SOVIET PHYSICS JETP VOLUME 33, NUMBER 2 AUGUST, 1971

THE ROLE OF QUANTUM FLUCTUATIONS OF THE GRAVITATIONAL FIELD IN GENERAL RELATIVITY THEORY AND COSMOLOGY

V. L. GINZBURG, D. A. KIRZHNITS, and A. A. LYUBUSHIN

P. N. Lebedev Physical Institute, U.S.S.R. Academy of Sciences

Submitted August 17, 1970

Zh. Eksp. Teor. Fiz. 60, 451–459 (February, 1971)

The role of quantum fluctuations of the gravitational field is discussed in terms of including them in the nonlinear equations of general relativity theory. A treatment is given for gravitational fields of arbitrary symmetry, although the explicit direct calculation of the vacuum contribution is made only for the closed Friedman cosmological model. Our results lead, first, to the assertion that the dropping of the Λ term is not justified. Second, when one uses appropriate ("observed") values of the gravitational constant $\kappa = 8\pi k/c^4$ and the cosmological constant Λ, the effect of the quantum (zero-point) fluctuations is extremely small as long as the matter density is small in comparison with the characteristic density $\rho_{cr} \sim (c^3\kappa^2\hbar)^{-1} \sim 10^{94}\ \mathrm{g\cdot cm^{-3}}$, so that the radius of curvature a is large compared with $l_g \sim (\kappa\hbar c)^{1/2} \sim 10^{-33}$ cm. Near a singularity, however, the fluctuation effects can in principle be very important. This fact may be of importance for some cosmological models now under discussion.

THE theory of the gravitational field (specifically, we shall be concerned with the general theory of relativity, referred to hereafter as GTR) is usually constructed on a classical (nonquantum) basis. This approach is quite natural, since quantum effects are extremely small in the case of "ordinary" astronomical problems in which the force of gravity plays a part. The situation can be different, however, if one considers solutions with singular points (cosmology, collapse). Besides this, quantum fluctuations of the gravitational field could in principle be important when included in the nonlinear field equations themselves in the form of some sort of vacuum "contribution" to the expression for the energy-momentum tensor or the Λ term.

The question of including quantum fluctuations, and quantum effects in general, in GTR has been repeatedly discussed in the literature (cf., e.g.,[1-8]), but it seem to us that the question has not been made sufficiently clear. This is primarily due to the appearance of diverging (and indeed nonrenormalizable) expressions, which have not as yet allowed the construction of any consistent quantum theory of the gravitational field.

In the present paper we do not claim any success in the solution of this fundamental problem. Nevertheless, it seems interesting to present here a rather simple analysis which indicates the possibility of in some fashion estimating the importance of quantum fluctuations in GTR.

1. We begin with the Einstein equations with the cosmological term[1)]

$$G_i^k \equiv R_i^k - {}^1\!/_2\delta_i^k R = \varkappa T_i^k + \Lambda\delta_i^k, \qquad \varkappa = 8\pi k/c^4, \tag{1}$$

which we shall regard as exact microscopic equations; in the description of quantum effects they are to be understood as equations for Heisenberg operators.

Suppose the dynamical quantities in these equations can have fluctuations around their average values. These fluctuations include: zero-point (vacuum) or actual fluctuations of the matter or the gravitational field, spatial nonuniformity in the distribution of the matter,[10] and so on. In this paper we shall consider only quantum fluctuations of a gravitational field of the tensor type, which in the linear approximation are not coupled with the fluctuations of the matter (cf.[11]). We shall regard the energy-momentum tensor $T_i^k = (\epsilon + p)u_i u^k - p\delta_i^k$ as a given nonfluctuating quantity satisfying the conservation law $T^k_{i;k} = 0$.

Taking the average over all the "modes" of the vacuum fluctuations of the gravitational field,[2)] we arrive at the equations

$$G_i^k = \varkappa T_i^k + \Lambda\delta_i^k + \Phi_i^k \tag{2}$$

where G_i^k is the Einstein tensor corresponding to the average metric and Φ_i^k is a tensor describing the contribution of the fluctuations.

2. We now ascertain the general structure of the tensor Φ_i^k, considering the case in which the tensors G_i^k, R^j_{klm}, etc., can in a certain sense be regarded as small. In the cosmological application this means that we consider comparatively late stages of the evolution. We make the assumption (see below, Sec. 4) that the tensor Φ_i^k can be expanded in a series in the average tensor R^i_{klm}

$$\Phi_i^k = \Phi_{i(0)}^k + \Phi_{i(1)}^k + \Phi_{i(2)}^k + \ldots, \tag{3}$$

where the index in parentheses shows the power to which the tensor R^i_{klm} occurs in the term. More exactly, we shall also include in $\Phi^k_{i(2)}$ terms of the type $R^{;m}_{;m}$, which, though linear in R^i_{klm}, also contain additional powers of a length in the denominator. Therefore the expansion (3) is actually an expansion in inverse lengths (the tensor R^i_{klm} itself has the dimensions of inverse length squared).

We make two remarks. The fact that the tensor Φ_i^k must be expressible only in terms of the tensor

[1)]Our notation throughout is essentially the same as in [9].

[2)]We do not introduce a special symbol for averages, since hereafter only averages will appear in the calculations.

R^i_{klm} and its derivatives is due to our neglect of the fluctuations of the matter. In this case the tensor T^k_i must appear in the expression for Φ^k_i as a whole, and can be simply expressed in terms of the tensor R^k_i by Eq. (1), in zeroth order in the fluctuations. Our second remark concerns a possible nonlocal dependence of Φ^k_i on R^i_{klm}. Taking (3) as an expansion in inverse powers of a length, we can reflect this nonlocality by the introduction of terms containing derivatives of the Rieman tensor.

We proceed to the determination of the forms of the tensors $\Phi^k_{i(0)}$, $\Phi^k_{i(1)}$, etc., in Eq. (3). The first of these quantities must be of the form const $\cdot\,\delta^k_i$, owing to covariance and the fact that it is independent of R^i_{klm}. Therefore the tensor $\Phi^k_{i(0)}$ leads to a renormalization of the cosmological term, and can be ignored hereafter.

The tensor $\Phi^k_{i(1)}$, which is linear in the Riemann tensor, can have the following structure:

$$\Phi_{i(1)}^{k} = AR_i^k + BR\delta_i^k \equiv A'G_i^k + B'R\delta_i^k,$$

where A, B, A′, B′ are constants. The term proportional to G^k_i gives a renormalization of the gravitational constant and can also be omitted. Hereafter we shall take κ and Λ to mean the renormalized values of these quantities, which are directly measurable. (From this we see in particular that it is not justified to assume that $\Lambda = 0$). As for the term $B'R\delta^k_i$, it must be absent, since it violates in this order of the expansion (3) the conservation law $\Phi^k_{i;k} = 0$, which follows from the equations $G^k_{i;k} = 0$ and $T^k_{i;k} = 0$ (see Sec. 1). Accordingly, we can take $\Phi^k_{i(0)} = \Phi^k_{i(1)} = 0$.

Finally, the general structure of the tensor $\Phi^k_{i(2)}$ is

$$\Phi_{i(2)}^{k} = CRR_i^k + DR_{ib}^{ka}R_a^b + ER_a^kR_i^a + FR_{bc}^{ka}R_{io}^{bc} + HR^2\delta_i^k + IR_a^bR_b^a\delta_i^k + JR_{;i}^{;k} + KR_{i;a}^{k;a} + LR_{;a}^{;a}\delta_i^k.$$

The constants C, D, E, F, H, I, K, J, and L are actually connected by a number of relations, which follow from the conservation law $\Phi^k_{i;k} = 0$ already mentioned. Namely, it turns out that in this order there are only two linearly independent conserved combinations:

$$\Phi_{i(2)}^{k} = \alpha\varphi_{1i}^{k} + \beta\varphi_{2i}^{k}, \tag{4}$$

where

$$\varphi_{1i}^{k} = -2RR_i^k + {}^1/_2R^2\delta_i^k + 2R_{;i}^{;k} - 2R_{;a}^{;a}\delta_i^k, \tag{5a}$$

$$\varphi_{2i}^{k} = -2R_b^aR_{ia}^{kb} + {}^1/_2R_a^bR_b^a\delta_i^k + R_{;i}^{;k} - R_{i;a}^{k;a} - {}^1/_2R_{;a}^{;a}\delta_i^k. \tag{5b}$$

It is essential to emphasize that these combinations become linearly dependent in the conformal-flat case, which in particular includes the Friedman metric. In this case, however, an additional conserved tensor appears,

$$\varphi_{2i}^{\prime k} = -{}^2/_3RR_i^k + R_i^aR_a^k + {}^1/_4R^2\delta_i^k - {}^1/_2R_a^bR_b^a\delta_i^k. \tag{5c}$$

Therefore in our Eq. (6) below we can take φ^k_{2i} to mean either (5b) or else (5c).[3)]

3. Accordingly, Eq. (1) takes the form

$$G_i^k = \kappa T_i^k + \Lambda\delta_i^k + \alpha\varphi_{1i}^k + \beta\varphi_{2i}^k. \tag{6}$$

In accordance with the assumptions we have made, we take the quantities α and β that appear here to be constants, although, as will be shown in Sec. 4, it is more probable that they have a weak (logarithmic) dependence on the curvature. The general treatment which we shall give does not allow us to exclude the possibility that $\alpha = 0$, $\beta = 0$, or $\alpha = \beta = 0$. We think, however, that such a situation is extremely improbable. One can most simply estimate the order of magnitude of the constants α and β from dimensional considerations, assuming, of course, that in a future "good" quantum theory of gravitation these constants will have finite values. Then α and β must be of the dimensions of the square of length, and will evidently be proportional to the only quantity of these dimensions that is characteristic for the quantum fluctuations of the gravitational field, $\kappa hc \equiv l_g^2 \sim (10^{-33}\ \text{cm})^2$.

Let us now examine the role of the fluctuation correction in Eq. (6) as applied to cosmology. We here confine ourselves to homogeneous and isotropic cosmological models. As is well known, for the solution of this sort of cosmological problem it is sufficient to use the (0, 0) component of Eq. (6). Substituting in Φ^k_i the curvature tensor expressed in zeroth approximation in terms of ϵ, p, and Λ, we get[4)]

$$G_0^0 = \kappa\varepsilon + \Lambda + l_r P_2(\kappa\varepsilon, \kappa p, \Lambda), \tag{7}$$

where P_2 is a homogeneous quadratic form with coefficients of the order of unity. At late stages of the evolution, to which the expression (7) derived from perturbation theory applies directly, we break up P_2 into a sum of three terms: $M\Lambda^2 + \Lambda Q_1(\kappa\epsilon, \kappa p) + Q_2(\kappa\epsilon, \kappa p)$, where M is a constant of the order of unity, and Q_1 and Q_2 are linear and quadratic forms. It is easily seen that the first two terms of this sum lead to effective changes of κ and Λ by quantities of the orders $l_g^2\Lambda\kappa$ and $l_g^2\Lambda^2$. These changes can be neglected, since at the present epoch $\kappa\epsilon \lesssim 10^{-56}\ \text{cm}^{-2}$ and $|\Lambda| \lesssim \kappa\epsilon$. Therefore instead of P_2 we can simply use Q_2 in (7).

At the present epoch the ratio of the fluctuation term in (7) to the main term is of the order of $l_g^2\kappa\epsilon \sim 10^{-120}$ and is altogether negligible. It is natural to ask when the fluctuation terms could have played an appreciable role. This would set a possible limit on the applicability of classical GTR. Continuing (7) formally into the region of large density, we get as an estimate of the ratio just mentioned the quantity $l_g^2\kappa\epsilon$. We see from this that the fluctuations could have been important for

$$\rho = \frac{\varepsilon}{c^2} > \rho_{cr} \sim \frac{1}{\kappa l_r^2 c^2} \sim \frac{c^5}{k^2\hbar} \sim 10^{+94}\ \text{g-cm}^{-3} \tag{8}$$

This conclusion is quite natural, since ρ_{cr} is the only quantity of the dimensions of density that can be con-

3)It is interesting to note that the two combinations (5a) and (5b) can be derived by varying functionals $\int d^4x\,(-g)^{1/2}\mathcal{L}$ with $\mathcal{L}$ equal to the respective quantities R^2 and $R_a{}^bR_b{}^a$. In this connection we note that the variation of functionals of this type automatically leads to conserved quantities (cf, e.g., [12]).

4)More exactly, there is an additional term of the form $l_g^2\kappa\epsilon/a^2$ in the right member of (7), owing to the presence of derivatives of the curvature in the tensor $\varphi_{1i}{}^k$. This then makes no change at all in our calculations to be given here in the case of the Friedman cosmological model, but in other cases terms of this form might be important.

structed from κ, $\hbar$, and c.[5)] It may be supposed that an analogous result holds also for more general cosmological models, but this question calls for special investigation.

We note that although we have used in the derivation of (8) the expression (7), which generally speaking holds only in the region of small densities, the result can nevertheless be regarded as correct in order of magnitude. The point is that precisely for $\rho \sim \rho_{cr}$ the cubic and more complicated terms neglected in (3) are of the same order as $\Phi^k_{i(2)}$.

Meanwhile, it is of course impermissible to use Eq. (7) for $\rho \gtrsim \rho_{cr}$, and particularly in the neighborhood of a singular point. In this connection we shall mention a paper of Ruzmaĭkina and Ruzmaĭkin,[13] in which an equation analogous to (7) was used to determine the effect of matter fluctuations on the behavior of the cosmological solution near the singular point. Besides the need to include cubic and higher terms, we must point out that the authors of[13] start from the proposition that classical GTR can be applied up to densities such that the radius of curvature of the world is of the order of l_g. Actually, as we have just shown, the limit of applicability of GTR is many orders of magnitude lower.

We also point out that the equation derived in[13] agrees with Eq. (7) only for a nonfriedman metric, since it contains the term φ^k_{2i} instead of φ'^k_{2i}. The absence of the latter quantity comes from the calculations of[4] applied to the matter fluctuations. In this connection we note that we do not think it obvious that the operation used in[4], of averaging over the matter fluctuations in the Lagrangian before varying the action function, is correct. Therefore it might be that a more correct averaging over the matter fluctuations would have led to the appearance of the term φ'^k_{2i}. As for the appearance of this term in our equations for the gravitational field, there can be no objections to it. Moreover, the term φ'^k_{2i} behaves in a radically different way from φ^k_{2i}, and does not vanish in the hot model when R = 0 (for $\Lambda = 0$).

4. To corroborate our formulas and, mainly, to check the assumption that the Φ^k_i are analytic functions of R^i_{klm}, we shall give below the result of a direct calculation of the coefficients α and β in (4). We shall use an approximate method analogous to the well known fluctuation method in the theory of the Lamb shift.[14] We introduce the fluctuations of the metric by formally replacing the metric tensor g_{ik} with $g_{ik} + \delta g_{ik}$, where g_{ik} is the mean value and the amplitude of the fluctuations will be determined subsequently from the condition that the energy of each vibrational degree of freedom be normalized to the zero-point energy of an oscillator. When we substitute the expression $g_{ik} + \delta g_{ik}$ for the metric tensor in (1) and work out the expansion of the tensor G^k_i, which is nonlinear in the g_{ik}, to and including the second degree in the δg_{ik}, we get correction terms of the first and second orders in the δg_{ik} (we shall denote them by $\delta_1 G^k_i$ and $\delta_2 G^k_i$). The equation $\delta_1 G^k_i = 0$ provides the determination of the form of the function δg_{ik}. Finding the average over all the "modes" of the fluctuations, which we shall indicate with a bar, we get $\Phi^k_i = -\overline{\delta_2 G^k_i}$.

The perturbation δg_{ik} of the gravitational field can be put in the form of a linear superposition of "harmonics" $\delta g_{ik(N)}$, $\delta g^*_{ik(N)}$ (N is the index which numbers the "harmonics"), which are described in detail in[11]:

$$\delta g_{ik} = \sum_N (C_N \delta g_{ik(N)} + C^*_N \delta g^*_{ik(N)}).$$

Substitution of this expansion in $\delta_2 G^k_i$ leads to an expression

$$\delta_2 G^k_i = {}^1/_2 \sum_N (C_N C^*_N + C^*_N C_N)\, \delta_2 G^k_{i(N)}, \qquad (9)$$

where $\delta_2 G^k_{i(N)}$ is the corresponding quantity for the N-th "harmonic." When we consider the gravitational waves as quantized, the coefficients C_N are regarded as operators.

As is well known, the quantum theory of the gravitational field encounters a number of difficulties, even apart from the problem of renormalizations.[15] In our case, however, we are concerned with the quantization of a weak field δg_{ik} with given average field g_{ik}. Under these conditions the quantization of the gravitational field is analogous to that of the electromagnetic field, for example. This means that the small fluctuations of the gravitational field formally correspond to the presence of a physical (tensor) quantized field in the world with geometry defined by the quantities g_{ik}. However, the quantization of the gravitational field has the specific feature that it must be carried out with the curvature of space-time taken into account.[6)] Moreover, because there is no true energy-momentum tensor and because the problem is nonstationary, it is impossible to make direct use of the standard Hamiltonian formalism. Nevertheless it turns out[16] that in this case too, with a suitable normalization of the harmonics $\delta g_{ik(N)}$, we can retain the usual commutation relations for the amplitudes:

$$[C_N, C^+_M] = \delta_{NM}, \quad [C_N, C_M] = 0.$$

Using this result and carrying out a quantum-mechanical averaging in (9), we get

$$\overline{\delta_2 G^k_i} = \sum_N [{}^1/_2 + \rho(N)]\, \delta_2 G^k_{i(N)},$$

5) The world radius of curvature a that corresponds to the density (8) is much larger than the quantity l_g (more exactly, by a we mean the quantity a or b that occurs in the Friedman solutions; see [9], Secs 104–106). In fact, supposing that at present $\kappa\epsilon = a_0 a^{-3}$ with $a_0 \sim 10^{28}$ cm, and referring relative values to the critical density (i.e., setting $\kappa\epsilon \sim \kappa c^2 \rho_{cr}$), we have $a_{cr} \sim (a_0 l_g)^{1/3} \sim 10^{-12}$ cm. This is of course a crude calculation, since at high densities the equation of state does not correspond to the "dustlike-matter" model we are using here. Therefore in reality $a_{cr} > 10^{-12}$ cm. If one uses the equations of state p = 0 for small densities and $p = \epsilon/3$ for large ones, one gets $a_{cr} \sim 10^{-5}$ to 10^{-6} cm. To find an upper limit on a_{cr} we can use the hard-limit equation of state $p = \epsilon$, for which $\kappa\epsilon = a_0^4 a^{-6}$ and $a_{cr} \sim (a_0^2 l_g)^{1/3} \sim 10^8$ cm. The inequality $a_{cr} \gg l_g$ simply reflects the fact that the quantum fluctuations become important not when the "size" of entire universe is of the order of l_g, but when a mass $M_g \sim (c\hbar/G)^{1/2} \sim 10^{-5}$ g is concentrated in a volume of the order of l_g^3. Of course, the largest components of the tensor R_{iklm} are then of the order of $1/l_g^2$.

6) As will be seen from what follows, we are mainly interested in precisely the terms that essentially depend on the quantity a/λ, where λ is the wavelength of a graviton.

where $\rho(N)$ are occupation numbers. Values $\rho(N) = 0$ correspond to the vacuum fluctuations. For the closed model a direct calculation, whose details we omit (cf.[16,11]) leads for $\rho(N) = 0$, after the necessary subtractions, to an expression for $\overline{\delta_2 G_i^k}$ of the type of Eq. (4) with

$$\alpha, \beta \sim l_r^2 \sum_{n=3}^{\infty} \frac{1}{n},$$

where n is a quantum number occurring in N and having the meaning of the ratio of the radius of curvature a of the world to the wavelength of the gravitation. Assuming that the divergent sum is "cut off" at a value $n = n_0$ which corresponds to an effective distance of the order of l_g, we get $n_0 \sim a/l_g$ and $\alpha, \beta \sim l_g^2 \ln(a^2/l_g^2)$. In order to write this expression in an explicitly covariant and local form, we must introduce in the argument of the logarithm a scalar constructed from the Riemann tensor. We have no way to determine it exactly, and can only indicate that possible quantities are $l_g^{-2} R^{-1}$, $l_g^{-2}(R_a^b R_b^a)^{-1/2}$, etc. The difference between them is beyond the scope of the logarithmic accuracy we are considering.

Accordingly, we conclude that the coefficients α and β in (4) are most likely not constants, but depend logarithmically on the curvature (see also[17]). This, however, has no effect at all on our previous conclusions about the role of quantum fluctuations in GTR.

5. One of the central questions in GTR and in cosmology is the following: What are the limits of applicability of the basic equations of the theory; specifically, what are the smallest sizes and matter densities for which one can use the equations of GTR and their cosmological solutions?

We here abstract from the possibility that there is need for some modification of GTR even in weak fields.[7)] Then, according to classical ideas, the limits of GTR are due to the fact that the field equations (1) are set up by using the requirement that derivatives of orders higher than the second not occur. If we use a series expansion, the introduction of higher derivatives leads to the appearance in the equations for the g_{ik} of terms of the type of $\Phi_{i(2)}^k$, and of higher orders in the tensor R_{klm}^i and its derivatives. In order of magnitude all of these terms contain additional factors $(l/a)^2$, $(l/a)^4$, and so on, as compared with the terms usually taken into account in GTR, where a is the characteristic radius of curvature of the g_{ik} field and 1 is some universal length. Such terms can in general persist even in the nonrelativistic limit $c \to \infty$, and correspond to replacing the equation $\Delta\varphi = 4\pi k\rho$ with an equation of the type

$$\Delta(1 + al^2\Delta + bl^4\Delta^2 + \ldots)\varphi = 4\pi k\rho,$$

where φ is the gravitational potential and $a \sim b \sim 1$ are constants.

It is interesting to see what limits are put on 1 by the observations (see also[18]). Let us assume, for example, that the correction $l^2\Delta\varphi$ amounts to no more than 1 percent of the usual relativistic corrections, which are of the order of $(\varphi/c^2)\varphi$. Then for the case of the sun's field, in which $|\varphi|/c^2 \lesssim 10^{-6}$ (at the surface of the sun $|\varphi| = kM_\odot/r_\odot \approx 2\cdot 10^{-6}c^2$) and $l^2\Delta\varphi \lesssim l^2\varphi/r_\odot^2$, we have

$$\frac{l^2}{r_\odot^2} \leqslant 10^{-2}\frac{kM_\odot}{r_\odot c^2} \sim 10^{-8} \text{ and } l \leqslant 10^{-4} r_\odot \sim 10^7 \text{ cm}$$

It is quite possible that a more detailed examination of the motion of the planets or of their satellites will make possible a considerable lowering of this limit on l. Equations with higher derivatives are also by no means always acceptable, owing to the possibility of the appearance of inadmissible new solutions, in particular solutions corresponding to negative energy.

We shall not dwell further on these questions, since it is most probable that a new fundamental length, if it exists, must be significant not only in the theory of gravitation, but also in electrodynamics and in general for all physical fields. At the same time the quantum-field-theory equations now used are valid at least down to distances of the order of 10^{-15} cm, and according to some arguments[19] even down to 10^{-20} cm. Consequently, we may suppose that $l \lesssim 10^{-16}$–10^{-20} cm, and it is this l that determines the limits of GTR. If, on the other hand, no new fundamental length exists in nature, then obviously the value that must be taken for l is the gravitational length $l_g \sim (\kappa\hbar c)^{1/2} \sim 10^{-33}$ cm.

Accordingly, from general considerations we may already suppose that the maximum breadth of the region of applicability of GTR is determined by the length l_g. In the foregoing this conclusion has, on one hand, been given a somewhat concrete form. On the other hand, it has been shown that for the average ("smoothed") g_{ik} field the corresponding correction terms [cf., e.g., Eq. (7)] appear even in the case when we adopt the Einstein equations (1) for the true g_{ik} field. Finally, it has been shown that the correction terms proportional to l_g^2 are already large (and consequently can radically alter the cosmological solutions) not at $a^2 \sim l_g^2$, but for densities $\rho \sim \rho_{cr} \sim (\kappa l_g^2 c^2)^{-1} \sim 10^{94}$ g·cm^{-3} corresponding to a radius of curvature $a_{cr} \gtrsim 10^{-12}$ cm $\gg l_g$. We recall also that we have here taken into account only the fluctuations of the gravitational field, while there can be some contribution to the correction terms (and possibly a still larger contribution) from the fluctuations of the matter, i.e., the tensor T_i^k. Thus there are no grounds for trusting the cosmological solutions of GTR [i.e., the corresponding solutions of Eq. (1)], not only as we approach the singular point $a \to 0$, or for $a \lesssim l_g$, but also everywhere for $a \lesssim a_{cr} \gg l_g$.

The question as to the minimum values a_{min}, down to which one can use the singular solutions of GTR in the case of the homogeneous and isotropic cosmological models or of the spherically symmetric collapse problem, is not an especially sharp one. Obviously the point is that in these cases the development of the system for $a > a_{min}$ does not depend so very strongly on its behavior for $a \le a_{min}$. The situation can be radically different in anisotropic and homogeneous models, or in the analysis of the question of the "initial" perturbations (inhomogeneities) against the

[7)] At present we see no real basis for such assumptions (for example, for replacing GTR with a tensor-scalar theory). We must not, however, lose sight of the fact that even in weak fields GTR has so far been verified experimentally only to low accuracy, not better than a few percent.

background of a homogeneous and isotropic solution (cf., e.g.,[20]).

What we have said is perhaps especially important in the treatment of cosmological models with "intermixing" (oscillations), which are now attracting much attention.[8,21,22] Such models can be of actual cosmological interest,[22] evidently, only if the corresponding GRT solutions are good on a scale (of the radius a) incomparably smaller even than the length $l_g \sim 10^{-33}$ cm. As we have seen, however, all known arguments bear strongly against such a possibility (the considerations given in Misner's paper,[8] so far as we can understand, relate to other aspects of the problem and do not change the conclusions reached here). It is true that this in itself does not logically exclude the possibility of the unlimited use of the classical singular solutions of GTR for applications to the cosmological problem, for example; our estimates and qualitative arguments are not rigorous enough for this. But also, on the contrary, one can be convinced, in our opinion, of the validity of solutions of the type of those in[8] and and[21], or any other classical singular solutions, for the analysis of the cosmological problem, only on the basis of the quantum-theory equations of the gravitational field and a sufficiently thorough investigation of these equations.

[1] W. H. McCrea, Proc. Roy. Soc. 206, 562 (1951).

[2] J. A. Wheeler, Einstein's Vision, Springer Verlag, 1968. R. U. Sexl, Fortschr. d. Phys. 15, 269 (1967).

[3] É. B. Gliner, Zh. Eksp. Teor. Fiz. 55, 2334 (1968) [Sov. Phys.-JETP 28, 1238 (1969)].

[4] A. D. Sakharov, Dokl. Akad. Nauk SSSR 177, 70 (1967) Sov. Phys.-Doklady 12, 1040 (1968).

[5] Ya. B. Zel'dovich, Usp. Fiz. Nauk 95, 209 (1968) [Sov. Phys.-Usp. 11, 381 (1968)].

[6] N. Kumar, J. Phys. A (Gen. Phys.) 2, 210 (1969); Prog. Theor. Phys. 41, 382 (1969).

[7] M. A. Markov, Zamknutost' Vselennoi i zakony sokhraneniya elektricheskogo, barionnogo, i leptonnykh zaryadov (The Closedness of the Universe and the Conservation of the Electric, Baryon, and Lepton Charges), Preprint OIYaI (Joint Inst. Nucl. Res.) D2-4534, 1969; Ann Phys. 59, 109 (1970).

[8] C. W. Misner, Phys. Rev. 186, 1319, 1328 (1969); Classical and Quantum Dynamics of Closed Universe, Preprint, 1969.

[9] L. D. Landau and W. M. Lifshitz, Classical Field Theory, Reading, Mass., Addison-Wesley, 1962.

[10] I. Z. Fisher and M. F. Shirokov, Astron. Zh. 39, 899 (1962).

[11] E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk 80, 391 (1963) [Sov. Phys.-Usp. 6, 495 (1964)]; Adv. in Phys. 12, 208 (1963).

[12] J. A. Schouten, Tensor Analysis for Physicists, Oxford Univ. Press, 1954.

[13] T. V. Ruzmaĭkina and A. A. Ruzmaĭkin, Zh. Eksp. Teor. Fiz. 57, 680 (1969) [Sov. Phys.-JETP 30, 372 (1970)].

[14] T. Welton, Phys. Rev. 74, 1157 (1948).

[15] P. G. Bergmann, Status of Canonical Quantization, Preprint, 1969.

[16] A. A. Lyubushin, Teor. i Matem. Fiz. (in press).

[17] B. S. De Witt, Phys. Rev. 162, 1239 (1967).

[18] N. M. Polebtkov-Nikoladze, Zh. Eksp. Teor. Fiz. 57, 2010 (1969) [Sov. Phys.-JETP 30, 1089 (1970)].

[19] D. A. Kirzhnits and V. A. Chechin, Yad. Fiz. 7, 431 (1968) [Sov. J. Nucl. Phys. 7, 275 (1968)].

[20] L. M. Ozernoĭ and G. V. Chibisov, Astron. Zh. 47, 769 (1970).

[21] E. M. Lifshitz and I. M. Khalatnikov, ZhETF Pis. Red. 11, 200 (1970) [JETP Lett. 11, 123 (1970)]; Phys. Rev. Lett. 24, 76 (1970).

[22] A. A. Doroshkevich and I. D. Novikov, Peremeshannyi mir i kosmologicheskaya problema (The Mixmaster Universe and the Cosmological Problem), Preprint IPM (Inst. Prikl. Mekh.) No. 15, 1970; Astron. Zh. 47, 948 (1970).

Translated by W. H. Furry
49