infinity. In particular, the solution (44) is also valid in the absence of a total current in a sheet, i.e., when $\Gamma = 0$. This means that a sheet may exist, in principle, when the external circuit is open because of backward currents. In this case, if $A(t) = 0$ as well, Eq. (52) uniquely determines the width b of the sheet through the parameters of the external field $\alpha(t)$ and $\beta(t)$.

Evidently, the backward currents in the region of a neutral line arise and are maintained not directly on account of the external electric field \mathbf{E} , but as a result of the nonuniform motion of the plasma caused by this field. It may be thought therefore that during slow motions and also for finite, albeit large, conductivity of the plasma, the magnitude of the backward currents is small and the picture is close to the special particular case (47) (see Figs. 2c and 3b). In this case (under the condition that $A(t) \equiv 0$), the width of the sheet is also completely determined by the external field.

Thus, the width of a current sheet may be determined directly from (52) in the two limiting cases $\Gamma = 0$ and $\Gamma = -\pi\alpha b^n$. The real value of Γ and the corresponding width of the sheet, according to (49) and (33), lie between these limits.

It is interesting to note here that in the case of a neutral point of the first order (when $n = 2$), the solution (44) without backward currents, i.e., satisfying the con-

dition (47), corresponds to the magnetic field of the exact self-similar solution found in^[8,9] for the finite plasma cylinder at the moment when this solution is cut-off. To verify this, it is sufficient to compare the expression given in^[9] for the potential of the field outside the plasma cylinder at the moment $t = t_c$ with the real part of the potential (44). The same picture of the field is given when $t \rightarrow \infty$ by the solution^[7] for a finite cylinder of an incompressible liquid although the width b of a sheet then tends to infinity as well.

The general case of a neutral line of n -th order considered, requires for $b > 2a$ a very high symmetry of the sources of the field and can hardly be realized under real cosmic space conditions. However, such a general consideration may be of interest for laboratory experiments in which a high symmetry may be artificially achieved. Thus, for example, in high-current discharges the symmetry of the field depends on the shape of the discharge chamber. If the chamber is not cylindrical but, for example, rectangular or polygonal, then during the initial stage of the discharge, the current develops not in the form of a uniform cylindrical shell, but mainly at the most widely spaced parts of the chamber, i.e., in the external corners of its cross-section. The mutual attraction between these currents leads in the next stage of the discharge to a noncylindrical asymmetric pinch effect in which, depending on the symmetry of the chamber, a singular neutral line of one or another order develops.

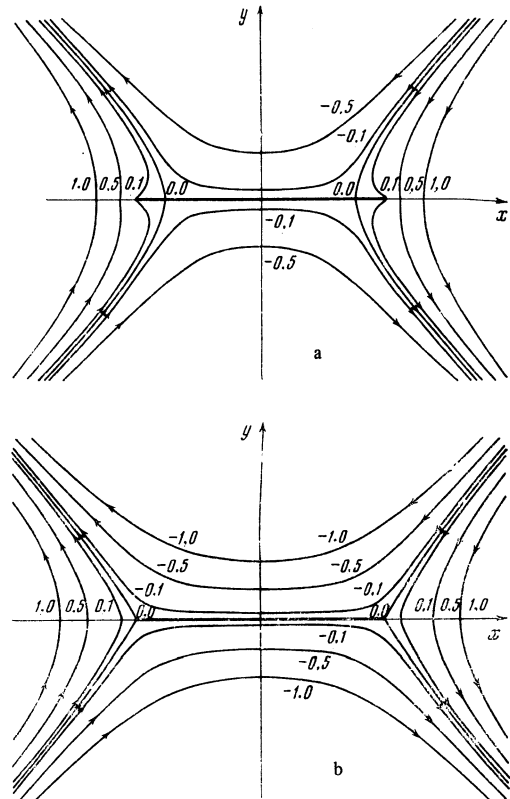


FIG. 3. Development of a current sheet from a singular neutral line of the first order: a depicts the general case, b—the case when backward currents are absent (cf. (47)). The heavy lines represent cuts. The thin curves—lines of magnetic force (i.e., the lines $A = \text{const}$). The values for A are indicated on the curves.

We may suppose that the distinctive geometrically correct pattern of plasma glow observed in^[16,17] during discharges in rectangular and hexahedral chambers is related to the process of development of current sheets at the locations of singular neutral lines considered above.

Under cosmic space conditions, and possibly in laboratory experiments with discharges in cylindrical chambers, during which asymmetries develop as a result of accidental perturbations (there is an indication to this in^[18]), the simplest first order neutral lines ($n = 2$) are usually realized. For them the potential of the developing current sheet and the intensity of the magnetic field are equal to

$$F(z, t) = \frac{\alpha}{2} z \sqrt{z^2 - b^2} + \frac{\Gamma}{2\pi} \operatorname{Ln} \frac{z + \sqrt{z^2 - b^2}}{b} + A(t) \quad (53)$$

and

$$\frac{dF}{dz} = -H_y - iH_z = \alpha \frac{z^2 - \frac{1}{2}b^2 + \Gamma/2\pi\alpha}{\sqrt{z^2 - b^2}}. \quad (54)$$

The pattern of the lines of force corresponding to the potential (53) in the case when $\Gamma/\pi\alpha b^2 = -0.2$ and $A(t) = 0$, is shown in Fig. 3a and in the special case when the condition $\Gamma = -\pi\alpha b^2$ (cf. (47)) is fulfilled—in Fig. 3b.

In conclusion, we note that the potentials (44) and (53) describe a current sheet developing at internal neutral points, i.e., neutral points which have been in existence from the very beginning inside the region under consideration. Current sheets, as is clear from the foregoing, may also develop from neutral lines located at the boundary of the region. The method expounded above for the determination of the field is applicable in this case as well. Examples of application to concrete problems of the physics of the solar atmosphere and of the magnetosphere will be considered in separate papers.

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