KINETIC EQUATION FOR SOLITONS

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The problem of a statistical description of a large number of solitary waves (solitons) on the basis of the Korteweg-de Vries equation is considered. The dynamics of soliton collisions is studied and it is shown that only paired collisions occur. Interaction with the nonsoliton part of the solution cannot change the amplitude or phase of the soliton. As a result the total soliton velocity distribution function does not depend on the time. A kinetic equation for solitons is derived in the form of a transport equation for the velocity distribution function. The instability of a system consisting of two periodic waves is studied on the basis of the kinetic equation.

1. It is well known that one-dimensional linear waves in media with weak dispersion are described by the Korteweg-de Vries (KDV) equation (see, for example,[1])

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

The KDV equation has a solution in the form of a solitary wave or soliton

$$u = 3s/\mathrm{ch}^2\frac{\sqrt{s}}{2}(x - st - x_0), \tag{2}$$

which is determined by two parameters—the velocity s and the position of the center x_0 (phase). The soliton is a stable formation and it is possible to raise the question of describing the interaction of a large number of solitons distributed in some manner with respect to s and x_0. To this end it is necessary to study the dynamics of the collision of solitons.

In the present paper we shall show that only paired collisions take place, and that there is no velocity exchange between the solitons in these collisions. Thus, there is in principle no change in the summary (or average) distribution function of the solitons with respect to the velocities s. Nonetheless, the kinetics of a "soliton gas" is, as we shall show, not quite trivial. It is possible within this framework, for example, to investigate the instability of certain velocity distributions of the solitons. We shall use the mathematical formalism developed for the KDV equation by Kruskal and his co-workers[2-4] and also by Lax[5]. A brief description of this formalism is given below.

2. We consider a self-adjoint differential operator $\hat{L}_t$ acting on the complex-valued functions $\psi(x)$ $(-\infty < x < \infty)$:

$$\hat{L}_t = \frac{d^2}{dx^2} + \frac{1}{6}u(x, t). \tag{3}$$

The operator $\hat{L}_t$ depends on the time t as a parameter.

It was shown in[2,5] that if u(x, t) is a solution of the KDV equation, then the operators $\hat{L}_t$ at different values of t are unitarily equivalent to one another. The spectrum of the operator $\hat{L}_t$ is conserved in time, and its eigenfunctions $\psi(x, t)$ (independently of the eigenvalue they belong to) satisfy the equation

$$\frac{\partial\psi}{\partial t} = -4\left(\frac{\partial}{\partial x}\right)^3\psi - \frac{1}{2}\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right)\psi. \tag{4}$$

Let u(x, t) tend rapidly to zero as $|x| \to \infty$. We consider the scattering problem for the operator (3):

$$\psi \to e^{-ikx} + S_k(t)e^{ikx}, \quad x \to +\infty,$$

$$\psi \to D_k e^{-ikx}, \quad x \to -\infty.$$

From (4) it follows that D_k does not depend on the time and, in addition,

$$S_k(t) = S_k(0)e^{8ik^3t}. \tag{5}$$

Let the operator $\hat{L}_t$ also have N discrete eigenvalues η_n which, as already noted, are integrals of the motion. The normalization of the corresponding eigenfunctions is also conserved, and their asymptotic form at large |x|,

$$\psi_n(x, t) \to M_n(t)e^{-\eta_n x}, \quad x \to \infty,$$

depends on the time, with

$$M_n(t) = M_n(0)\exp(4\eta_n^3 t). \tag{6}$$

Knowing $S_k(t)$, $M_n(t)$, and η_n, it is possible to reconstruct u(x, t) at any instant of time, by solving the inverse scattering problem.

Let us construct the function

$$F(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_k(t)e^{ikx}dk - \sum_{n=1}^{N} M_n^2(t)e^{-\eta_n x}. \tag{7}$$

The quantity u(x, t) can be found by solving the Marchenko equation[6,7] relative to the function K(x, y, t):

$$K(x, y, t) = F(x + y, t) + \int_x^{\infty} K(x, s, t)F(y + s, t)\,ds,$$

$$u(x, t) = 12\frac{d}{dx}K(x, x, t). \tag{8}$$

From (5)–(7) it follows also that F(x, t) satisfies the linear equation

$$\frac{\partial F}{\partial t} + 8\frac{\partial^3 F}{\partial x^3} = 0. \tag{9}$$

3. Let us consider the problem of the interaction of a finite number of solitons. Kruskal and Zabusky[8] have shown that the amplitudes of the solitons are not altered by the interaction, but the phases are. Let us investigate the character of this alteration.

We introduce the concept of a pure soliton solution of the KDV equation as a solution for which $S_k(t) \equiv 0$. We seek a solution of the KDV equation in the form

$$K(x, y, t) = \sum_{n=1} K_n(x, t) e^{-\eta_n y}.$$

After substituting in (8) we have a system of linear algebraic equations

$$K_n(x, t) = -M_n^2 e^{-\eta_n x} - M_n^2 e^{-\eta_n x} \sum_{m=1}^{N} \frac{K_m(x, t) e^{-\eta_m x}}{\eta_n + \eta_m}.$$

We introduce $A_n(x, t) = K_n(x, t) e^{-\eta_n x}$. The system is transformed into[1])

$$A_n(x, t) \exp(2\eta_n \xi_n) + \sum_{m=1}^{N} \frac{A_m(x, t)}{\eta_n + \eta_m} = -1. \tag{10}$$

Here $\xi_n = x - 4\eta_n^2 t - x_n$; $x_n = -2 \ln|M_n(0)|$. We note that

$$u(x, t) = 12 \frac{d}{dx} \sum_{n=1}^{N} A_n(x, t).$$

The matrix $1/(\eta_i + \eta_k)$ is positive-definite. Therefore the matrix of the system (10) is nondegenerate, and the same pertains to all other matrices appearing in this section. The solutions of the system (10) are bounded by a certain constant that does not depend on x or t. We arrange the eigenvalues η_n in increasing order and investigate the asymptotic form of the solution of the system (10) as $t \to \pm\infty$ on the straight line $x = 4\eta_k^2 + x_0$, where η_k is one of the numbers η_n. As $t \to +\infty$, the value of A_n tends to zero if $n < k$, and to a certain value $A_n^{(0)}$ that does not depend on the time if $n \geq k$, where $A_n^{(0)}$ satisfies the system

$$\begin{gathered} A_k^{(0)} \left(\exp(2\eta_k \xi_k) + \frac{1}{2\eta_k} \right) + \sum_{m=1}^{k-1} \frac{A_m^{(0)}}{\eta_n + \eta_m} = -1, \\ \sum_{m=1} \frac{A_m}{\eta_n + \eta_m} + \frac{A_k^{(0)}}{\eta_n + \eta_k} = -1. \end{gathered} \tag{11}$$

In addition, as $|x_0| \to \infty$, all the $A_n^{(0)} \to$ const and accordingly $u(x, t) \to 0$. The next terms of the asymptotic expression have an exponential character in time. Analogously, as $t \to -\infty$, we have $A_n \to 0$ when $n \geq k$ and $A_n \to A_n^{(0)}$ when $n < k$. For $A_n^{(0)}$ we have the system

$$\begin{gathered} A_k^{(0)} \left(\exp(2\eta_k \xi_k) + \frac{1}{2\eta_k} \right) + \sum_{m=1}^{k-1} \frac{A_m^{(0)}}{\eta_n + \eta_m} = -1, \\ \sum_{m=1}^{k} \frac{A_m^{(0)}}{\eta_n + \eta_m} + \frac{A_k^{(0)}}{\eta_n + \eta_k} = -1 \end{gathered} \tag{12}$$

Shabat[10] showed that any purely soliton solution breaks up as $t \to \pm\infty$ into solitons moving with velocities $s_n = 4\eta_n^2$. Thus, both the system (11) and the system (12) describe a soliton having the same velocity $s_k = 4\eta_k^2$, but, in general, different phases[2]). To calculate the shift of these phases let us consider the interaction of two solitons. This case corresponds to the system (10) for N = 2. We have from the system (10)

$$u(x, t) = 12 \frac{d}{dx} \frac{Z_1}{Z_2},$$

$$Z_1 = e^{2\eta_1 \xi_1} + e^{2\eta_2 \xi_2} + \frac{1}{2\eta_1} + \frac{1}{2\eta_2} - \frac{2}{\eta_1 + \eta_2},$$

$$Z_2 = \left(e^{2\eta_1 \xi_1} + \frac{1}{2\eta_1} \right) \left(e^{2\eta_2 \xi_2} + \frac{1}{2\eta_2} \right) - \frac{1}{(\eta_1 + \eta_2)^2}, \quad \eta_2 > \eta_1.$$

We choose M_1^2 and M_2^2 such that when $t \to -\infty$ we have

$$u(x, t) \to 3s_1/\mathrm{ch}^2 \frac{\sqrt{s_1}}{2}(x - s_1 t) + 3s_2/\mathrm{ch}^2 \frac{\sqrt{s_2}}{2}(x - s_2 t).$$

Here $s_1 = 4\eta_1^2$, $s_2 = 4\eta_2^2$, As $t \to +\infty$ we obtain

$$u(x, t) \to 3s_1/\mathrm{ch}^2 \frac{\sqrt{s_1}}{2}(x - s_1 t - \delta_1) + 3s_2/\mathrm{ch}^2 \frac{\sqrt{s_2}}{2}(x - s_2 t - \delta_2),$$

$$\delta_1 = -\frac{1}{\eta_1} \ln \left| \frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} \right|, \quad \delta_2 = \frac{1}{\eta_2} \ln \left| \frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} \right|. \tag{13}$$

As a result of the scattering, the solitons acquire phases δ_1 and δ_2, with the faster soliton acquiring a positive phase and the slower one a negative one.

Assume now that we have N solitons. The corresponding solution of the system (13) breaks up as $t \to \pm\infty$ into the same solitons. Obviously, as $t \to +\infty$ the fastest soliton will be propagated in front, with the solitons following one another in decreasing order of velocity. As $t \to -\infty$ the arrangement of the soliton is reversed, and thus as the time changes from $-\infty$ to $+\infty$ every soliton will collide with every other one. If all the solitons are sufficiently far from one another, then the total phase shift of each soliton as it propagates along the entire straight line is equal to the sum of the phase shifts in paired collisions. We note, however, that the solutions of the systems (11) and (12), which represent the limiting states of the soliton as $t \to \pm\infty$, depend only on the amplitudes of the remaining solitons and do not depend on their positions, which are determined by their phases. Thus, the total phase shift of the soliton is equal to the sum of the phase shifts in paired collisions also in the general case, and it can be assumed in a certain sense that only paired collisions of the solitons take place. This explains to some degree the conservation of the amplitudes of the solitons: no velocity exchange takes place in paired collisions of any one-dimensional particles.

4. In the general case, $S_k(t) \neq 0$, and theoretically there remains the possibility of change in the amplitude and phase of the solitons as a result of their interaction with the "non-soliton" part of the solution. We shall show that this does not take place. We represent the function F(x, t) in the form

$$F(x, t) = F_0(x, t) + F_1(x, t),$$

$$F_0(x, t) = -\sum M_n^2(t) e^{-\eta_n x},$$

$$F_1(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_k(t) e^{ikx} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_k(0) e^{8ik^3 t + ikx} dk. \tag{14}$$

Both F_0 and F_1 satisfy Eq. (9). Let $u(x, t)|_{t=0}$ be a sufficiently smooth function that decreases rapidly as $|x| \to \infty$. The same is valid also for the function $F_1(x, t)|_{t=0}$. For $t \to +\infty$ and $x > 0$ we obtain from the integral (14) the estimate

1)The system of equations (10) (in a somewhat different form) was obtained in the paper of Kay and Moses [9].

2)The solution of the systems (11) and (12) is a solution of the KDV equation and depends only on $\xi_k = x - 4\eta_k^2 t - x_n$, decreasing as $|\xi_k| \to \infty$. Only a soliton with the parameter η_k can be such a solution.

$$|F_1(x,t)| < \frac{c}{t^{1/3}} \exp\left\{-\frac{1}{3\sqrt{6}} \frac{x^{3/2}}{t^{1/2}}\right\}. \qquad (15)$$

Let us estimate the correction to the purely soliton solution as $t \to +\infty$:

$$K(x,y,t) = \sum_{n=1}^{N} K_n(x,t) e^{-\eta_n y} + \delta K(x,y,t).$$

We put

$$\delta K(x,y,t) - \int_x^\infty \delta K(x,s,t) F_0(y+s,t)\, ds = \delta G(x,y,t),$$

For δK we have

$$\delta G(x,y,t) = F_1(x+y,t) + \sum_{n=1}^{N} K_n(x) \int_x e^{-\eta_n s} F_1(y+s,t)\, ds. \qquad (16)$$

We neglect here the term proportional to $F_1 \delta K$. For δG we have the estimate

$$|\delta G| < \frac{c}{t^{1/3}} \left(1 + \sum_{n=1}^{N} \frac{|A_n(x)|}{\eta_n}\right) < \frac{c_1}{t^{1/3}} \exp\left\{-\frac{1}{3\sqrt{6}} \frac{x^{3/2}}{t^{1/2}}\right\}, \qquad (17)$$

since all the A_n are bounded when $-\infty < x < \infty$. We seek the solution of (16) in the form

$$\delta K(x,y,t) = \sum_{n=1}^{N} \delta K_n(x,t) e^{-\eta_n y} + \delta G(x,y,t).$$

We obtain

$$\dot{\delta A}_n e^{2\eta_n \xi_n} + \sum_{n=1}^{N} \frac{\delta A_m}{\eta_n + \eta_m} = e^{\eta_n x} \int_x^\infty \delta G(x,s,t) e^{\eta_n s}\, ds = f_n. \qquad (18)$$

For f_n we have the estimate

$$|f_n| < \frac{c_1}{\eta_n t^{1/3}} \exp\left\{-\frac{1}{3\sqrt{6}} \frac{x^{3/2}}{t^{1/2}}\right\}.$$

At any straight line $x = \lambda t + x_0$ the coefficients of the matrix of the system (18) tend to constant values as $t \to \infty$, and the right-hand side vanishes like

$$\frac{c_1}{n_1 t^{1/3}} \exp\left(-\frac{\lambda^{3/2}}{3\sqrt{6}} t\right);$$

thus, $A_n \to 0$, including the case when $\lambda = 4\eta_k^2$. It follows therefore that

$$\delta K(x,x,t) = \sum \delta A_n(x,t) \to 0$$

On any straight line $x = \lambda t + x_0$ as $t \to \infty$.

From the foregoing proof it follows that an arbitrary solution of the KDV equation tends to a pure soliton solution as $t \to +\infty$. Similar reasoning proves that as $t \to -\infty$ any arbitrary solution of the KDV equation tends to a purely soliton solution, and obviously to the same solution as when $t \to +\infty$. Thus, the "nonsoliton" part of the solution does not change the amplitude and phase of the solitons and does not influence the process of their scattering.

5. Let us consider now the propagation of an individual soliton in a "gas" (the interaction with the nonsoliton part, by virtue of the foregoing, can be neglected). Interaction with other solitons leads to a change in the average velocity of the solitons as a result of the successive jumps of the phase in the collisions. We introduce the distribution function $f(\eta, x_0)$ of the solitons with respect to the parameters η and the positions of the centers x_0. We then obtain for the velocity of the "trial" soliton

$$\tilde{s}(\eta) = 4\eta^2 + \frac{4}{\eta} \int_0^\infty \ln\left|\frac{\eta_1 + \eta}{\eta_1 - \eta}\right| (\eta^2 - \eta_1^2) f(x_0, \eta_1)\, d\eta_1. \qquad (19)$$

Here $4\eta^2 = s(\eta)$ is the velocity of the soliton with parameter η in "empty" space.

Formula (19) takes into account the paired collisions of the solitons and is valid under the condition that the correction to $s(\eta)$ is small, i.e., under the condition

$$\int f(\eta)\, d\eta \ll \eta_0, \qquad (20)$$

where η_0 is the characteristic value of the parameter. Condition (20) is the criterion by which a "gas" of solitons can be regarded as rarefied.

The KDV equation admits of a solution in the form of a wave constituting a periodic sequence of solitons with a period much larger than the dimension of the soliton. Such a wave can be visualized as a monochromatic "beam" of solitons. If the soliton parameter is η_0 and the period of the wave is L, then the distribution function corresponding to the wave is

$$f(\eta) = \delta(\eta - \eta_0)/L \quad (L \gg 1/\eta_0).$$

When two such soliton waves interact, the solitons of one of the waves collide in sequence with the solitons of the other, as a result of which the wave velocities become renormalized. Let the periods of the interacting waves be L_1 and L_2 and let the soliton parameters be η_1 and η_2 ($\eta_1 < \eta_2$). We have for the renormalized velocities

$$\tilde{s}_1 = 4\eta_1^2 - \frac{4}{\eta_1 L_2} (\eta_2^2 - \eta_1^2) \ln \frac{\eta_1 + \eta_2}{\eta_2 - \eta_1},$$

$$\tilde{s}_2 = 4\eta_2^2 + \frac{4}{\eta_2 L_1} (\eta_2^2 - \eta_1^2) \ln \frac{\eta_1 + \eta_2}{\eta_2 - \eta_1}.$$

(We note that the renormalization of the wave velocity as a result of the interaction of its own solitons is exponentially small like $\exp(-L\eta)$. We can consider the interaction of a large number of periodic waves analogously.

6. Formula (19) makes it possible to write a kinetic equation for solitons. Inasmuch as when solitons collide with one another their parameters η remain unchanged, the following continuity equation should hold for the function f

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_0} \tilde{s}(\eta) f = 0, \qquad (21)$$

where $\tilde{s}(\eta)$ is given by (19). This equation, obviously, conserves the quantity

$$\Phi(\eta) = \int_{-\infty}^{\infty} f(\eta, x_0)\, dx_0.$$

We note that the fact of conservation of $\Phi(\eta)$ should take place also for any more exact theory, since it follows directly from the general Kruskal-Lax theory. Indeed, the quantity $\Phi(\eta)\, d\eta$ is the number of discrete eigenvalues of the operator $\hat{L}_t$ contained in the interval from η to $\eta + d\eta$, and by virtue of the conservation of the spectrum of the operator $\hat{L}_t$ it should be a conserved quantity[3)].

We note also that since only paired collisions exist, the kinetic equation (21) is valid with exponential accuracy with respect to the rarefaction parameter $\int f(\eta)d\eta/\eta_0$, since the higher-order terms of the expansion in this parameter, which corresponds to many-particle collisions, are identically equal to zero.

By way of an example of an application of (21), let us consider the stability of a system with two interacting periodic waves. We seek a solution of (21) in the form

$$f(\eta, x_0) = \frac{n_1(x_0, t)}{L_1}\delta(\eta - \eta_1) + \frac{n_2(x_0, t)}{L_2}\delta(\eta - \eta_2).$$

We have

$$\frac{\partial n_1}{\partial t} + \frac{\partial}{\partial x_0}(4\eta_1^2 - q_1 n_2) n_1 = 0,$$

$$\frac{\partial n_2}{\partial t} + \frac{\partial}{\partial x_0}(4\eta_2^2 + q_2 n_1) n_2 = 0;$$

$$q_1 = \frac{4}{\eta_1 L_2}(\eta_2^2 - \eta_1^2)\ln\frac{\eta_2 + \eta_1}{\eta_2 - \eta_1},$$

$$q_2 = \frac{4}{\eta_2 L_1}(\eta_2^2 - \eta_1^2)\ln\frac{\eta_2 + \eta_1}{\eta_2 - \eta_1}.$$

Linearizing $n_1 = 1 + \delta n_1$, $n_2 = 1 + \delta n_2$ and putting $\delta n_1, \delta n_2 \sim e^{-i\omega t + ikx}$, we get

$$\left(-\frac{\omega}{k} + 4\eta_1^2 - q_1\right)\left(-\frac{\omega}{k} + 4\eta_2^2 + q_2\right) + q_1 q_2 = 0.$$

Considering waves with close amplitudes, such that

$$4\eta_1^2 - q_1 = 4\eta_2^2 + q_2 = s_0,$$

we verify that there is instability with an increment

$$\omega / k = s_0 + i\sqrt{q_1 q_2}.$$

Similar instabilities can arise also for systems made up of three and more periodic waves. The development of such instabilities leads to regular or quasiregular oscillations of the soliton-gas density.

Since no redistribution of the energy over the degrees of freedom takes place when the solitons collide, the kinetics of the soliton gas differs in principle from the usual kinetics of waves in nonlinear media and is "reversible." It is not clear at present whether this property is exclusive for KDV systems or whether it takes place also for some other nonlinear systems.

In conclusion, the author thanks A. B. Shabat for certain valuable remarks.

3)The kinetic equations obtained in [11,12] for solitons differ from (21) and do not have the property of conserving the total distribution function. The discrepancy is due to the fact that an incorrect approximation was used in [11,12] for the solution of the KDV equation. In the cited papers, the solution was approximated by a linear superposition of solitons [11] or of periodic waves [12] with slowly varying parameters. Since the soliton collisions are rare, a fictitious small parameter was obtained in the problem, with respect to which expansion was carried out. However, because of the strong nonlinearity of the problem, the function u(x, t) differs strongly from a superposition of solitons in regions where the solitons collide and in which the entire effect "accumulates." In these regions it is necessary to use the exact solution of the KDV equation, as was done in the present paper.

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