SKIN EFFECT ON A ROUGH SURFACE

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Submitted September 14, 1970

Zh. Eksp. Teor. Fiz. 60, 838–845 (February, 1971)

The surface impedance of metals with a non-ideal boundary is calculated. In the case of an extremely anomalous skin effect the impedance tends to a value corresponding to specular reflection of the electrons from the surface.

1. INTRODUCTION

To study different surface phenomena in metals, the samples are prepared with sufficiently smooth surfaces. Usually the surface is given an optical mirror finish, and consequently the dimension of the natural roughnesses does not exceed the corresponding wavelength of light. It can probably be assumed that the roughness of the single-crystal samples has an atomic scale, comparable with the characteristic wavelength of the conduction electrons in the metal. For this reason it is customary to assume that the conduction electrons, in the main, are diffusely reflected from the boundary of the metal. This conclusion was reinforced also by a comparison of data on the surface impedance with the theory of Reuter and Sondheimer[1], according to which in the case of the extremely anomalous skin effect, when the ratio of the mean free path to the depth of the skin layer is $l/\delta \to \infty$, the values of the impedance in diffuse and specular reflection differ by a factor 8/9.

Recently, however, magnetic surface levels were observed in a number of metals[2]. These levels cannot exist in the case of diffuse reflection. This means that at small glancing angles corresponding to the observed surface levels, the electrons are reflected spectrally. Similar electrons with glancing angles on the order of δ/l play a decisive role in the anomalous skin effect.

The noted contradiction has resulted in a more careful attention to the phenomenological boundary condition used by Reuter and Sondheimer for the distribution function. The boundary condition obtained in[3] takes into account, in the Born approximation, the scattering of electrons by the roughnesses and differs appreciably from the former. It is valid for sufficiently low temperatures $T \ll T_D(\delta/l)^{1/4}$, where T_D is the Debye temperature. With increasing temperature, processes of phonon emission and absorption in collisions between electrons and the surface come into play, making the degree of diffuseness dependent on the temperature.

The present article is devoted to the theory of the skin effect with allowance for scattering of electrons by surface roughnesses. A preliminary report of the results was published earlier[4].

2. ELECTRIC CURRENT

Let us find the distribution of the electric field E(x) $\sim e^{-i\omega t}$ of frequency ω in a metal occupying the half-space $x > 0$. We confine ourselves for simplicity to a quadratic electron dispersion law $\epsilon = p^2/2m$. In this case the field and the current are parallel and their direction is chosen to be the y axis.

It is required to solve the Maxwell equation[1)]

$$E''(x) = -4\pi i\omega j(x) \tag{1}$$

in conjunction with the kinetic equation

$$v_x \frac{df}{dx} + (\tau^{-1} - i\omega)(f - f_0) = v_y E(x) \tag{2}$$

for the distribution function f, which determines the current:

$$j(x) = -\frac{1}{4\pi^3}\int d^3p\, v_y f \frac{df_0}{d\varepsilon} = \frac{1}{4\pi^3}\int d^2p\, f \frac{v_y}{|v_x|}.$$

The last integration over the tangential components p_y and p_z (the two-dimensional vector will henceforth be denoted by the letter p) is carried out over the Fermi surface $p_x^2 + \mathbf{p}^2 = 2m\epsilon_0 \equiv p_0^2$. In the kinetic equation (2) the collisions with the volume defects are taken into account by means of the time between the collisions τ.

The boundary condition at x = 0 for the equation (2) is of the form[3]

$$f^>(p) = \left[1 - p_x \int \frac{d^2p'}{\pi^2} p_x' \xi_2(\mathbf{p} - \mathbf{p}')\right] f^<(p) + p_x \int \frac{d^2p'}{\pi^2} p_x' \xi_2(\mathbf{p} - \mathbf{p}') f^<(p'). \tag{3}$$

Here $p_x = +(p_0^2 - p^2)^{1/2}$, $f^>$ and $f^<$ are the values of the distribution function for electrons traveling from the surface and to it, respectively; $\xi_2(p)$ is the Fourier component of the binary correlation function of the roughnesses. In the isotropic case $\xi_2(p)$ is characterized by two parameters (Fig. 1): the correlation radius d^{-1}, i.e., by the radius of the region on the p plane in which ξ_2 differs noticeably from zero, and by the value at p = 0: $\xi_2(0) \sim a^2d^2$. The parameters a and d have the following meaning: a^2 is the mean squared deviation of

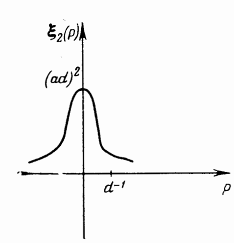

FIG. 1

1) In the intermediate formulas $c = \hbar = e = 1$.

the surface from the plane x = 0, and d is the average magnitude of the flat sections of the surface.

In the derivation of (3) it was assumed that the increments to unity in the right-hand side of the equation are small. For roughnesses of atomic scale, this requirement is satisfied by virtue of the smallness of the glancing angle $p_x/p_0 \sim \delta/l \ll 1$ of the electrons that make their main contribution to the current in the anomalous skin effect. In the normal skin effect, there is no analogous small parameter. However, the tangential components of the current are determined predominantly by the electrons traveling parallel to the surface, making it possible to use the condition (3) for order of magnitude estimates. The condition for the applicability of (3) improves for smooth surfaces with large d (see (13′) and (18′)).

We introduce the complex length $l = v_0/(\tau^{-1} - i\omega)$ ($v_0 = p_0/m$ is the Fermi velocity), which coincides with the mean free path for $\omega\tau \ll 1$, and in the opposite limiting case $|l|$ is the path traversed by the electron during one period of the field.

The solution of the kinetic equation (2)

$$f - f_0 = \int_{C(p)}^{x} dx' \frac{v_y E(x')}{v_x} \exp\left\{\frac{(x'-x)v_0}{l v_x}\right\}$$

is determined, accurate to an arbitrary function C(p), which for $v_x < 0$ is determined from the condition that f be finite at $x \to \infty$:

$$f^{<} - f_0 = \int_{\infty}^{x} dx' \frac{v_y E(x')}{v_x} \exp\left\{\frac{(x'-x)v_0}{l v_x}\right\},$$

and for $v_x > 0$ from the condition (3):

$$\int_{C(p)}^{0} dx' \frac{p_y}{p_x} E(x') e^{x'\eta} = \left(1 - p_x \int \frac{d^2p'}{\pi^2} p_x'\xi_2\right) \int_{\infty}^{0} dx' \frac{p_y}{-p_x} E(x') e^{-x'\eta} \quad (4)$$

$$+ p_x \int \frac{d^2p'}{\pi^2} p_x'\xi_2 \int_{\infty}^{0} dx' \frac{p_y'}{-p_x'} E(x') e^{-x'\eta'},$$

where $p_x = +(p_0^2 - p^2)^{1/2}$, $\eta = p_0/lp_x$, $\eta' = p_0/lp_x'$; for the equilibrium distribution function f_0, which depends only on the energy, the condition (3) is satisfied identically.

We calculate the current

$$j(x) = \frac{1}{4\pi^3} \int d^2p \frac{p_y}{p_x}\left(\int_{C(p)}^{x} dx' \frac{p_y}{p_x} E(x') e^{(x'-x)\eta} - \int_{\infty}^{x} dx' \frac{p_y}{p_x} E(x') e^{-(x'-x)\eta}\right).$$

We break up the integral from C(p) to x into two (from C(p) to 0 and from 0 to x) and use Eq. (4). In the integral resulting from the term with unity in the right-hand side of (4), we continue the electric field in even fashion to the region $x < 0$. We obtain

$$j(x) = \frac{1}{4\pi^3} \int d^2p \frac{p_y}{p_x}\left\{\int_{-\infty}^{+\infty} dx' \frac{p_y}{p_x} E(x') e^{-|x-x'|\eta}\right.$$

$$\left. - \int_{0}^{\infty} dx' E(x') e^{-x\eta} \int \frac{d^2p'}{\pi^2} \xi_2(\mathbf{p}-\mathbf{p}') \left[p_y p_x' e^{-x'\eta} - p_y' p_x e^{-x'\eta'}\right]\right\}. \quad (5)$$

We change over to Fourier components with respect to x, continue the current, just as the field, in even fashion. Then the connection between the current and the field (5) takes the form

$$j(k) = \sigma(k)\mathscr{E}(k) + \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \sigma(kk')\mathscr{E}(k'), \quad (6)$$

where

$$\sigma(k) = \frac{p_0 l}{2\pi^3} \int \frac{d^2p}{p_x} \frac{p_y^2}{p_0^2 + (klp_x)^2}, \quad (7)$$

$$\sigma(kk') = \frac{p_0^2 l^2}{2\pi^5} \int d^2p\, d^2p' \frac{p_x p_x' p_y \xi_2(\mathbf{p}-\mathbf{p}')}{p_0^2 + (klp_x)^2}$$

$$\times\left[\frac{p_y'}{p_0^2 + (k'lp_x')^2} - \frac{p_y}{p_0^2 + (k'lp_x)^2}\right]. \quad (8)$$

The integral term in (6) takes into account the diffuseness and is missing in the case of specular reflection ($\xi_2 = 0$).

3. ASYMPTOTIC FORM OF ELECTRIC CONDUCTIVITY

Formula (7) makes it possible to obtain known limiting expressions for the electric conductivity in the case of specular reflection:

$$\sigma(k) = \begin{cases} e^2 p_0^2 l/3\pi^2\hbar^3, & |kl| \ll 1, \quad (9) \\ e^2 p_0^2/4\pi\hbar^3|k|, & |kl| \gg 1. \quad (10) \end{cases}$$

Substituting these expressions in (1), we obtain the characteristic distance over which the electric field in the metal changes. Formula (9) leads to the usual depth of the skin layer:

$$\delta = \frac{c}{ep_0}\left|\frac{3\pi\hbar^3}{2l\omega}\right|^{1/2}, \quad (11)$$

and (10) to the depth

$$\delta = \frac{\hbar}{\omega^{1/3}}\left(\frac{c}{ep_0}\right)^{2/3}, \quad (12)$$

corresponding to the anomalous skin effect. In the latter case, as seen from (7), the electric conductivity $\sigma(k)$ at $k \sim \delta^{-1}$ is determined by electrons with small glancing angles $p_x/p_0 \sim \delta/|l| \ll 1$.

To calculate the asymptotic forms of $\sigma(kk')$ we turn to Fig. 2. The circle with radius p_0 shows the region of integration with respect to $\mathbf{p}'$ in (8). On the dashed periphery $p_x' = 0$. The circle of radius d^{-1} with center at the point $\mathbf{p}$ bounds the region in which the function $\xi_2(\mathbf{p}-\mathbf{p}')$ differs from zero, and its value here is of the order of a^2d^2.

For a sufficiently smooth surface, the parameter d is large and ξ_2 is a rapidly decreasing function. In this case the integrand in (8) must be expanded in powers of $\mathbf{p}-\mathbf{p}'$. The first-order term vanishes because ξ_2 is even, and formula (8) reduces to

$$\sigma(kk') = \frac{p_0^2 l^2}{2\pi^3} I_1 \int d^2p \frac{p_x p_y}{p_0^2 + (klp_x)^2} \Delta_p \frac{p_x p_y}{p_0^2 + (k'lp_x)^2}, \quad (13)$$

where the integral

$$I_1 = \int \frac{d^2p'}{2\pi^2} (\mathbf{p}'-\mathbf{p})^2 \xi_2(\mathbf{p}'-\mathbf{p}) \sim (a/d)^2 \quad (13')$$

depends only on the properties of the surface; Δ_p is the two-dimensional Laplacian.

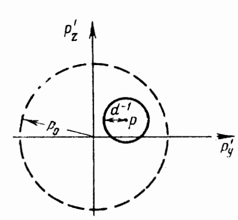

FIG. 2

The main contribution to the integral (13) is made by small p_x. Therefore, integrating in (13) by parts, it is necessary to differentiate only p_x:

$$\sigma(kk') = -\frac{p_0^2 l^2}{2\pi^3} I_1 \int d^2p \frac{p_y^2 p^2 [p_0^2 - (klp_x)^2][p_0^2 - (k'lp_x)^2]}{p_x^2 [p_0^2 + (klp_x)^2]^2 [p_0^2 + (k'lp_x)^2]^2}. \quad (14)$$

Integration with respect to the angle (14) entails no difficulty, and on changing over to the variable p_x there arises an integral that diverges logarithmically at the lower limit. The minimum value of p_x is determined from the condition that the circles be tangent (Fig. 2):

$$p_{x\,min} = [2p_0(p_0 - p)]_{min}^{1/2} = (2p_0/d)^{1/2}.$$

Under conditions of the normal skin effect, interest attaches to small k, $k' \sim \delta^{-1} \ll |l|^{-1}$. In this case we obtain with logarithmic accuracy

$$\sigma(kk') = -\left(\frac{p_0 l}{2\pi}\right)^2 I_1 \ln p_0 d, \quad p_0 d \gg 1, \quad k, k' \ll |l|^{-1}. \quad (15)$$

In the anomalous skin effect, the characteristic values are k, $k' \sim \delta^{-1} \gg |l|^{-1}$ and the integral (14) is determined by small p_x/p_0 that lie in the interval $|kl|^2, |k'l|^2 \gg p_0 d \gg 1$. It is therefore necessary to replace p in (14) by p_0, and the upper limit with respect to p_x must be set equal to infinity. After integration we obtain

$$\sigma(kk') = -\left(\frac{p_0 l}{2\pi}\right)^2 I_1 \left[\frac{k^2(k^4 - 3k'^4 - 6k^2k'^2)}{(k^2 - k'^2)^3} \ln \frac{p_0 d}{|kl|^2} - \left(\frac{k^2 + k'^2}{k^2 - k'^2}\right)^2 + (k \rightleftarrows k')\right], \quad (16)$$
$$p_0 d \gg |kl|^2, \quad |k'l|^2 \gg 1.$$

When $k = k'$, formula (16) simplifies:

$$\sigma(kk) = -\left(\frac{p_0 l}{2\pi}\right)^2 I_1 \ln \frac{p_0 d}{|kl|^2}, \quad p_0 d \gg |kl|^2 \gg 1. \quad (17)$$

With decreasing parameter d, the region of applicability of (16) and (17) becomes narrower, and when $|kl|^2$, $|k'l|^2 \gg p_0 d \gg 1$ the function ξ_2 is smoother than the pole factors in (8). In this case the principal role is assumed by the second term in the square brackets of (9), since the absence of the pole factor with p'_x extends the region of integration with respect to the primed variable. Formula (8) reduces to the form

$$\sigma(kk') = -\frac{p_0^2 l^2}{2\pi^3} I_2 \int \frac{d^2p\, p_x p_y^2}{[p_0^2 + (klp_x)^2][p_0^2 + (k'lp_x)^2]}, \quad (18)$$

where only the integral

$$I_2 = \int \frac{d^2p'}{\pi^2} p_x' \xi_2(\mathbf{p} - \mathbf{p}') \sim a^2 (p_0/d)^{1/2} \quad (18')$$

depends on the quality of the surface when $p_x \ll p_0$ and $p_0 d \gg 1$.

The integral (18) is determined by the values $p_x/p_0 \sim \delta/|l| \ll 1$. In view of its rapid convergence, the upper limits of integration with respect to p_x can be set equal to infinity, and we get

$$\sigma(kk') = \frac{-p_0^3}{4\pi l k k'(k + k')} I_2, \quad (19)$$
$$|kl|^2, \ |k'l|^2 \gg p_0 d \gg 1, \ k, \ k' > 0.$$

Finally, at sufficiently small Fermi momenta $p_0 d \ll 1$ the conductivity $\sigma(kk')$ in the region of the anomalous skin effect is determined by expression of the type (19), in which there appears in place of I_2

$$I_3 = \xi_2(0) \int \frac{d^2p'}{\pi^2} p_x' \sim a^2 d^2 p_0^3,$$

and in the region of the normal skin effect

$$\sigma(kk') = -\frac{l^2}{2\pi^3 p_0^2} I_3 \int d^2p\, p_x p_y^2 \sim -\frac{l^2 p_0^3}{2\pi^2} I_3. \quad (20)$$

4. SURFACE IMPEDANCE

In the experiment one measures the impedance

$$Z = E(0) \Big/ \int_0 j(x)\,dx. \quad (21)$$

With the aid of Maxwell's equation, the definition of the impedance can be rewritten in the form

$$Z = 4\pi i\omega E(0)/E'(0) = \frac{4\pi i\omega}{E'(0)} \int_0^\infty \frac{dk}{\pi} \mathscr{E}(k), \quad (22)$$

and the last equality takes into account the fact that $\mathscr{E}(k)$ is even.

The problem of determining the impedance consists of solving Maxwell's equation (1), which in terms of the Fourier component is given by

$$k^2 \mathscr{E}(k) + 2E'(0) = 4\pi i\omega j(k), \quad (23)$$

with the connection between the current and the field determined by (6). We shall solve it by iteration with respect to $\sigma(kk')$ in the sense of the expansion in terms of $p_x a$, which was used in the derivation of the boundary condition (3).

In the zeroth approximation we obtain the field

$$\mathscr{E}_0(k) = 2E'(0)[4\pi i\omega\sigma(k) - k^2]^{-1}$$

and the known expression for the impedance in specular reflection

$$Z_0 = 8i\omega \int_0^\infty dk [4\pi i\omega\sigma(k) - k^2]^{-1}. \quad (24)$$

The main contribution to the integral (24) arises on going around the poles. The values of k_0 at the poles give the previous definitions (11) and (12) of the skin-layer depth $\delta \sim |k_0|^{-1}$.

With the aid of the asymptotic forms (9) and (10) we obtain the well known limiting values for the impedance:

$$Z_0 = \begin{cases} \dfrac{(\pi\hbar)^{3/2}}{ecp_0}\left(\dfrac{6\omega}{l}\right)^{1/2}(1 - i), & \tilde{\delta} \gg |l|, \quad (25) \\[2ex] \dfrac{8\pi\hbar}{3^{3/2}}\left(\dfrac{\omega}{ec^2p_0}\right)^{2/3}(1 - i3^{1/2}), & \delta \ll |l|. \quad (26) \end{cases}$$

The impedance increment linear in $\sigma(kk')$ is

$$\Delta Z = 2^5\omega^2 \int_0^\infty \frac{dk\,dk'\sigma(kk')}{[k^2 - 4\pi i\omega\sigma(k)][k'^2 - 4\pi i\omega\sigma(k')]}. \quad (27)$$

The asymptotic expressions obtained in the preceding section enable us to estimate the integral (27). Breaking up the integral into regions in which the asymptotic forms of $\sigma(kk')$ are valid, we can verify that the main contribution is always made by the vicinity of the poles of the integrand (27).

In the case of the normal skin effect ($\tilde{\delta} \gg |l|$)

$$\Delta Z = -{}^1\!/_2 Z_0^2 \sigma(00), \quad (28)$$

where $\sigma(00)$ is given by formulas (15) and (20), and Z_0 by (25). We note that when $\omega\tau \gg 1$ the imaginary parts of l and Z_0 are larger than the real ones by a factor $\omega\tau$, whereas the increment ΔZ is the main real. An order of magnitude estimate yields

$$\left|\frac{\Delta Z}{Z_0}\right| \sim \begin{cases} (p_0^2 ad)^2 |l|/\tilde{\delta}, & p_0 d \ll 1, \quad (29) \\ (a^2|l|/d^2\tilde{\delta}) \ln p_0 d, & p_0 d \gg 1. \quad (30) \end{cases}$$

The factor $l/\tilde{\delta}$ arises here simply as the ratio of the depth l, to which the electrons carry information on the surface, to the thickness $\tilde{\delta}$, within which the current flows mainly.

In the anomalous skin effect, the relative increment to the impedance, due to the diffuseness, is ($\delta \ll |l|$)

$$\frac{\Delta Z}{Z_0} \sim \begin{cases} (p_0^2 ad)^2 \delta/l, & p_0 d \ll 1, \quad (31) \\ (p_0 \dot{a})^2 \delta/(p_0 d)^{1/2} l, & 1 \ll (p_0 d)^{1/2} \ll |l|/\delta, \quad (32) \\ (al/d\delta)^2 \ln |p_0 d \delta^2 / l^2|, & |l|/\delta \ll (p_0 d)^{1/2}, \quad (33) \end{cases}$$

where we have left out complex coefficients of the order of unity.

A plot of $\Delta Z/Z_0$ against l/δ at $p_0 d \gg 1$ is shown in Fig. 3. As $l \to \infty$ we have $\Delta Z \to 0$, i.e., the impedance tends to its specular value. This is explained by the fact that with increasing l/δ there decreases the glancing angle of the electrons that determine the value of the current, and at the same time the influence of the roughness also decreases. In the theory of Reuter and Sondheimer[1], the "specular" and "diffuse" values of the impedance differed by a numerical factor, since the phenomenological boundary condition used by them did not depend on the glancing angle.

With decreasing ratio l/δ, the increment increases, reaching a maximum $(\Delta Z/Z_0)_{max} \sim (ap_0)^2/p_0 d$ at $l/\delta \sim (p_0 d)^{1/2}$ (if $p_0 d \gg 1$), and then again begins to decrease. The latter is connected with the fact that the glancing angle δ/l of the effective electrons that make the main contribution to the current becomes here large compared with the angle interval $\Delta p_x/p_0 \sim (\Delta p_x/p_0)^{1/2} \sim (p_0 d)^{-1/2}$, in which the electron can be scattered by collisions with the surface, and the effective electrons turn out to be inside the skin layer after the scattering.

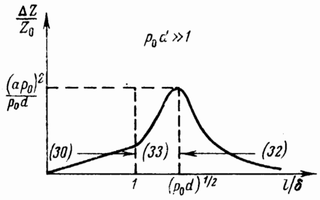

FIG. 3

The results (29)—(33) are valid in the approximation linear in $\sigma(kk')$, i.e., if $\Delta Z/Z_0 \ll 1$. Since $\Delta Z/Z_0$ decreases with increasing l, the approximation in question is sufficient in the region $l/\delta \gg (ap_0)^2/(p_0 d)^{1/2}$. However, the maximum value $(\Delta Z/Z_0)_{max}$ can be calculated in this manner only for smooth surfaces for which $(ap_0)^2/p_0 d \ll 1$.

I take the opportunity to thank A. A. Abrikosov and M. Ya. Azbel' for a discussion and remarks.

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Translated by J. G. Adashko
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