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NONLINEAR DRIFT WAVES IN AN INHOMOGENEOUS MAGNETIC FIELD

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A nonlinear equation for drift oscillations of a weakly ionized inhomogeneous plasma in an inhomogeneous magnetic field is derived in the hydrodynamic approximation. For the case of small supercriticality, a nonstationary solution is obtained, which describes the time variation of the amplitude of the nonlinear drift wave. The stationary convection plasma flow across the magnetic field is calculated.

1. INTRODUCTION

ONE of the important forms of instability of an inhomogeneous plasma placed in a magnetic field is, as is well known, the "gravitational" instability due to the curvature of the magnetic force lines (see, for example, the review[1]). The development of this instability leads to convection of the plasma across the magnetic field; to describe the convection it is necessary to solve the nonlinear problem. The convection of a fully ionized plasma in toroidal systems, connected with gravitational dissipative instability on trapped drift waves localized in the region of the maxima of the magnetic field, was investigated by Pogutse and one of the authors[2]. The convection of a weakly-ionized plasma, which is connected with instability on perturbations of the flute type, was investigated in the quasilinear approximation in[3-5].

In the present paper we consider "gravitational" laminar convection of an inhomogeneous weakly-ionized plasma due to the instability on perturbations that are inhomogeneous along a magnetic field. Just as in[2], the stationary state in the plasma is established as a result of competition of linear buildup of mode instability and nonlinear damping due to generation of its higher stable harmonics. Unlike in[2], however, in a weakly ionized plasma, the stability of the higher harmonics is ensured not by ion viscosity but by collisions of the ions with the atoms of the neutral gas.

In Sec. 2 of the present paper we derive, in the hydrodynamic approximation, a nonlinear equation describing the drift oscillations of a weakly-ionized magnetized inhomogeneous plasma. In Sec. 3 the van der Pol method is used to find a nonstationary solution of this equation under conditions of small supercriticality, when the "gravitational" buildup only slightly exceeds the damping due to the atom collisions; this solution determines the time evolution of the initially unstable perturbation. In Sec. 4 we derive an equation of the stationary state of the plasma for arbitrary supercriticality, and consider its solution in the limit when the supercriticality is small. In Sec. 5 we calculate the convection flow of a weakly-ionized plasma across the magnetic field; this flow is connected with the "gravitational" instability on drift waves.

2. FORMULATION OF PROBLEM. EQUATION FOR DRIFT WAVES OF FINITE AMPLITUDE

Let us consider a flat layer of a weakly-ionized low-pressure plasma, the density of which varies in a direction perpendicular to the external constant magnetic field **H**. The inhomogeneity of the magnetic field, or, more accurately, the curvature of the magnetic force lines, will be imitated with the aid of an effective gravitational field directed along the density inhomogeneity. Assuming the plasma to be strongly non-isothermal, we shall neglect effects of ion pressure and of the finite Larmor radius of the ions. The only dissipative mechanism to be taken into account is the friction of the charged particles against the neutrals.

Under these conditions we describe the slow quasineutral motions of the plasma by using the following system of hydrodynamic equations:

$$e\nabla\varphi-\frac{e}{c}[\mathbf{v}_e\mathbf{H}]-T\frac{\nabla n}{n}-\nu_e m\mathbf{v}_e=0, \tag{1*}$$

$$-e\nabla_\perp\varphi+\frac{e}{c}[\mathbf{v}_{i\perp}\mathbf{H}]-\nu_i M\mathbf{v}_{i\perp}+M\mathbf{g}=0, \tag{2}$$

$$M\partial v_{i\parallel}/\partial t=-e\nabla_\parallel\varphi, \tag{3}$$

$$\partial n/\partial t+\operatorname{div} n\mathbf{v}_e=0 \tag{4}$$

$$\partial n/\partial t+\operatorname{div} n\mathbf{v}_i=0. \tag{5}$$

Here φ is the electric potential, $\mathbf{v}_{e,i}$ the average velocity of the electronic and ionic components, n the plasma density, m and M the masses of the electron and of the ion, T the electron temperature, $\nu_{e,i}$ the frequency of collision between the charged particles and the neutral atoms, **g** the acceleration of the effective gravitational field ($g \approx T/MR_0$, R_0 is the radius of curvature of the force lines), and $-e$ the charge of the electron. In the system (1)–(5) we have retained the equation of the ionic longitudinal motion (3) (in which only the principal terms were retained), since, as will be shown later, the main nonlinear effects are connected precisely with the longitudinal motion of the ions. We note that in Eq. (1) we have left out a term with the effective gravitational field, since the velocity of the electron drift connected with this field is smaller by a factor M/m than the phonon drift velocity. We choose a coordinate system in which the z axis is directed along the magnetic field **H** and the x axis along the gravitational field **g**. For simplicity we assume that the change of the density n along the coordinate x is exponential, i.e., $-\partial \ln n/\partial x \equiv \kappa = \text{const} > 0$, and φ and $\mathbf{v}_{e,i}$ do not depend on x.

Neglecting the transverse diffusion and the mobility of the electrons, we obtain from (1) and (4)

$$\left(\frac{\partial}{\partial t}-D_e\frac{\partial^2}{\partial z^2}\right)\ln n=-\left(v_{\rm dr}\frac{\partial}{\partial y}+D_e\frac{\partial^2}{\partial z^2}\right)\psi, \tag{6}$$

*$[\mathbf{V}_e\mathbf{H}] \equiv \mathbf{V}_e\times\mathbf{H}$.

where $\psi = e\varphi/T$; $D_e = T/m\nu_e$ is the electron diffusion coefficient; $v_{dr} = cT\kappa/eH$ is the Larmor drift velocity. In deriving (6) we have omitted from the electronic continuity equation the nonlinear term $v_{ez}\partial \ln n/\partial z$, since its contribution is proportional to the deviation of the density distribution from the Boltzmann distribution; this deviation in turn is due to friction in the electronic component. Assuming the contribution from the dissipation to be small, we shall henceforth take into account its influence only with the aid of the principal linear terms. For the ions we have from (2) and (5)

$$\left(\frac{\partial}{\partial t}-\frac{g}{\Omega_i}\frac{\partial}{\partial y}\right)\ln n+\left(v_{\mathrm{dr}}\frac{\partial}{\partial y}-\nu_i r_H^2\frac{\partial^2}{\partial y^2}\right)\psi+v_{iz}\frac{\partial \ln n}{\partial z}=0, \quad (7)$$

Here $\Omega_i = eH/Mc$ is the ion cyclotron frequency; $r_H = c_s/\Omega_i$ is the Larmor radius of the ions at the electron temperature; $c_s = \sqrt{T/M}$ is the velocity of the ion sound. The last term in (7) is connected with allowance for the longitudinal motion of the ions, and it is precisely this term which determines the nonlinear effect in the model in question. In the derivation of (7) we have omitted, for simplicity, the term $\partial v_{iz}/\partial z$ from the linear part. Physically this means that we take into account only the contribution of the longitudinal motion of the ions to the amplitude of the drift wave, neglecting its contribution to the frequency.

Eliminating the quantities ψ and ln n from (3), (6), and (7), we obtain for v_{iz} a nonlinear partial differential equation

$$D_e\frac{\partial^2}{\partial z^2}\left[\frac{\partial}{\partial t}+\left(v_{\mathrm{dr}}-\frac{g}{\Omega_i}\right)\frac{\partial}{\partial y}\right]v_{iz}$$
$$-\left(\nu_i D_e\frac{\partial^2}{\partial z^2}+\varkappa g\right)r_H^2\frac{\partial^2 v_{iz}}{\partial y^2}+\frac{1}{2}D_e\frac{\partial^3 v_{iz}^2}{\partial z^3}=0. \quad (8)$$

Here in linear approximation the term with the square bracket determines the oscillation frequency, the term with κg the "gravitational" buildup of the oscillations, and the term with ν_i the damping.

3. NONLINEAR DEVELOPMENT OF UNSTABLE PERTURBATION IN TIME

From (8) it is easy to obtain the frequency and the increment for a flat wave of infinitesimally small amplitude, propagating in a weakly-ionized plasma in an inhomogeneous magnetic field:

$$\operatorname{Re}\omega=\omega_*,\quad \operatorname{Im}\omega=(\varkappa g-\nu_i k_z^2 D_e)/\omega_s, \quad (9)$$

where $\omega_* = k_y v_{dr}$ is the drift frequency; $\omega_s = (k_z^2/k_y^2)(M/m)(\Omega_i^2/\nu_e)$. In the derivation of (9) it is assumed for simplicity that $\omega_* \gg k_y g/\Omega_i$.

According to (9), the condition for the instability of the system relative to the buildup of drift oscillations in a weakly-ionized plasma can be represented in the form $L^2/a_0 R_0 > b_e/b_i$, where L and a_0 are the characteristic longitudinal and transverse dimensions of the system and $b_{e,i}$ are the mobilities of the charged particles.

Thus, from the expression for the imaginary part of the frequency we see that when $\kappa g > \nu_i k_z^2 D_e$ the drift mode with given k_y and k_z is unstable, but its higher harmonics, which result from nonlinear distortions, will be damped starting with a certain harmonic. Under these conditions it is natural to expect a stationary wave of finite amplitude to become established in the plasma (see, for example,[6]).

Accordingly, let us consider the solutions of Eqs. (8) under conditions of small supercriticality, when the buildup of the fundamental mode only slightly exceeds its damping, i.e.,

$$R\equiv(\varkappa g-\nu_i k_z^2 D_e)/\nu_i k_z^2 D_e \ll 1. \quad (10)$$

In this limiting case it turns out to be possible to establish the time variation of the amplitude of the fundamental mode and of its harmonics. It is natural to assume (as will be confirmed by the result) that the main contribution to the formation of the profile of the wave of finite amplitude is made by the first several harmonics. Pursuing further in the spirit of the van der Pol method, we seek a solution of Eq. (8), confining ourselves for simplicity to two harmonics, in the form

$$v_{iz}=v_1(t)\exp\{i(k_y y+k_z z-\omega_* t)\}$$
$$+v_2(t)\exp\{2i(k_y y+k_z z-\omega_* t)\}+\text{c.c.}, \quad (11)$$

where $v_{1,2}$ are slowly varying amplitudes; k_y and k_z are the average wave numbers. Strictly speaking, it is necessary in the van der Pol method to write the quantity $\omega(t)$, which is a slowly varying function of the time, in the expansion (11) in place of ω_*. This would correspond to allowance for the influence of the nonlinear effects on the frequency, which, as already noted, we neglect for simplicity. Substituting (11) in (8) and equating the coefficients of the corresponding exponentials to zero, we obtain

$$\partial w_1/\partial\tau-w_1+iR^{-1}w_1^* w_2=0, \quad (12)$$

$$R\,dw_2/d\tau+3w_2+iw_1^2=0, \quad (13)$$

where $w = \omega_s k_z v/\kappa g$, $\tau = \gamma_1 t$, $\gamma_1 = (\kappa g - \nu_i k_z^2 D_e)/\omega_s$ is the linear increment of the fundamental mode (9); the asterisk to w denotes the complex conjugate. Omitting for simplicity the small term with the derivative from (13), we obtain from (12) and (13) the following equation for the square of the amplitude of the fundamental mode:

$$\frac{1}{2}\frac{dA_1}{d\tau}-A+\frac{1}{3R}A_1^2=0, \quad (14)$$

where $A_1 = w_1 w_1^*$. The solution of this nonlinear equation is well known:

$$A_1(\tau)=A_{1\infty}A_{10}e^{2\tau}/[A_{1\infty}+A_{10}(e^{2\tau}-1)], \quad (15)$$

where $A_{10} = A_1(\tau = 0)$ is the initial value of the square of the amplitude of the fundamental mode, and the stationary value of $A_{1\infty}$ with allowance for (10) is

$$A_{1\infty}=3R. \quad (16)$$

As expected, the amplitude of the fundamental mode of the nonlinear drift wave in the stationary state turns out to be proportional to the square root of the supercriticality, which in turn is proportional to the linear increment. For the square of the amplitude of the second harmonic $A_2 = w_2 w_2^*$ we obtain

$$A_2(\tau)={}^1/_9 A_1^2(\tau), \quad (17)$$

whence we get for the stationary value

$$A_{2\infty}=R^2. \quad (18)$$

Taking (15) into account, we can readily show that the term with the derivative, which was discarded from

(13), is smaller by a factor R^{-1} than the second term.

It is seen from (15) that as $t \to \infty$, the stationary value of the amplitude is reached for any (nonzero) initial value of the wave amplitude—a result typical for self-oscillating systems. Thus, the obtained stationary solution (16), (18), describes a stable state, which the system assumes as a result of the temporal evolution of an unstable perturbation.

4. STATIONARY DRIFT WAVES

In the preceding section, for the case of small supercriticality, we considered the time variation of the amplitude of the drift oscillations. However, from (8) it is easy to obtain a nonlinear equation describing the stationary state of the plasma and valid for any supercriticality. Indeed, for $t \to \infty$ we shall seek a solution of Eq. (8) in the form of a stationary traveling wave

$$v_{iz} = v(\xi), \quad \xi = k_y y + k_z z - \omega t, \tag{19}$$

where ω is an unknown real frequency. Then, substituting (19) in (8), we obtain

$$\frac{1}{R+1}\frac{d^2w}{d\xi^2} - (\Delta + w)\frac{dw}{d\xi} + w = 0, \tag{20}$$

where

$$w = \omega_s k_z v / \varkappa g, \quad \Delta = \omega_s(\omega_* - \omega') / \varkappa g$$
$$\omega' = \omega + k_y g / \Omega_i.$$

For the case of small supercriticality, putting

$$w = w_1 e^{i\xi} + w_2 e^{2i\xi} + w_3 e^{3i\xi} + \text{c.c.},$$

we obtain from (20) a system of three algebraic equations for the determination of the amplitudes w_1, w_2, and w_3 (see (12) and (13)):

$$\begin{aligned} &(\Delta + iR)w_1 + w_2 w_1^* + w_2^* w_3 = 0, \\ &(2\Delta - 3i)w_2 + w_1^2 + 2w_3 w_1^* = 0, \\ &(3\Delta - 7i)w_3 + 3w_1 w_2 = 0. \end{aligned} \tag{21}$$

From (21), taking into account the fact that $w_1 w_1^*$ is real, we obtain

$$\omega' = \omega_*, \tag{22}$$

$$|w_1|^2 = 3R, \quad |w_2|^2 = R^2, \quad |w_3|^2 = {}^{27}/_{64}R^3. \tag{23}$$

We note that the result (22) is in general exact for Eq. (20) and is not connected with the assumption that the supercriticality is small (see[7], where it is shown that an equation of the type (20) has a periodic solution only if the coefficient of the linear term with the first derivative vanishes). From (23) we see that the amplitudes of the harmonics of the fundamental mode are quantities of higher order of smallness with respect to the supercriticality R:

$$|w_1|^2 : |w_2|^2 : |w_3|^2 \approx 1 : R : R^2.$$

5. CONVECTION FLOW OF WEAKLY IONIZED PLASMA

Knowing the amplitude of the drift oscillations, we can easily calculate the convection flow of a weakly ionized plasma in the direction of the inhomogeneity. For its calculation it is convenient to introduce a generalized potential Q by means of the relation[2]

$$v_{iz} = c_s^2\left(\frac{\partial}{\partial t} - D_e\frac{\partial^2}{\partial z^2}\right)\frac{\partial Q}{\partial z}. \tag{24}$$

We then have from (3) and (6)

$$\ln n = \left(D_e\frac{\partial^2}{\partial z^2} + v_{\text{dr}}\frac{\partial}{\partial y}\right)\frac{\partial Q}{\partial t}, \tag{25}$$

$$\psi = \left(D_e\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t}\right)\frac{\partial Q}{\partial t}. \tag{26}$$

By definition, the convection flow in the direction of the inhomogeneity is equal to

$$I = \langle n' v_x' \rangle, \tag{27}$$

where the angle brackets denote averaging over the time and n' and v_x' denote the perturbations of the density ($n = n_0 + n'$, where n_0 is the unperturbed density), while x denotes the component of the ion velocity. From (27), (taking (2), (22), (25), and (26) into account as well as the stationarity (19)), we obtain

$$I = n_0 g k_y^2 r_H^2 \omega_*^2 k_z^2 D_e \langle (d^3Q / d\xi^3)^2 \rangle \tag{28}$$

or, using the definition (24)

$$I = n_0\frac{g}{\omega_s}\left(\frac{\omega_*\varkappa g}{\omega_s k_z^2 c_s^2}\right)^2 \langle w^2 \rangle \tag{29}$$

or

$$I = (n_0 g / \omega_s)\langle (n' / n_0)^2 \rangle. \tag{30}$$

Comparison with the Bohm flow yields

$$\frac{I}{I_B} = 16\frac{k_y g}{\omega_* \omega_s}\left\langle\left(\frac{n'}{n_0}\right)^2\right\rangle, \quad I_B = \frac{1}{16}n_0 v_{\text{dr}}. \tag{31}$$

We note that the expressions obtained for the convection currents (28)—(31) do not depend on the assumption concerning the magnitude of the supercriticality of the system. In our case of small supercriticality, substitution of the amplitude (23) in (29) yields

$$\frac{I}{I_B} = 96\frac{k_y}{\varkappa}\left(\frac{b_e}{b_i}\right)^2\frac{k_z^2 c_s^2 \omega_*}{\omega_s^3}\left(\frac{\varkappa g}{k_z^2 c_s^2} - \frac{b_e}{b_i}\right). \tag{32}$$

The theory developed in the present paper can be used to interpret experiments on anomalous diffusion in a weakly ionized plasma using sufficiently short discharge tubes in the form, say, of a section of a torus, on the ends of which are placed electrodes (see, for example,[8]). Under these conditions, when the presence of the electrodes leads, generally speaking, to a weakening of the flute instability, a decisive role will be played by the "gravitational" instability on the drift waves. To identify the mechanism of convection it is necessary, simultaneously with the measurement of the current, to carry out an analysis of the spectrum of the oscillations, the presence of which can offer evidence of the development of the instability in question.

In the present paper, the solution of Eq. (20) was obtained under the assumption that the supercriticality of the system is low. The limiting case of large supercriticality will be considered by us in a different paper.

6. CONCLUSION

Let us note certain features of the results obtained in this paper.

1. The oscillatory system considered by us is not conservative: energy builds up at wavelengths with wave vector **k** satisfying the condition for linear buildup, and energy is dissipated on smaller scales. As already noted in the Introduction, the existence of a stationary wave under these conditions is ensured by the mechanism of competition between the linear buildup of the

mode with given **k** and the nonlinear damping due to the generation of its higher harmonics. In this sense we can speak of our system as being self-oscillating. Because of the linearity of the dispersion law, the mechanism of generation of higher harmonics corresponds exactly to the mechanism of nonlinear distortion of the profile of a sound wave with finite amplitude in ordinary gasdynamics. We see that the picture indicated here differs significantly from the process of formation of a stationary wave in a collisionless plasma (see, for example, the review[9]), for which the decisive factor is the deviation from the linear dispersion law. We recall that oscillations in a collisionless plasma are conservative.

2. Equation (20), which describes the stationary state of the plasma, has in the case under consideration ($\omega' = \omega_*$) a continuum of periodic solutions. We have to choose one of them that corresponds to the state assumed by the system as a result of the evolution of the initial perturbation. Such a solution will obviously be a solution having the same period as the initial perturbation (i.e., period $2\pi/k$ in terms of the spatial variable or period 2π in terms of the variable ξ). The solution obtained in Sec. 4 corresponds precisely to this condition. As expected, it coincides with the asymptotic solution of the temporal problem in Sec. 3.

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