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IMPOSSIBILITY OF MIXING IN THE BIANCHI TYPE IX COSMOLOGICAL MODEL

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Detailed estimates are made of the distance covered by light in a homogeneous anisotropic model of the Bianchi IX type. It is proven rigorously that in models with a diagonal metric and randomly prescribed parameters, light cannot circle the universe in any direction with a probability close to unity. Effective mixing is possible only if the model parameters are chosen in a certain manner.

RECENTLY much interest has been aroused in the "mixmaster universe" model as a possible model of earlier stages of expansion of the real universe. The corresponding solution of the Einstein equations was obtained by E. Lifshitz, Khalatnikov, and Belinskiĭ[1,2] and applied to cosmology by Misner[3]. The three-dimensional space of this model is a homogeneous anisotropic space of Bianchi type IX. The universe has different dimensions in three orthogonal directions, characterized by functions $a(t)$, $b(t)$, and $c(t)$.

Misner's idea reduces to the notion that in this model light and sound have time to go around the universe several times during the early stages. Thus, in this model, unlike in the Friedmann model, there is no optical horizon, and this produces conditions for equalization of the inhomogeneities. The physical properties in such a universe were considered by Zeldovich[4]. Misner has assumed that on approaching the singularity (going into the past)[1)] there is an infinite repetition of a situation wherein two functions (for example $a(t)$ and $b(t)$ oscillate for a long time with small amplitude, while the third, $c(t)$, decreases monotonically and stays much smaller than a and b.

With respect to the propagation of light, the universe during this stage is a perturbed Taub solutions[1] and the light then has time to go around the world many times along the smallest axis c. This then occurs along a second axis, etc.

However, E. Lifshitz, I. Lifshitz, and Khalatnikov[5] and Doroshkevich and Novikov[6] found the conditions for the occurrence of "Taub" periods and showed that (for a random choice of the model parameters) the aforementioned periods are never encountered in the solution, with a probability equal to unity[2)].

The question arises: is mixing (multiple circling of the universe by light and sound) possible in the absence of "Taub periods" in the solution?

Formulas for the number of circuits of light around the universe were obtained in[6], and the above question was answered there in the negative: with practically unity probability, light starting from a true singularity can never have time to circle around the universe. However, in[6] there is no proof of this statement. The purpose of the present paper is to present a more detailed analysis of this question.

We present below estimates of the distance that the light traverses, starting with a certain arbitrary instant of time (during the stage of strong anisotropy) up to the instant of singularity. The analysis pertains directly to the diagonal model of the mixmaster universe, as do all the papers cited above, with a uniform co-moving space.

The evolution of the model consists of an infinite number of cycles, in each of which two functions oscillate with large amplitude, unlike the perturbed Taub solution, where the amplitude of the oscillations is small, and the third decreases monotonically.

In the next cycle, the monotonic function begins to oscillate, and one of the functions that oscillated before decreases monotonically, etc.[3)]. We shall number the

[1)] We shall consider throughout motion towards the singularity—collapse. For the cosmological problem of expansion it is necessary to reverse the time throughout.

[2)] We are considering only the sharply anisotropic stage, when the gravitation of matter can be neglected and the model is far from approaching the isotropic solution of Friedmann.

cycles starting with one arbitrarily chosen in the direction toward the singularity (into the past for the cosmological problem). The number of circuits of the light around the universe in the given cycle, under the condition $a \gg c$ and $b \gg c$, is determined, as shown in[6], by two parameters): $Q_s = a^2/b^2$ at the point of the maximum of the function a, and the quantity $u_s = k_s + x_s > 1$[4]. The integer part of u_s, $k_s = [u_s]$, is equal to the number of oscillations in the s-th cycle, and the fractional part of u_s, x_s, defines the parameter u for the next cycle: $u_{s+1} = 1/x_s$[5]. The connection between Q_s and Q_{s+1} is approximately described by the formula[6]

$$Q_{s+1} \approx Q_s^{u_s^2}, \tag{1}$$

Thus, Q_s increases rapidly on approaching the singularity.

Let us consider the propagation of light along three principal directions of the model, the lengths of which are characterized by the functions a, b, and c. In each cycle, the monotonically varying function is always much smaller than the oscillating ones, and the most favorable conditions for the light circling the universe obtain for a ray moving along this axis.

The number of circuits of light around the universe along the c axis during the time of the large cycle is determined by the formula[5]

$$N_s \approx u_s / 2\pi\sqrt{Q_s}, \quad u_s \gg 1. \tag{2}$$

For the number of circuits along the a or b axis we have

$$N_{a,b} < Q_s^{-1/2}.$$

Thus, regardless of the direction of the light, the number of circuits around the universe N_s during the s-th cycle satisfies the inequality

$$N_s \lesssim u_s / \pi\sqrt{4Q_s} \equiv N_s^*. \tag{3}$$

For the complete number of circuits of light around the universe, N, along a fixed direction during all the time, starting with the singularity and up to the cycle with number unity, the following inequality is valid:

$$N < \sum_{s=1}^{\infty} N_s^*.$$

We shall show that for an arbitrary choice of the parameters of the model, this sum is small compared with unity with a probability that barely differs from unity.

The proof is broken up into two stages. During the first stage we prove that the sum of a finite but sufficiently large number s_1 of the first terms of the series is small. During the second stage we show that the infinite series

$$\sum_{s=s_1}^{\infty} N_s^* \tag{4}$$

has a small sum starting with sufficiently large s_1.

We shall prove that if $Q_1 \gg 1$, then the probability of violating the inequality $N_s^* < N_0 = \text{const} \ll 1$ is small. The concrete value of N_0 is determined by the requirement that the sum of the first s_1 terms of the series be small, i.e., $s_1 N_0 \ll 1$. The proof is constructed by analogy with the probabilistic procedure developed in[5][6]. From the condition $N_s^* < N_0$ and from the definition of N_s^* (see (3)) there follows the inequality

$$u_s < (N_0\pi)(4Q_s)^{1/2}. \tag{5}$$

From (1) and (5) it follows that in order for this inequality to be violated it is necessary that x_{s-1} fall in an interval of width

$$\delta x_{s-1} \approx (N_0\pi)^{-1}(4Q_s)^{-1/2}. \tag{6}$$

The quantity δx_{s-1} is connected with δx_1 by the relation[5]

$$\delta x_{s-1} \approx k_2^2 k_3^2 \ldots k_{s-1}^2 \delta x_1.$$

Thus, in order for the inequality $N_s^* < N_0$ to be violated for at least one of the first s_1 terms of the sequence it is necessary that x fall in the following fraction of the unit interval:

$$W_0 = \frac{1}{2\pi N_0}\left(\frac{1}{\sqrt{Q_1}} + \sum_{s=2}^{s_1}\sum_k \frac{1}{\sqrt{Q_s}\,k_1^2\ldots k_{s-1}^2}\right). \tag{7}$$

The internal sum is taken over all k from 1 to ∞. This is the estimate of the probability of violating the condition $N_s^* < N_0$ at least once.

Using the relations

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

and $Q_s > Q_1^s$, which follows from (1) with $u_s > 1$, we obtain

$$W_0 < \frac{1}{2N_0\pi\sqrt{Q_1}}\left(1 + \sum_{s=2}\left(\frac{\pi^2}{6\sqrt{Q_1}}\right)^{s-1}\right) \approx \frac{1}{2N_0\pi\sqrt{Q_1}}. \tag{8}$$

We continued the summation over s to infinity.

The last expression is much smaller than unity if

$$N_0 \gg 1/\pi\sqrt{4Q_1}. \tag{9}$$

It is necessary that the sum of the first s_1 terms be much smaller than unity, i.e., $s_1 N_0 \ll 1$, and consequently

$$s_1 \ll 1/N_0 \ll \pi\sqrt{4Q_1}. \tag{10}$$

As will be seen subsequently, it is conveneint to take s_1 of the order of 10. Then $Q_1 > 10^3$. This estimate can be reinforced (this will not be done here). Thus,

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[3]The question of applicability of the equations of general relativity theory near the singularity is considered in [6]. Misner assumes that in spite of the known limitations connected with quantum fluctuations at curvatures $l_g \approx 10^{-33}$ cm, it is possible in the case of the mixmaster-universe model to employ the solution without limit up to the true singularity. In [6] it is stated that the limitation $l > l_g$ is also significant in the mixmaster-universe model. Here, however, we do not consider this limitation and analyze the situation under the assumption that general relativity theory is applicable without limit up to the true singularity.

[4]s is the number of the cycle. The quantities Q_s and u_s are determined from the first oscillation of the cycle.

[5]Rougher estimates, which, incidentally, are already sufficient for acquisition of the results of the present article, are given in [6].

[6]The mean value $\overline{N_s^*}$ calculated from the distribution (11) diverges. In this case, however, the mean value $\overline{N_s^*}$ has no definite meaning, since the variance $\overline{N_s^{*2}} - \overline{N_s^{*}}^2$ diverges even more rapidly.

the sum of the first s_1 terms of the series satisfies the inequality (10) with a probability that differs from unity by the small amount (8).

We shall now prove that in the probabilistic sense the sum of the infinite series (4) is small. The probability for u (see[5]) is

$$W(u) = 1/u(u+1)\ln 2, \tag{11}$$

and for large u

$$W \sim 1/u^2 \ln 2. \tag{12}$$

The idea of the proof consists of majoring the series $\sum_{s_1}^{\infty} N_s^*$ by the series

$$\sum_{s_1}^{\infty} \exp\left[-\frac{s}{2}\ln Q_1\right]$$

and estimating the probability of violation of this majorization.

The probability that in the s-th period we have

$$N_s^* > \exp\{-{}^1/_2 s \ln Q_1\}, \tag{13}$$

is determined by the formula ($s \geq 2$)

$$W_s = \int_1^{\sqrt{s}} f_s(x)\,dx + \int_{\sqrt{s}}^{\infty} f_s(x)\exp\{s - x^2)\ln Q_1^{1/2}\}\,dx, \tag{14}$$

$$W_s < \frac{(\ln\sqrt{s})^{s-2}}{(s-2)!}\left(1 - \frac{1}{\sqrt{s}}\right) + \frac{\exp\{(s-2)(\ln(s-2) - 1 - \ln 2)\}}{2\sqrt{s}\ln Q_1^{1/2}(s-2)!}, \tag{15}$$

where

$$f_s(x)\,dx = \frac{(\ln x)^{s-1}}{(s-1)!}\frac{dx}{x^2}, \tag{16}$$

determines the probability that the product $u_1 \ldots u_s$ lies in the interval $x - x + dx$. From (15) it follows that when $s \geq s_1 \gg 1$ we have

$$W_s < 1/2^{s-1} s \ln Q_1. \tag{17}$$

Thus, the total probability of violating the inequality (13) in at least one of the periods, starting with $s = s_1 \gg 1$ and up to the singularity, satisfies the inequality

$$W < \sum_{s_1}^{\infty} 1/2^{s-1} s \ln Q_1 < 1/2^{s_1-2} s_1 \ln Q_1 \ll 1 \tag{18}$$

for $s_1 \gg 1$ and $Q_1 \gg 1$.

On the other hand, when the inequality (13) is satisfied we obtain the following estimate for the total number of circuits of the light around the universe, starting with the cycle s_1,

$$N < \sum_{s_1} \exp\left\{-\frac{s}{2}\ln Q_1\right\} \approx \exp\left(-\frac{s_1}{2}\ln Q_1\right) \ll 1. \tag{19}$$

We have thus proved that without an extremely specialized choice of the parameters of the model the light will never have time to circle around the universe in any direction during the entire time, starting from the singularity and ending with the anisotropic stage when Q is large. During the period of isotropization of the model, when the self-gravitation of ordinary matter comes into play, the light may possibly have time to go around the universe once (although preliminary calculations show that this is not so), but this is patently insufficient for realizing Misner's idea of mixing.

We recall once more, in conclusion, that we did not consider in the mixmaster-universe model the possible role of directional streams of matter, which lead to the appearance of non-diagonal terms in the matrix for the metric of the model. This is a separate problem (see[7,8]). However, preliminary estimates show that in this more complicated case light likewise does not have time to circle around the universe and no mixing takes place.

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