A SIMILARITY HYPOTHESIS IN THE STRONG INTERACTIONS. II. CASCADE PRODUCTION OF HADRONS AND THEIR ENERGY DISTRIBUTION ASSOCIATED WITH e^+e^- ANNIHILATION

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The energy distribution of the hadrons in the process of e^+e^- annihilation is studied on the basis of the hypothesis of similarity. The result contradicts the Bjorken formulas which were used in articles [3,4]. The law of similarity turns out to be valid only for the moments of the energy distribution (formula (3.14)). A qualitative description corresponding to the hypothesis of similarity is proposed. The question of the self-consistency of this hypothesis is discussed.

1. INTRODUCTION

In processes involving the lepton-hadron interaction, the leptons play the role of a "test body," with the aid of which one can investigate the internal structure of the hadron. Increasing the energy and momentum transferred from the leptons to the hadrons, one is able to investigate increasingly smaller space-time regions inside the hadron.

At the same time the properties of the strong interactions over small distances have scarcely been investigated theoretically. An attempt was made in the articles by Wilson[1] and by the author[2] (cited below as I) to use the hypothesis of scale invariance, which had previously been used in the theory of turbulence and in the theory of phase transitions, in order to describe such interactions. In I it was shown that in application to the process $e^+e^- \to$ hadrons the hypothesis enables us to make a number of predictions about the cross sections for the production of a given number of hadrons. The basic quantities considered in I were the amplitudes of the interaction of strongly virtual particles, which are self-similar functions of their own arguments.

At the same time, a knowledge of the amplitudes described above is inadequate for the solution of many problems associated with the lepton-hadron interactions.

Information about the interaction of virtual particles with real particles is required for the analysis of the essentially inelastic scattering of electrons by protons, for the investigation of the energy distribution in the e^+e^- annihilation process, and in certain other cases. In the present article the similarity problem is solved as applied to the calculation of the energy distribution of the hadrons associated with e^+e^- annihilation. The obtained results enable us to clarify the physical picture of the production of hadrons in this process and qualitatively reproduce the fundamental result of article I—namely, the power-law dependence of the multiplicity on the energy.

The problem considered in the present article was previously investigated in articles by Drell, Levy, and Yan,[3] and by Pestieau[4] with the aid of the assumption about the existence of the Bjorken limit.[5] Our results differ substantially from the results of articles [3,4] and, thus, the Bjorken conjecture[5] is not satisfied in a theory with scale invariance.

2. THE INTERACTION BETWEEN REAL AND VIRTUAL PARTICLES

The probability for the production upon e^+e^- annihilation of n hadrons with momenta $k_1, \ldots, k_n$ is obviously proportional to the quantity (see I)

$$dW_n^{\alpha\beta} = \frac{1}{n!}\Gamma_n^{\alpha}(k_1\ldots k_n)\dot{\Gamma}_n^{\beta}(k_1\ldots k_n) \times \delta\left(q-\sum k_i\right)\prod_j \delta(k_j^2-m_j^2)d^4k_j, \quad (2.1)$$

where q denotes the total momentum of the e^+e^-, and $\Gamma_n^\alpha = \langle 0 | J^\alpha | k_1 \ldots k_n \rangle$ is the electromagnetic current of the hadrons. Two quantities suitable for observation are easily expressed in terms of the distribution (2.1): the cross section for annihilation provided that a given hadron (for example, a proton) with a given momentum p is detected, and the probability to detect a particle with momentum p. For the corresponding cross section we have[3]

$$\frac{d^2\sigma}{dE\,d\cos\vartheta} = \frac{4\pi\alpha^2}{(q^2)^4}\frac{M^2\nu}{\sqrt{q^2}}\sqrt{1-\frac{q^2}{\nu^2}}\Big[2W_1(q^2,\nu) + \frac{2M\nu}{q^2}\left(1-\frac{q^2}{\nu^2}\right)\frac{\nu W_2}{2M}\sin^2\vartheta\Big], \quad (2.2)$$

where $M\nu = (pq)$ and W_1 and W_2 are defined by the relation

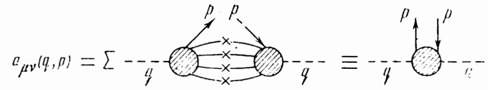
$$\sum_n\int\sum_{i=1}^n\delta(p-k_i)\,dW_n^{\mu\nu} \equiv a_{\mu\nu}(q,p) = -W_1\left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)+\frac{W_2}{M^2}\left(p_\mu-\frac{(pq)}{q^2}q_\mu\right)\left(p_\nu-\frac{(pq)}{q^2}q_\nu\right). \quad (2.3)$$

The normalized probability for the detection of a hadron with a given momentum p is expressed by the same formula (2.2), but with the functions $\widetilde{W}_1$ and $\widetilde{W}_2$ defined by the amplitude

$$\tilde{a}_{\alpha\beta}(q,p) = \left(\frac{8\pi\alpha^2}{q^2}\sum_n\int dW_n^{\mu\mu}\right)^{-1}\sum_n\frac{1}{n}\int\sum_{i=1}^n\delta(p-k_i)\,dW_n^{\alpha\beta}. \quad (2.4)$$

instead of (2.3).

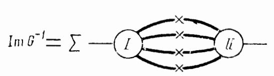
In diagram notation the relation appears in the following form:

 (2.5)

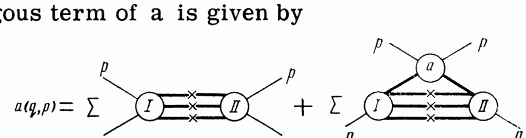
and therefore $a_{\alpha\beta}$ is related to the amplitude for the interaction of a virtual photon with a real hadron. In order to calculate this quantity it is convenient to first determine the amplitude for the interaction of a real and a virtual hadron, which is defined in analogy with expression (2.5).

The quantity a(q, p) is proportional to the probability to find a hadron with momentum p among the products of the decay of a virtual particle with momentum q. According to I, one can represent the decay of a virtual particle into real hadrons as the formation of several other virtual particles (preliminary decay into "jets") and their subsequent decay into real hadrons. Therefore, a(q, p) is equal to the probability for the decay of the virtual particle (jet) with momenta $k_1, \ldots, k_N$ ($\Sigma k_i = q$), multiplied by $\sum_{i=1}^{N} a(k_i, p)$, that is, by the sum of the probabilities to find the appropriate hadron among the i-th jet. As was shown in I, the probability of the decay into jets is a self-similar function of its own arguments and, as will be shown below, this property enables us to find a number of relations for a(q, p).

In accordance with what has been said above, one can anticipate that the expansion of a(q, p) in jets is constructed according to the following rule: Any term of the expansion of Im G^{-1} by jets is taken (see I) and each line, corresponding to Im G, is either broken or replaced by $|G|^2 a(q, p)$. In other words, if some term of the expansion of Im G^{-1} has the form

 (2.6)

(where the blocks I and II are constructed according to the rules given in article I out of amplitudes that cannot be divided by cutting a single line; Im G is associated with the heavy lines containing a cross), then the analogous term of a is given by

 (2.7)

(where the left heavy lines without any crosses are associated with G(k), and the right ones with G*(k)). It is not difficult to see that it follows from the rules for the construction of blocks I and II that the lower block in (2.7) cannot be cut with respect to two lines, and Eq. (2.7) is the Bethe-Salpeter equation, where the contribution of the broken lines forms the inhomogeneous term.

In analytic form (2.7) appears as follows:

$$a(q,p) = V(q,p) + \int V(q,k)\,|G(k)|^2 a(k,p)\,d^4k, \tag{2.8}$$

where

$$V(q,k) = \sum \frac{1}{(N-1)!}\int \Gamma_{\rm I}(q,k,k_s)\Gamma_{\rm II}^*(q,k,k_s)\cdot$$
$$\times\,\delta\Big(q-k-\sum k_s\Big)\prod_{s=1}^{N-1} d^4k_s\,\mathrm{Im}\,G(k_s).$$

Introducing the amplitude $\alpha(q, p) = (\mathrm{Im}\,G^{-1})^{-1} a(q, p)$, one can rewrite relation (2.8) in the form

$$\alpha(q,p) = \frac{V(q,p)}{\mathrm{Im}\,G^{-1}(q)} + \sum \frac{1}{N!}\int \Gamma_{\rm I}(q,k_i)\Gamma_{\rm II}^*(q,k_i) \tag{2.9}$$
$$\times\,\delta\Big(q-\sum k_i\Big)\sum_j \alpha(p,k_j)\prod_{s=1}^{N} d^4k_s\,\mathrm{Im}\,G(k_s)\,\frac{1}{\mathrm{Im}\,G^{-1}(q)}.$$

The quantity $\alpha(q, p)$ satisfies a series of sum rules which make it possible to solve Eq. (2.9) for large values of q^2 and (pq). The first two sum rules follow directly from the definition of α and have the form

$$\int \alpha(q,p)\,\delta(p^2-m^2)\frac{d^4p}{(2\pi)^3} = \bar n(q^2) \backsim (q^2)^\delta,$$
$$\int \alpha(q,p)\,p_\alpha\,\delta(p^2-m^2)\frac{d^4p}{(2\pi)^3} = q_\alpha. \tag{2.10}$$

Here $\bar n \backsim (q^2)^\delta$ is the average multiplicity of the production of hadrons, which was introduced in I.

For the proof let us substitute the definition of α into (2.10)

$$\alpha(q,p) = \frac{1}{\mathrm{Im}\,G^{-1}}\sum_n \frac{1}{(n-1)!}\int |\Gamma(k_1\ldots k_n,p)|^2$$
$$\times\,\delta\Big(q-p-\sum k_i\Big)\prod_{j=1}^{n-1}\delta(k_j^2-m_j^2)\,d^4k_j$$

and in the first equation of (2.10) we write $[(n-1)!]^{-1} = n/n!$, and in the second equation, by using the symmetry of the quantity $\Gamma(k_1\ldots k_n)$ we substitute the quantity

$$\frac{1}{n}\Big(p+\sum_1^{n-1} k_i\Big)_\alpha = \frac{q_\alpha}{n}.$$

instead of p_α. By comparing the obtained expressions with the unitarity condition for Im G^{-1}, we arrive at formulas (2.10).

The meaning of the formulas (2.10) becomes quite clear if we recall that $\alpha(q, p)$ is the average number of real hadrons with momentum p which are produced as the result of the decay of a virtual hadron with momentum q. In this connection the second sum rule in (2.10) indicates that the average energy of the produced hadrons is equal to the energy of the initial hadron as a consequence of a conservation law.

In order to obtain detailed information about $\alpha(q, p)$ let us calculate the higher moments of this distribution. Since the conservation laws do not restrict the energy fluctuations, it is necessary to use the hypothesis of similarity in order to solve this problem.

We define the average value of the product of the hadron momenta $T^{(n)}_{\alpha_1\ldots\alpha_n}$ by the formula

$$T^{(n)}_{\alpha_1\ldots\alpha_n}(k)\,\mathrm{Im}\,G(k) = k_{\alpha_1}\ldots k_{\alpha_n}\delta(k^2-m^2)$$
$$+\,|G(k)|^2\int p_{\alpha_1}\ldots p_{\alpha_n}\,a(k,p)\,\delta(p^2-m^2)\,d^4p. \tag{2.11}$$

It is obvious that, as a consequence of (2.10) the following equations are valid for $k^2 > m^2$:

$$T^{(0)}(k) = \bar n(k^2),\quad T^{(1)}_\alpha(k) = k_\alpha.$$

For the higher moments $T^{(n)}$ the equation

$$T^{(n)}_{\alpha_1\ldots\alpha_n}(q)\,\mathrm{Im}\,G^{-1}(q) = \int V(q,k)\,\mathrm{Im}\,G(k)\,T^{(n)}_{\alpha_1\ldots\alpha_n}(k)\,d^4k. \tag{2.12}$$

follows from (2.8). It is convenient to investigate Eqs. (2.12) by introducing quantities B_n which are linear combinations of the $T^{(n)}$, transforming according to the irreducible representations of the Lorentz group:

$$B_n(k)U_n\left(\frac{k_0}{\sqrt{k^2}}\right) = U_n\left(\frac{k_0}{\sqrt{k^2}}\right)\delta(k^2-m^2) \quad (2.13)$$
$$+|G(k)|^2\int U_n\left(\frac{p_0}{m}\right)a(k,p)\delta(p^2-m^2)d^4p.$$

Here the $U_n(x)$ are the Tchebichef polynomials of the second kind. Let us demonstrate that the B_n are Lorentz invariant. We express the vectors k and p in (2.12) in terms of the quantities α, β, and ν according to the formulas

$$k = \sqrt{k^2}(\mathrm{ch}\,\alpha,\ \mathbf{n}\,\mathrm{sh}\,\alpha),\quad p = m(\mathrm{ch}\,\beta,\ \mathbf{n}'\,\mathrm{sh}\,\beta),$$
$$(pk) = m\sqrt{k^2}(\mathrm{ch}\,\alpha\,\mathrm{ch}\,\beta - \mathrm{sh}\,\alpha\,\mathrm{sh}\,\beta\cos\vartheta),\quad \cos\vartheta = \mathbf{nn}'. \quad (2.14)$$

In terms of these variables one can write the integral in (2.13) in the form

$$\int U_n(\mathrm{ch}\,\beta)a(k^2, \mathrm{ch}\,\alpha\,\mathrm{ch}\,\beta - \mathrm{sh}\,\alpha\,\mathrm{sh}\,\beta\cos\vartheta)\mathrm{sh}^2\beta\, d\beta\, d(\cos\vartheta). \quad (2.15)$$

Instead of integrating over cos θ in (2.15), one can integrate with respect to the quantity

$$\mathrm{ch}\,\gamma = \mathrm{ch}\,\alpha\,\mathrm{ch}\,\beta - \mathrm{sh}\,\alpha\,\mathrm{sh}\,\beta\cos\vartheta,$$

taking into consideration that, by definition, $|\alpha-\beta| < \gamma < \alpha+\beta$. In this connection expression (2.15) takes the form

$$\int \mathrm{sh}\,\gamma\, d\gamma\, a(k^2, \mathrm{ch}\,\gamma)\frac{1}{\mathrm{sh}\,\alpha}\int \mathrm{sh}\,\beta\, d\beta\, U_n(\mathrm{ch}\,\beta)\theta(\alpha+\beta-\gamma)\theta(\gamma-|\alpha-\beta|). \quad (2.16)$$

Now let us use a formula which can be verified by elementary integration:

$$\int \mathrm{sh}\,\beta\, d\beta\, U_n(\mathrm{ch}\,\beta)\theta(\alpha+\beta-\gamma)\theta(\gamma-|\alpha-\beta|)$$
$$= \int \mathrm{sh}(n+1)\beta\, d\beta\,\theta(\alpha+\beta-\gamma)\theta(\gamma-|\alpha-\beta|) \quad (2.17)$$
$$= \frac{\mathrm{sh}\,\alpha\,\mathrm{sh}\,\gamma}{n+1}U_n(\mathrm{ch}\,\alpha)U_n(\mathrm{ch}\,\gamma).$$

Substituting (2.17) into (2.16) we reduce this integral to the form

$$\frac{1}{n+1}U_n(\mathrm{ch}\,\alpha)\int \mathrm{sh}^2\gamma\, d\gamma\, a(k^2,\mathrm{ch}\,\gamma)U_n(\mathrm{ch}\,\gamma). \quad (2.18)$$

Comparing expressions (2.18) and (2.13) we verify that B_n only depends on k^2.

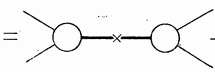
Now let us return to Eq. (2.12) in which we shall use the quantities $B_n(k^2)$. This equation has the form

$$B_n(w) = \frac{1}{\mathrm{Im}\,G^{-1}(w)}\int_0^w dw_1\, w_1 V_n(w,w_1)\,\mathrm{Im}\,G(w_1)B_n(w_1), \quad (2.19)$$

where

$$(n+1)V_n(w,w_1) = \int \mathrm{sh}^2\alpha\, d\alpha\, U_n(\mathrm{ch}\,\alpha)V(w,w_1,\mathrm{ch}\,\alpha),$$
$$\mathrm{ch}\,\alpha = (qk)/\sqrt{q^2k^2},\quad w = q^2,\quad w_1 = k^2.$$

So far, our analysis has been purely kinematical. Now let us discuss the properties of the function $V(w, w_1)$ which appears in (2.9). In the region $w_1 \sim w$ this function, which according to (2.7) can be represented by a series of diagrams

$V =$  $+ \cdots;$ (2.20)

has, therefore, an order of magnitude given by

$$V \sim \Gamma^2 G \sim w^{2\eta-2},$$

if $G \backsim w^{-\eta}$. By virtue of the similarity hypothesis we have

$$V = w^{2\eta-2}\Phi\left(\frac{w_1}{w}, \mathrm{ch}\,\alpha\right). \quad (2.21)$$

Thus, in the region $w_1 \sim w$ the amplitude V has the same properties as the total amplitude of the interaction of two particles. However, for $w_1 \ll w$ these quantities differ strongly from each other. In fact, let us compare the sum rules (2.10) for V and a:

$$\int V(q,k)k_\alpha \mathrm{Im}\,G(k)d^4k = q_\alpha \mathrm{Im}\,G^{-1}(q), \quad (2.22)$$
$$\int a(q,k)k_\alpha\delta(k^2-m^2)d^4k = q_\alpha\,\mathrm{Im}\,G^{-1}(q).$$

If expansion (2.20) is substituted into the first of these equations, an expansion of Im G^{-1} by jets is obtained, and in this expansion, as was shown in I, all terms are of the same order and the intrinsic mass of each jet is of the order of the mass of the initial virtual particle. Thus, if the similarity hypothesis is valid then values $k^2 \sim q^2$ are essential in the first equation, but the region $k^2 \sim m^2$ gives a small contribution. At the same time, in the second equation the integration takes place in the region $k^2 = m^2$, and a is so large in this region that its integral is the same as in the first case. The qualitative reason for the different behavior of V and a for $k^2 \ll q^2$ lies in the fact that a large number (increasing with q^2) of diagrams participate in the production of a real particle with $k^2 = m^2$, the contribution from each of these diagrams being small. Discarding the diagrams with two-particle fissions (or any other group of diagrams, for example, the diagrams containing three-particle fissions), we reduce the number of diagrams and impair the balance between their large number and their small magnitude, which led to the second sum rule.

Equations (2.22) refer to the first moment B_1. With increasing values of n, the quantities B_n for $q^2 \gg m^2$ grow, as will be shown, and consequently the region $k^2 \sim q^2$ will be all the more important in the integral (2.19). Substituting (2.21) into (2.19) we obtain

$$B_n(w) = \int K_n\left(\frac{w_1}{w}\right)B_n(w_1)\frac{dw_1}{w}, \quad (2.23)$$

where

$$K_n\left(\frac{w_1}{w}\right) = \mathrm{const}\cdot\left(\frac{w_1}{w}\right)^{1-\alpha}\int \mathrm{sh}^2\alpha\, d\alpha\,\Phi\left(\frac{w_1}{w},\mathrm{ch}\,\alpha\right)U_n(\mathrm{ch}\,\alpha). \quad (2.24)$$

It is obvious that the solution (2.23) has the form

$$B_n(w) = B(n)w^{\gamma(n)}, \quad (2.25)$$

where γ is determined from the equation

$$1 = \int K_n(x)x^{\gamma(n)}dx. \quad (2.26)$$

The particular values of γ associated with n = 0, 1 are determined by the rules (2.10) and are given by

$$\gamma(0) = \delta,\quad \gamma(1) = 0. \quad (2.27)$$

One can also obtain formulas (2.27) directly from (2.26).

Now let us show how from the known $B_n(k^2)$ one can obtain the amplitude a, which is proportional to the probability for creating a hadron with a given momentum p. Let us consider the most interesting case, when the selected hadron is ultrarelativistic. Let us introduce the variable $\epsilon = 2(pq)/q^2$. The energy E of the hadron in the center of mass system is given by the formula $E = \epsilon\sqrt{w}/2$ (we recall that $w = q^2$). It is obvious that $0 \le \epsilon \le 1$ since the following relation must hold:

$$(q-p)^2 \approx q^2 - 2pq = w(1-\varepsilon) > 0.$$

If ultrarelativistic hadrons are in fact most important, then in formula (2.18) the values

$$\operatorname{ch}\gamma = \frac{(qp)}{m\sqrt{q^2}} = \varepsilon\frac{\sqrt{w}}{2m} \gg 1. \tag{2.28}$$

are most important. In the region (2.28) one can assume that

$$U_n(\operatorname{ch}\gamma) \approx (\sqrt{w}/2m)^{n+1}\varepsilon^{n+1}.$$

Changing to an integration over ϵ, we obtain the following formula for $B_n(w)$:

$$B_n(w) = \left(\frac{\sqrt{w}}{2m}\right)^{n+2}\int_0^1 a(w,\varepsilon)\varepsilon^{n+1}\,d\varepsilon. \tag{2.29}$$

From Eqs. (2.29) and (2.25) it follows that

$$a(w,\varepsilon) = \text{const}\cdot\int_{n_0-i\infty}^{n_0+i\infty} dn\,B(n)\frac{w^{\rho(n)}}{\varepsilon^{n+1}},$$
$$\int_0^1 a(w,\varepsilon)\varepsilon^{n+1}\,d\varepsilon = \text{const}\cdot B(n)w^{\rho(n)}, \tag{2.30}$$
$$\rho(n) = \gamma(n) - (n+2)/2.$$

Below formulas analogous to (2.30) will be derived for the observable amplitudes W_1 and W_2; therefore the second formula admits experimental verification. We note that in our notation the results of articles [3,4] are written in the form

$$a(w,\varepsilon) = F(\varepsilon)$$

and contradict all of the sum rules (2.30) except the rule with n = 1.

One can show that the integral in the first formula of (2.30) is determined by the saddle point, which is given by the equation

$$\frac{d\rho}{dn} + n\frac{\ln(1/\varepsilon)}{\ln(w/m^2)} = 0, \tag{2.31}$$

and with logarithmic accuracy one finds

$$a(\varepsilon, w) = \frac{\text{const}}{\sqrt{\tau}}f\left(\frac{\xi}{\tau}\right)e^{\tau g(\xi/\tau)} \tag{2.32}$$

where

$$\xi = \ln(1/\varepsilon), \quad \tau = \ln(w/m^2).$$

Formula (2.32) is inconvenient for comparison with experiment; therefore we shall not discuss its proof.

3. ALLOWANCE FOR THE SPIN

The amplitude a(p, q) which was discussed in the previous section is not an observable quantity. In order to make a comparison with experiment, it is necessary to know the amplitude $a_{\alpha\beta}$, as formulas (2.2) and (2.3) indicate. Since this amplitude is not a Lorentz-invariant quantity, the examination given in Sec. 2 becomes a little complicated. In the present section we obtain directly a set of sum rules for the amplitude $a_{\alpha\beta}(q, p)$. In principle, each of these sum rules admits experimental verification.

First of all we note that for $a_{\alpha\beta}$ the analog of the rigorous sum rules (2.10) will be the relations

$$\begin{gathered}\int a_{\alpha\beta}(q,p)\delta(p^2-m^2)d^4p = \text{const}\cdot\bar{n}(q^2)\operatorname{Im}\Pi_{\alpha\beta}(q),\\ \int p_\mu a_{\alpha\beta}(q,p)\delta(p^2-m^2)d^4p = q_\mu\operatorname{Im}\Pi_{\alpha\beta}(q).\end{gathered} \tag{3.1}$$

Here Im $\Pi_{\alpha\beta}$ is the imaginary part of the polarization operator which, as is shown in I, is proportional to q^2 for $q^2 \gg m^2$.

In order to extract information about the functions W_1 and W_2 from Eqs. (3.1), we take in (3.1) the trace with respect to the subscripts α and β.

As a result we obtain

$$\begin{gathered}\int\{1, p_\nu\}\left[3W_1 + \left(\frac{(pq)^2}{m^2q^2} - 1\right)W_2\right]\delta(p^2-m^2)d^4p\\ = \{(q^2)^{1+\delta}, q_\nu q^2\}\cdot\text{const.}\end{gathered} \tag{3.2}$$

It is quite clear that since the quantity $a_{\alpha\alpha}$ is Lorentz invariant, it satisfies all of the relations obtained for a. In terms of the variables $\epsilon = 2(pq)/q^2$ and $w = q^2$ we have

$$\begin{gathered}\int_0^1\left\{W_1 + \frac{\varepsilon^2w}{12m^2}W_2\right\}\varepsilon^{n+1}\,d\varepsilon = \text{const}\cdot A(n)w^{\alpha(n)},\\ \alpha(0) = \delta, \quad \alpha(1) = 0.\end{gathered} \tag{3.3}$$

However, relations (3.3) are not sufficient for the determination of the two functions W_1 and W_2. In order to obtain a second relation it is necessary to form from the components $a_{\alpha\beta}$ combinations which transform under the Lorentz group in analogy to helicity amplitudes under the rotation group. In order to determine such combinations, let us consider the quantities

$$\int\delta(p^2-m^2)U_n(up)a_{\alpha\beta}(q,p)d^4p.. \tag{3.4}$$

Here u denotes the unit vector in a time-like direction:

$$u = (\operatorname{ch}\alpha, \operatorname{sh}\alpha, 0, 0), \tag{3.5}$$

and q will be assumed to be given by

$$q = \sqrt{q^2}(1,0,0,0). \tag{3.6}$$

Let us parametrize the momentum p of integration in the following manner:

$$p = m(\operatorname{ch}\beta, \operatorname{sh}\beta\cos\vartheta, \operatorname{sh}\beta\sin\vartheta\cos\varphi, \operatorname{sh}\beta\sin\vartheta\sin\varphi). \tag{3.7}$$

Let us introduce the combinations

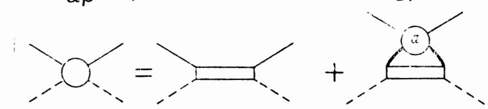
$$\begin{gathered}A^{(0)} = A_{xx} + A_{yy} + A_{zz} \propto W_1 + \operatorname{sh}^2\beta\,W_2,\\ A^{(2)} = 2A_{zz} - A_{xx} - A_{yy} \propto \operatorname{sh}^2\beta\,W_2P_2(\cos\vartheta),\end{gathered} \tag{3.8}$$

where P_2 is the Legendre polynomial. For these combinations one can write the integral in (3.4) in the form

$$\begin{gathered}A_n{}^l(k^2)C_{n-l}^{l+1}(\operatorname{ch}\alpha)\operatorname{sh}^{l+1}\alpha \equiv \int\operatorname{sh}^2\beta\,d\beta\,d(\cos\vartheta)\\ \times U_n(\operatorname{ch}\alpha\operatorname{ch}\beta - \operatorname{sh}\alpha\operatorname{sh}\beta\cos\vartheta)\\ \times A^{(l)}(\operatorname{ch}\beta)P_l(\cos\vartheta) = \text{const}\cdot C_{n-l}^{l+1}(\operatorname{ch}\alpha)\\ \times\operatorname{sh}^{l+1}\alpha\int d\beta\operatorname{sh}^{2+l}\beta\,C_{n-l}^{l+1}(\operatorname{ch}\beta)A^{(l)}(\operatorname{ch}\beta).\end{gathered} \tag{3.9}$$

Here l = 0, 2, and the C_{n-l}^{l+1} are Gegenbauer polynomials; the addition theorem for Tchebichef polynomials [6] and the orthogonality of Legendre polynomials has been used.

Now let us consider the analog of Eq. (2.8) for the amplitudes $a_{\alpha\beta}$ or, what is the same thing, for $a^{(l)}$:

 (3.10)

From this equation, by proceeding in the same way as in the derivation of (2.19) we obtain

$$A_n^l(w)w = \int_0^w V_n^l(w, w_1) \operatorname{Im} G(w_1) B_n(w_1) w_1 \, dw_1, \quad (3.11)$$

where the $B_n(w)$ are defined by formula (2.13), the $A_n^l(w)$ by formula (3.9), and

$$V_n^l(w, w_1) = \text{const} \cdot \int \operatorname{sh}^{2+l}\alpha \, d\alpha \, V^l(w, w_1, \operatorname{ch}\alpha) C_{n-l}^{l+1}(\operatorname{ch}\alpha). \quad (3.12)$$

The amplitudes V^l are related to $V_{\alpha\beta}(q, k)$ by the following relationship: if

$$V_{\alpha\beta} = V_1\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right) + V_2\left(k_\alpha - \frac{(kq)}{q^2} q_\alpha\right)\left(k_\beta - \frac{(kq)}{q^2} q_\beta\right)$$

then

$$V^{(0)} = 3V_1 + V_2\left(k^2 - \frac{(kq)^2}{q^2}\right), \quad V^{(2)} = V_2\left(k^2 - \frac{(kq)^2}{q^2}\right).$$

From Eq. (3.11) it follows that

$$A_n^l(w) = F_l(n) w^{\gamma(n)}. \quad (3.13)$$

Taking into consideration that the ultrarelativistic region plays the major role in the integral for $A_n^l(w)$, we arrive at the final formulas which generalize (2.30) for the spin case:

$$\int_0^1 \varepsilon^{n+1} W_1 \, d\varepsilon = F_0(n) w^{\alpha(n)}, \quad (3.14)$$

$$\int_0^1 \varepsilon^{n+3} W_2 \, d\varepsilon = F_2(n) w^{\alpha(n)-1},$$

$$\alpha(0) = \delta, \quad \alpha(1) = 0.$$

These formulas admit direct experimental verification in experiments involving e^+e^- annihilation.

For $n \neq 1$ the sum rule (3.14) contradicts the Bjorken conjecture,[3,5] according to which one would have

$$W_1 = F_1(\varepsilon), \quad W_2 = \frac{1}{\varepsilon w} F_2(\varepsilon).$$

4. QUALITATIVE PICTURE OF e^+e^- ANNIHILATION

In article I and in the preceding sections of the present article, the e^+e^- annihilation process has been studied on the basis of formal equations, which express the laws of similarity. In the present section a qualitative picture, lying at the foundation of these laws, will be described, and from this picture follow many (although not all) of the results obtained formally.

We represent the annihilation process in the form of the sequential fission of a heavy virtual quantum into lighter fragments, until finally real hadrons are produced (see the accompanying figure, where for simplicity it is assumed that each fragment divides in two). The physical content of the similarity hypothesis consists in the fact that a small number of fragments are predominantly produced in each fission process, where the masses of these fragments are comparable with the mass of the antecedent fragments. Using this hypothesis it is not difficult to find the average number of real hadrons, i.e., the multiplicity of the process.

We call the age of a hadron the number of fissions which it has experienced prior to becoming a real hadron. If the (unique) broken line connecting a given real hadron with the initial virtual quantum is isolated on the given tree-type diagram, then the age is simply the number of kinks in this line.

We shall trace how the mass changes along the selected line. If w denotes the mass of the initial quantum, then the mass w_n of the fragment of age n will obviously be given by

$$w_n \sim c^{-n} w, \quad (4.1)$$

where $c > 1$. From here it follows that the average age L of a real hadron is determined by the relation

$$c^{-L} w \sim m^2, \quad L = \ln(w / m^2) / \ln c. \quad (4.2)$$

It is obvious that if the ages of the majority of particles in a diagram of the tree type is of the order of L, then the number of these particles is given by N $\sim \exp(AL)$. Hence it follows that with an increase of w the quantity N increases according to a power law, $N \sim w^\delta$. This result was obtained in I on the basis of formal considerations.

Now let us proceed to the energy distribution of the hadrons. If each fission process is considered in its own reference system—the center of mass system of the fragments of a given fission process, then the indicated fragments will not be ultrarelativistic although their velocities will be of the order of the velocity of light. In this system the energy $\tilde{E}_n$ of the n-th fragment is of the order of $\sqrt{w_n}$. However, if the problem is considered in the reference frame corresponding to the center of mass of the total system, then the energy will be multiplied in each fission by a Lorentz factor of the order of unity. Therefore

$$E_n \sim b^n \tilde{E}_n \sim (b / \sqrt{c})^n \sqrt{w}. \quad (4.3)$$

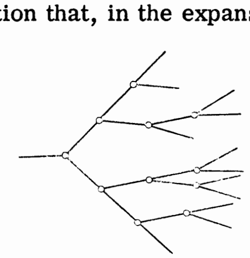
On the average the real hadron will have an energy given by

$$E \sim (b / \sqrt{c})^L \sqrt{w} \sim w^{1/2 - \gamma},$$

$$\gamma = \ln c / \ln(b / \sqrt{c}).$$

Since the quantity b fluctuates with an amplitude of the order of unity (because of the fact that the Lorentz factor each time depends on the direction of flight of the fragments), the quantity γ is not exactly determined. This is explained by the fact that (see Eq. (3.14)) different moments of the distribution depend on w differently, although the power law is valid for all of the moments.

The picture discussed here is equivalent to our fundamental assumption that, in the expansion of Im G^{-1} in



terms of jets, for $w \gg m^2$ a small (of the order of unity) number of terms are important, and in each of these terms the range of internal masses of the order of the external mass is essential. The question of the validity of such an assumption arises, since it corresponds to the following strange property.[1)]

The amplitude for the production of particles contains N! terms which are connected with the symmetrization of a tree of the type shown in the figure:

$$\Gamma(k_1 \ldots k_n) = \sum_{P} \tilde{\Gamma}(k_{i_1} \ldots k_{i_n}). \tag{4.4}$$

N! terms arise in connection with the evaluation of Im G^{-1}:

$$\operatorname{Im} G^{-1} = \sum \frac{1}{N!} \int |\Gamma|^2 d\tau_N = \sum_N \sum_p \int \Gamma(k_1 \ldots k_n) \Gamma^*(k_{i_1} \ldots k_{i_n}) d\tau_N. \tag{4.5}$$

Taking account of the small number of terms in (2.6) corresponds to selecting from (4.5) those terms which correspond either to the identity permutation or else correspond to a small number of permutations of a type which will be described below. Thus, the number of terms taken into consideration is very much smaller than the number of terms discarded.

In order to prove the self-consistency of our qualitative picture, it is necessary to explain why the interference of the particles is very improbable, that is, why the neglected terms are small. The smallness arises from the fact that, as has been indicated, according to Eq. (4.3) ultrarelativistic fragments (in the reference system corresponding to the center of mass of the total system) result from the sequential fission processes. In this connection, the fragments begin to be emitted in increasingly narrower cones due to the relativistic shrinkage of the angles. Therefore the particles are well separated in space, and interference between them is impossible. From a formal point of view the smallness originates in the following way. In diagrams without interference the last fragments have masses $\sim m$ and ultrarelativistic energies. Kinematics guarantees the fulfilment of these conditions without any restrictions on the angles of flight of the fragments in the reference system corresponding to their centers of mass. However, if the two hadrons 1 and 2 exchange places, then in the exchange diagram it is necessary to ensure the condition $(p_1 + p_2)^2 \sim m^2$, that is

$$p_1 p_2 \sim E_1 E_2 \theta^2 \sim m^2, \quad \theta^2 \ll 1.$$

This abruptly decreases (by a factor of θ^2) the phase volume in which the particles are created.

In connection with what has been said, it is necessary to make one reservation. If two fragments are not ultrarelativistic with respect to each other, then they can interfere. In order to determine what kind of interferences are possible, we introduce the concept of the "degree of the relationship" l_{12} between two fragments. On the figure we trace two broken lines connecting two spcificed fragments with the initial quantum, and we shall measure the ages of the fragments from the point where these lines merge together. We call the sum of the ages thus defined the "degree of the relationship." It is obvious that in the diagram one can exchange the places of any two fragments with $l_{12} \sim 1$, and this does not change the order of its magnitude. One can verify that taking such a type of interference into account exactly corresponds to taking account of the small number of terms in (2.6).

We note that our line of reasoning pertains to processes which give the major contribution to the cross section. In regard to very improbable processes in which $N \gg w^\delta$ particles are produced, it may turn out to be invalid, and for $N \geq N_0 \gg w^\delta$ all N! terms in this sum may give a contribution in Eq. (4.5). A mathematically similar phenomenon occurs with the partition function of an ideal Bose gas at the point of Bose condensation. This problem requires additional investigation.

In conclusion we emphasize that although in our formalism the expansion (2.6) with regard to Im G(k) has been utilized, and thus the one-particle Green's function has played a special role, this distinguishing feature is only apparent. With equal success one can use an expansion, not in terms of the functions $\langle \varphi(x_1) \varphi(x_2) \rangle$ but in terms of the more complicated Wightman functions $\langle \varphi(x_1) \ldots \varphi(x_n) \rangle$. If the similarity laws are valid, then all of these expansions are equivalent. It would be desirable to develop a formalism, based on the explicitly equally-justified participation of all Wightman functions; however, up to the present time it has not been possible to achieve success in this direction.

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[1)] This question was posed by V. N. Gribov.