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QUASILINEAR RELAXATION OF AN ULTRARELATIVISTIC ELECTRON BEAM IN A PLASMA

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We investigate the kinetic stage of quasilinear relaxation of an ultrarelativistic electron beam both in a homogeneous and in an inhomogeneous plasma. We show that in a homogeneous plasma the beam releases approximately half of its initial energy. The inhomogeneity of the plasma becomes significant under the condition $\mu\Lambda \equiv \Lambda(c/\omega_p L) \times (E/mc^2)(n/n') \gg 1$, where Λ is the Coulomb logarithm, L is the characteristic scale of the inhomogeneity, c is the speed of light, ω_p is the electron plasma frequency, $E \gg mc^2$ is the initial energy of the beam electrons, n is the plasma concentration, and n′ is the beam concentration. If the initial angle scatter of the beam $\Delta\theta_0$ is large, $\Delta\theta_0 > (\mu\Lambda)^{-1}$, then there is no relaxation. On the other hand, if $\Delta\theta_0 < (\mu\Lambda)^{-1}$, then the relaxation proceeds but the beam releases into the plasma only a negligible $(\sim(\mu\Lambda)^{-1})$ fraction of its initial energy.

1. INTRODUCTION

THE main question to be answered by the theory of relaxation of an ultrarelativistic electron beam in a plasma is what is the fraction of the initial energy lost by the beam as a result of two-stream instability, and over what length does this loss occur. Important results in this direction were obtained by Faĭnberg, Shapiro, and Shevchenko[1], who considered both the hydrodynamic and the kinetic stages of the instability for the case of a homogeneous plasma. We shall investigate in greater detail the kinetic stage of the instability, paying principal attention to the effects connected with the inhomogeneity of the plasma.

The important role of the plasma inhomogeneity in the problem of electron-beam relaxation is due to the specific character of the dispersion of the Langmuir oscillations: even a small change in the plasma concentration leads to an appreciable change in the phase velocity of these oscillations and to violation of the conditions for resonant interaction between the oscillations and the beam particles[2-4]. As will be shown in the present paper, it is precisely the presence (or absence) of inhomogeneity which determines the effectiveness of the interaction between the beam and the the plasma.

A complete theory of relaxation should undoubtedly include an allowance for the nonlinear processes. However, as the first step it is reasonable to confine oneself to an investigation of the quasilinear equations, so as to make use of the corresponding solution as the starting point in the construction of the general theory. In addition, the quasilinear solution is of considerable interest in itself, for in sufficiently weak beams the nonlinear effects are negligible. It is difficult at present to estimate quantitatively the limits of applicability of the quasilinear approximation, since the nonlinear processes with participation of Langmuir oscillations are apparently also very sensitive to the influence of inhomogeneity. The authors propose to present the corresponding estimates in a forthcoming paper.

2. FORMULATION OF THE PROBLEM AND FUNDAMENTAL EQUATIONS

We confine ourselves to an investigation of electron beams with sufficiently large angle spread in momentum space:

$$\Delta\theta > mc^2/E \tag{1}$$

where $E \gg mc^2$ is the energy of the beam electrons. Under this condition we can neglect the difference between the modulus of the beam-electron velocity and c, and we can assume that $\mathbf{v} = c\mathbf{p}/p$, where p is the momentum of the electron. If furthermore

$$\Delta\theta > \left(\frac{n'}{n}\frac{mc^2}{E}\right)^{1/4}, \tag{2}$$

where n′ and n are respectively the concentrations of the beam and of the plasma, then the instability is kinetic[1)].

To get an idea of the orders of magnitude of the quantities, we use the beam and plasma parameters given in the paper of Winterberg[5] in connection with experiments proposed there for heating of plasma of a solid target: $n' \sim 10^{18}\ \mathrm{cm^{-3}}$, $n \sim 10^{22}\ \mathrm{cm^{-3}}$, $E \sim 10$ MeV. Substituting these values of the parameters in (1) and (2) we find that, as applied to such experiments, our results will be valid already at $\Delta\theta > 0.05$.

Assuming inequalities (1) and (2) to be satisfied, let us consider the stationary boundary problem of injection of a beam into a half-space $z > 0$ filled with plasma. With respect to the beam and plasma parameters we assume that they depend only on z, and the plasma concentration increases in the direction of the beam injection, and the scale L over which it changes by an amount of the order of unity is large compared with the wavelength of the oscillations excited by the beam: $L \gg c/\omega_p$ (ω_p is the electron plasma frequency). As will be shown later, in those cases when

[1)]We note that in [4] there is indicated a somewhat more stringent condition than the inequality (2). Actually, the instability is kinetic already if the inequality (2) is satisfied.

relaxation is possible at all, it occurs over a scale much smaller than L. Therefore n(z) can be approximated by a linear function: $n = n_0(1 + z/L)$.

Under the conditions indicated above, the system of quasilinear equations can be written in the form[6-8]:

$$\frac{3v_T^2 k_z}{\omega_p}\frac{\partial W}{\partial z} - \frac{\omega_p}{2L}\frac{\partial W}{\partial k_z} = 2\gamma W, \tag{3}$$

$$c\frac{p_z}{p}\frac{\partial f}{\partial z} = \frac{\partial}{\partial p_\alpha}\mathscr{D}_{\alpha\beta}\frac{\partial f}{\partial p_\beta}, \tag{4}$$

where $\omega_p = (4\pi n_0 e^2/m)^{1/2}$ is the electron plasma frequency at the point $z = 0$, $v_T = (T_e/m)^{1/2}$ is the thermal velocity of the plasma electrons, $f \equiv f(z, p)$ is the beam-electron distribution function, $W \equiv W(z, \mathbf{k})$ is the spectral energy density of the Langmuir oscillations, $\gamma \equiv \gamma(z, \mathbf{k})$ is the instability increment, and $\mathscr{D}_{\alpha\beta} \equiv \mathscr{D}_{\alpha\beta}(z, p)$ is the diffusion tensor in momentum space.

Since it will be shown below that the angle spread of the beam remains small all the time during the relaxation process compared with unity, we can write down the left-hand side of (4) in the form $c\partial f/\partial z$. This means in particular, that the bean concentration does not vary along z and remain equal to its initial value n_0'.

For the actual calculation of the increment γ and of the tensor $\mathscr{D}_{\alpha\beta}$ it is convenient to use the spherical coordinates (p, θ, φ) in momentum space and (k, θ', φ') in wave-vector space (the angles θ and θ' are reckoned from the z axis). The results of the corresponding calculations are given in the Appendix.

The distribution function of the beam entering the plasma will be assumed, for concreteness, to be monoenergetic

$$f_0 = \frac{n_0' g_0(\theta)}{2\pi p_0^2}\delta(p - p_0). \tag{5}$$

Such an assumption corresponds to the experimental formulation of the problem in those cases when the beam is produced by electrostatic sources.

To abbreviate the notation, it is convenient to make the following substitutions in Eqs. (3) and (4):

$$z \to \frac{3v_T^2 n_0 p_0}{mc^2\omega_p n_0'}z, \quad \mathbf{k} \to \frac{\omega_p}{c}\mathbf{k}, \quad \mathbf{p} \to p_0\mathbf{p}, \quad f \to \frac{n_0'}{p_0^3}f,$$
$$W \to \frac{c^6 p_0 n_0'}{3\omega_p^3 v_T^2}W, \quad \gamma \to \omega_p\frac{n_0' mc}{n_0 p_0}\gamma, \quad \mathscr{D}_{\alpha\beta} \to \frac{mc^3 p_0 \omega_p n_0'}{3v_T^2 n_0}\mathscr{D}_{\alpha\beta}, \tag{6}$$

where the quantities z, k, p, f, W, γ, and $\mathscr{D}_{\alpha\beta}$ on the right sides are already dimensionless. In terms of the new (dimensionless) variables, the system of equations (3) and (4) takes the form

$$\cos\theta'\frac{\partial W}{\partial z} - \mu\left(\cos\theta'\frac{\partial W}{\partial k} - \frac{\sin\theta'}{k}\frac{\partial W}{\partial\theta'}\right) = 2\gamma W, \tag{7}$$

$$\frac{\partial f}{\partial z} = \frac{1}{p^2}\frac{\partial}{\partial p}p^2\left(\mathscr{D}_{pp}\frac{\partial f}{\partial p} + \frac{1}{p}\mathscr{D}_{p\theta}\frac{\partial f}{\partial\theta}\right) + \frac{1}{p\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\left(\mathscr{D}_{p\theta}\frac{\partial f}{\partial p} + \frac{1}{p}\mathscr{D}_{\theta\theta}\frac{\partial f}{\partial\theta}\right). \tag{8}$$

We have used here the spherical coordinate system (p, θ, φ) and (k, θ', φ') introduced above.

The dimensionless parameter μ, defined by the relation

$$\mu = \frac{1}{2}\frac{c}{L\omega_p}\frac{p_0}{mc}\frac{n_0}{n_0'}, \tag{9}$$

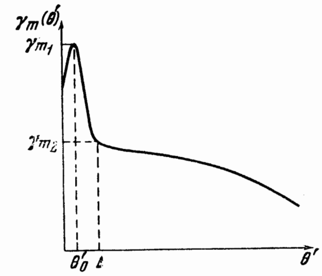

FIG. 1

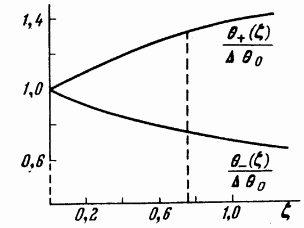

FIG. 2

FIG. 1. Dependence of the increment, maximized with respect to k, on θ'; $\Delta\theta$—angle spread of the beam.

FIG. 2. Result of numerical integration of Eqs. (17) and (18).

characterizes the role of the inhomogeneity of the plasma.

3. RELAXATION IN A HOMOGENEOUS PLASMA

The limiting case of a homogeneous plasma corresponds to vanishing of the parameter μ in Eq. (7). The resultant system of equations coincides, apart from the notation, with that already considered in[1]. Nevertheless, we shall pay some attention to it, for this will yield a number of estimates important for what follows, and, in addition, in the case of one simple model, an analytic solution of this system can be obtained and shows that as a result of relaxation there appears an appreciable number of electrons with energy exceeding the initial energy of the beam electrons.

The main result concerning relaxation of a beam in a homogeneous plasma was obtained already by Faĭnberg, Shapiro, and Shevchenko, and consists in the fact that the relaxation is almost one-dimensional, i.e., the energy loss by the beam occurs without an appreciable increase of its angle spread.

This circumstance can be seen from an analysis of the expressions for the increment and for the diffusion coefficients, given in the Appendix. Indeed, the dependence of the increment, maximized with respect to k, on the propagation angle θ' has the form shown schematically in Fig. 1. The width of the maximum on the curve $\gamma_m(\theta')$ is approximately equal to $\Delta\theta$, and the ratio γ_{m1}/γ_{m2} is of the order of unity (for the notation see the figure). On the basis of the results of[9] it can be stated that the spectrum of the oscillations will be concentrated in a narrow region of width of the order of $\Delta\theta/\sqrt{\Lambda}$ (Λ is a quantity of the order of the Coulomb logarithm) about the point $\theta' = \theta' \sim \Delta\theta$, where the function $\gamma_m(\theta')$ has a maximum.

Knowing the position and the width of the spectrum of the oscillations, it is possible to estimate with the aid of formula (A.6) the ratio of the diffusion coefficients $\mathscr{D}_{pp}$ and $\mathscr{D}_{\theta\theta}$: $\mathscr{D}_{pp}/\mathscr{D}_{\theta\theta} \sim \Delta\theta^{-2} \gg 1$. It follows therefore that the diffusion with respect to momentum is more rapid than the diffusion with respect to the angle, and within the time during which the angle spread of the beam $\Delta\theta$ increases by an amount on the order of its initial value $\Delta\theta_0$, the beam already releases into the plasma an appreciable fraction of its initial energy (on the order of unity). This occurs over a distance

$$z \sim \Lambda / \gamma_m \sim \Lambda \Delta\theta_0^2.$$

A complete quantitative description of the process of relaxation is very difficult, since its concrete details depend essentially on the form of the initial distribution function. We shall therefore consider a simple model, in which the spectrum of the oscillations is assumed to be exactly one-dimensional[2]: $W(k, \theta') = \tilde{W}(k)\delta(1 - \cos\theta')$. In the investigation of this model it is convenient to regard f as a function of the independent variables θ and $p_\perp = p \sin\theta$, and W as a function of the variable $\eta = [2(k-1)]^{1/2}$.

Using relations (A.2) and (A.7), and also the condition $\theta \ll 1$, we rewrite the system of equations (7) and (8) in terms of the new variables:

$$\frac{\partial}{\partial z} W(z,\theta) = -\frac{\pi}{\theta} W(z,\theta) \frac{\partial}{\partial \theta} h(z,\theta), \tag{11}$$

$$\frac{\partial}{\partial z} f(z,\theta,p_\perp) = \frac{2\pi^2\theta^2}{p_\perp^2} \frac{\partial}{\partial\theta}\left[\theta^2 W(z,\theta)\frac{\partial}{\partial\theta} f(z,\theta,p_\perp)\right],$$

$$h = 2\pi \int_0^\infty f p_\perp \, dp_\perp. \tag{12}$$

We indicate here two integrals of this system, the validity of which can be verified directly:

$$\int_0^\infty \frac{d\theta}{\theta^2} f(z,\theta,p_\perp) = \frac{g_0(p_\perp)}{2\pi p_\perp}, \tag{13}$$

$$\int_0^\infty p_\perp^3 f(z,\theta,p_\perp)\,dp_\perp + \theta^2 \frac{\partial}{\partial\theta}\theta^3 W(z,\theta) = \frac{\theta^4}{2\pi} g_0(\theta). \tag{14}$$

The first of them is the law of conservation of the number of particles with specified $p_\perp$, and the second is the analog of the well-known quasilinear integral[6,7].

We obtain a solution of the system (11) and (12) corresponding to the "step" function g_0:

$$g_0(\theta) = \begin{cases} 2/\Delta\theta_0^2, & \theta < \Delta\theta_0, \\ 0, & \theta > \Delta\theta_0. \end{cases}$$

The solution method is analogous to that employed in the theory of one-dimensional relaxation of a nonrelativistic beam[10]: in this region of values of θ where the spectral density of the oscillation energy greatly exceeds the thermal level, the distribution function can be regarded as independent of θ ("plateau"); on the other hand, where the oscillation energy is close to thermal, the distribution function is equal to its initial value. In other words, the distribution function can be written in the form

$$f(z,\theta,p_\perp) = \begin{cases} \dfrac{1}{\pi\Delta\theta_0^2}\delta\left(\dfrac{p_\perp}{\theta} - 1\right), & \theta < \theta_-(z), \\ \mathcal{P}(p_\perp, z), & \theta_-(z) < \theta < \theta_+(z), \\ 0, & \theta > \theta_+(z), \end{cases}$$

where the function $\mathcal{P}(p_\perp, z)$ determines the height of the plateau, and the functions $\theta_-(z)$ and $\theta_+(z)$ determines its boundaries, with $\mathcal{P}(p_\perp, z)$ expressed in terms of θ_- and θ_+ with the aid of the integral (13):

$$\mathcal{P}(p_\perp, z) = \begin{cases} 0, & p_\perp < \theta_-, \\ \dfrac{1}{\pi\Delta\theta_0^2 p_\perp}\dfrac{\theta_+\theta_-}{\theta_+ - \theta_-}, & \theta_- < p_\perp < \Delta\theta_0, \\ 0, & p_\perp > \Delta\theta_0. \end{cases}$$

[2]Such a formation of the problem is perfectly correct if a strong magnetic field parallel to the z axis exists in the plasma.

As to the spectrum of the oscillations, it can be easily found with the aid of the integral (14):

$$W(z,\theta) = \frac{1}{2\pi\theta^3}\left[\int_{\theta_-}^{\theta} \theta^2 g_0(\theta)\,d\theta - \frac{\theta_-^{-1} - \theta^{-1}}{\theta_-^{-1} - \theta_+^{-1}}\int_{\theta_-}^{\theta_+} \theta^2 g_0(\theta)\,d\theta\right]. \tag{15}$$

It now remains to determine the position of the boundaries of the plateau. To this end we integrate both parts of Eq. (11) with respect to θ from $\theta_- - 0$ to $\theta_- + 0$:

$$\frac{d\theta_-}{dz}\ln\frac{W(\theta_- + 0)}{W(\theta_- - 0)} = \frac{\pi}{\theta_-}[h(\theta_- + 0) - h(\theta_- - 0)]. \tag{16}$$

Recognizing that the logarithm of the ratio $\tilde{W}(\theta_- + 0)/\tilde{W}(\theta_- - 0)$ is equal with good accuracy to Λ (see[10]) and substituting in the right-hand side of (16) the explicit expressions for $h(\theta_+ + 0)$ and $h(\theta_- - 0)$, we obtain the following equation:

$$\frac{d\theta_-}{d\zeta} = -\theta_- + \frac{\theta_+}{\theta_+ - \theta_-}(\Delta\theta_0 - \theta_-), \quad \zeta = \frac{2\pi}{\Lambda\Delta\theta_0^2}z. \tag{17}$$

An analogous result is obtained also for θ_+ by integrating (11) over the interval $[\theta_+ - 0, \theta_+ + 0]$:

$$\frac{d\theta_+}{d\zeta} = \frac{\theta_-}{\theta_+ - \theta_-}(\Delta\theta_0 - \theta_-). \tag{18}$$

Since the instability increment calculated from the initial distribution function is positive only at one point ($\theta = \Delta\theta_0$), it is not difficult to write down the initial conditions for the system (17) and (18): $\theta_+(0) = \theta_-(0) = \Delta\theta_0$. In spite of the relatively simple form of Eqs. (17) and (18), they can be integrated only numerically. The results of the corresponding calculations are shown in Fig. 2.

The obtained solution of the initial system of equations (11) and (12) describes only the initial stage of the relaxation ($\zeta < 0.77$). The point is that the spectral density of the oscillation energy, formally calculated in accordance with (15), turns out to be negative at $\zeta > 0.77$ in the vicinity of the point $\theta = \theta_-$.

It is impossible to find the distribution function at $\zeta > 0.77$, but it is nevertheless possible to follow qualitatively the course of the relaxation. To this end we turn to Fig. 3. In the figure the circular arc of unit radius represents that part of momentum space, where the initial distribution function of the particles differs from zero, while the horizontal straight lines represent the line $p_\perp$ = const, along which diffusion equalization of the distribution function takes place. It is seen from the figure that during the relaxation process there appear "accelerated" electrons (with momentum $p > 1$), and their number is comparable with the number "slowed down" electrons (with momentum $p < 1$). Thus, at the end of the first stage of the relaxation, the fraction of the "accelerated" electrons is equal to 0.13 of the total number of beam electrons.

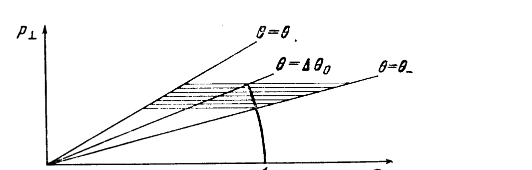

FIG. 3. Beam relaxation for one-dimensional spectrum of the oscillations.

4. RELAXATION IN AN INHOMOGENEOUS PLASMA (QUALITATIVE CONSIDERATION)

When a Langmuir oscillation propagates along a plasma that is inhomogeneous in the direction of the z axis, the longitudinal component of its wave vector changes with time[3)]:

$$\frac{dk_z}{dt} = -\frac{\partial \omega_p}{\partial z} = -\frac{\omega_r}{2L}.$$

On the other hand, the only oscillations that can interact with the beam (and consequently can become amplified) are those for which k_z lies in a narrow interval about the point $k_z = \omega_p/c$:

$$\left| k_z - \frac{\omega_p}{c} \right| \leqslant \frac{\omega_p}{c}\Delta\theta^2 + k_\perp \Delta\theta$$

(the last circumstance can be readily seen, e.g., from formula (A.1)). Therefore in an inhomogeneous plasma each oscillation interacts with the beam during a short time interval

$$\Delta t \sim \frac{L}{c}\left(\Delta\theta^2 + \frac{k_\perp c}{\omega_p}\Delta\theta\right),$$

which can turn out to be insufficient to satisfy the condition[4)]

$$\gamma \Delta t \gtrsim \Lambda, \tag{19}$$

under which the oscillation can exert a noticeable reaction on the beam. It is precisely for this reason that the relaxation in an inhomogeneous plasma can be much less effective than in a homogeneous one.

Recognizing that the increment of the two-stream instability can be estimated from the formula (see, (A.3)):

$$\gamma \sim \frac{\omega_p}{\Delta\theta^2}\frac{n_0'}{n_0}\frac{mc}{p_0}\frac{\omega_p^2}{\omega_p^2 + k_\perp^2 c^2}$$

we write relation (19) in the form

$$\frac{1 + k_\perp c/\omega_p \Delta\theta}{1 + (k_\perp c/\omega_p)^2} \gtrsim \mu\Lambda, \tag{20}$$

where the parameter μ is determined from formula (9). We see therefore that the inhomogeneity has different effects on oscillations with different $k_\perp$. For those values of $k_\perp$ for which the criterion (20) is satisfied, the role of the inhomogeneity is immaterial (since the corresponding oscillations have time to become strongly amplified in the time during which they interact with the beam). Conversely, if for certain values of $k_\perp$ it is satisfied, then such oscillations are actually not excited at all.

As was shown in Sec. 3, the relaxation of a beam in a homogeneous plasma is due to excitation of oscillations with $k_\perp \sim \Delta\theta k \sim \Delta\theta\omega_p/c$. For these, the criterion (20) takes the form

$$\mu\Lambda \lesssim 1. \tag{21}$$

Attention is called to the fact that now this criterion does not contain $\Delta\theta$ at all. This means that the influence of the plasma inhomogeneity on the relaxation of the beam is determined completely by the quantity $\mu\Lambda$: if it satisfies the inequality (21), then the role of the inhomogeneity is negligible; on the other hand, if the inverse inequality is satisfied

$$\mu\Lambda \gg 1, \tag{22}$$

then oscillations with $k_\perp \sim \Delta\theta\omega_p/c$ are not excited, and the entire dynamics of the relaxation changes.

For the experiments proposed by Winterberg[5] ($n_0 \sim 10^{22}$ cm^{-3}, $n_0' \sim 10^{18}$ cm^{-3}, E ~ 10 MeV, L ~ 0.2 cm), the product $\mu\Lambda$ is equal to 30, and the role of the inhomogeneity is very important.

Let us stop therefore to dwell on the case $\mu\Lambda \gg 1$ in greater detail. It is easy to see that when $\mu\Lambda \gg 1$ and

$$\Delta\theta \gtrsim 1/\mu\Lambda \tag{23}$$

the inequality (20) is not satisfied for any value of $k_\perp$. Consequently, if the initial angle spread of the beam is sufficiently large, $\Delta\theta_0 > 1/\mu\Lambda$, then there is no relaxation[5)].

On the other hand, when

$$\Delta\theta \ll 1/\mu\Lambda \tag{24}$$

relaxation is possible, since there exists an interval of values of $k_\perp$,

$$\frac{\omega_p}{c}\mu\Lambda\Delta\theta \lesssim k_\perp \lesssim \frac{\omega_p}{c}\frac{1}{\mu\Lambda\Delta\theta}, \tag{25}$$

for which the inequality (20) is satisfied. But now, unlike in a homogeneous plasma, only such oscillations are excited, for which the angle θ' is large in comparison with $\Delta\theta$ ($\theta' \gtrsim \mu\Lambda\Delta\theta$), i.e., the relaxation is essentially three-dimensional. It is clear that the relaxation terminates when the angle spread of the beam becomes of the order of $(\mu\Lambda)^{-1} \ll 1$.

5. RELAXATION IN AN INHOMOGENEOUS PLASMA (QUANTITATIVE CONSIDERATION)

In seeking the analytic solution of the problem, we shall assume that conditions (22) and (24) are satisfied; these mean that, on the one hand, the role of the inhomogeneity is already appreciable, and on the other hand, relaxation is still possible. Under these conditions, the beam excites in the plasma only such oscillations, for which the angle θ' lies in the interval

$$\mu\Lambda\Delta\theta \ll \theta' \ll 1. \tag{26}$$

The lower limit on θ' is connected with the fact that the longitudinal oscillations appear as a result of the inhomogeneity effect, and the upper limit is connected with the fact that increment of the instability decreases with increasing θ'.

The presence of the inequality (26) makes it possible to calculate the increment from the formula (A.4). Substituting in it (7) and integrating with respect to the characteristics, we can easily obtain W:

$$W(z, \theta', \varepsilon) = W_T \exp \nu(z, \theta', \varepsilon), \tag{27}$$

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3)In this section we use dimensionless variables for clarity.

4)In fact, the relaxation can occur also under the condition $\gamma\Delta t < \Lambda$ (including also on thermal noise), but the relaxation length becomes too large in this case.

5)It follows therefore that under the conditions of the experiments of the type of [5] the initial angle spread of the beam should be in any case smaller than 2–3 degrees—a very stringent limitation.

where

$$\nu(z,\theta',\varepsilon)=\left(1-\frac{\theta'^2}{2}\right)\int_0^z \xi\, dz'\int_{|\xi|}^{\infty} d\theta \frac{\partial g(z',\theta)/\partial\theta}{\sqrt{\theta^2-\xi^2}} \qquad (27')$$

$$\xi=\varepsilon+\frac{\mu}{\theta'}(z-z'),\quad g(z,\theta)=2\pi\int_0^{\infty} pf(z,\theta,p)\,dp,$$

and W_T is the spectral density of the energy of the thermal oscillations. Here and below we shall assume W to be a function of the variables z, θ', and $\epsilon = \theta' - [2(k-1)]^{1/2}$.

Formula (27) is valid only when $\nu > 0$. On the other hand, if $\nu < 0$, then $W \approx W_T$. Since we disregard in any case the influence of the thermal oscillations on the relaxation, we shall assume henceforth that W = 0 when $\nu < 0$.

As will be shown later, the beam releases in the plasma only an insignificant fraction of its initial energy. This means, in particular, that the change of the momentum of the beam electrons is small compared with their initial momentum, which in dimensionless notation is equal simply to unity. Introducing therefore into (8) a new variable q = p − 1 and assuming that $|q| \ll 1$, we write this equation in the form

$$\frac{\partial f}{\partial z}=\frac{\partial}{\partial q}\left(\mathscr{D}_{pp}\frac{\partial f}{\partial q}+\mathscr{D}_{p\theta}\frac{\partial f}{\partial\theta}\right)+\frac{1}{\theta}\frac{\partial}{\partial\theta}\theta\left(\mathscr{D}_{p\theta}\frac{\partial f}{\partial q}+\mathscr{D}_{\theta\theta}\frac{\partial f}{\partial\theta}\right).$$

Integrating it with respect to q and recognizing that, at the assumed accuracy we have

$$g(z,\theta)=2\pi\int_0^{\infty} f(z,\theta,q)\,dq,$$

we obtain

$$\frac{\partial g}{\partial z}=\frac{1}{\theta}\frac{\partial}{\partial\theta}\theta\mathscr{D}_{\theta\theta}\frac{\partial g}{\partial\theta}, \qquad (28)$$

where the quantity $\mathscr{D}_{\theta\theta}$ in accordance with the inequalities (26), can be calculated from the formula (A.8):

$$\mathscr{D}_{\theta\theta}=\frac{2\pi}{\theta^2}\int_0^{\infty}\theta'^3 d\theta'\int_{-\theta}^{\theta}\frac{W(z,\theta',\varepsilon)\,\varepsilon^2 d\varepsilon}{\sqrt{\theta^2-\varepsilon^2}}. \qquad (29)$$

Assuming that the angle spread of the beam on entering the plasma is small compared with the value that it reaches at the end of the relaxation, we shall use the following boundary condition for Eq. (28):

$$g\Big|_{z=0}=\frac{1}{\theta}\delta(\theta). \qquad (30)$$

The system (27) and (28) contains two unknown functions, namely, $W(z, \theta', \epsilon)$ and $g(z, \theta)$, and is a closed system. To solve it, we note that the diffusion coefficient $\mathscr{D}_{\theta\theta}$ is appreciably different from zero only when $\theta \gtrsim \epsilon_0$, where ϵ_0 denotes the characteristic value of ϵ. It follows therefore that ϵ_0 should be small compared with $\Delta\theta$:

$$\varepsilon_0 \ll \Delta\theta, \qquad (31)$$

for in the opposite case an appreciable fraction of the distribution functions would not be captured by the quasilinear diffusion and there would be no self-suppression of the instability. Assuming therefore that $\epsilon_0 \ll \Delta\theta$, we can represent (29) in the form

$$\mathscr{D}_{\theta\theta}=\frac{F(z)}{\theta^3},\quad F(z)=2\pi\int_0^{\infty}\theta'^3 d\theta'\int_{-\infty}^{+\infty} W(z,\theta',\varepsilon)\,\varepsilon^2 d\varepsilon. \qquad (32)$$

It is now easy to find for (28) a solution satisfying the boundary condition (30):

$$g(z,\theta)=\frac{2}{\Gamma(7/5)\,[\Delta\theta(z)]^2}\exp-\left[\frac{\theta}{\Delta\theta(z)}\right]^5, \qquad (33)$$

$$\Delta\theta(z)=\left[25\int_0^z F(z')\,dz'\right]^{1/5}. \qquad (34)$$

From the known function $g(z, \theta)$ it is easy to calculate the quantity $\nu(z, \theta', \epsilon)$ characterizing the spectrum of the oscillations (see (27)):

$$\nu(z,\theta',\varepsilon)=\left(1-\frac{\theta'^2}{2}\right)\int_0^z \frac{\xi}{[\Delta\theta(z')]^3}G\left[\frac{\xi}{\Delta\theta(z')}\right]dz' \equiv \left(1-\frac{\theta'^2}{2}\right)I, \qquad (35)$$

where ξ is defined in (27′) and

$$G(\xi)=\frac{10}{\Gamma(7/5)}\int_{|\xi|}^{\infty}\frac{\xi'^4 e^{-\xi'^5}}{\sqrt{\xi'^2-\xi^2}}\,d\xi'.$$

For the subsequent solution of the problem we shall use the following reasoning. Let the maximum of the function ν with respect to the variables θ' and ϵ lie at the point $\theta' = \theta_0'(z)$, $\epsilon = \epsilon_0(z)$ and let it equal $\nu_0(z)$, and let the corresponding value of W be $W_0(z)$. Then it follows from (27) that $\nu_0 = \ln W_0/W_T$. But since the ratio W_0/W_T is very large, it follows that its logarithm is insensitive to W_0 and is almost constant (approximately equal to the Coulomb logarithm Λ). We thus arrive at the conclusion that the maximum value of ν (with respect to the variables θ' and ϵ) should be equal to Λ independently of z.

The form of the spectrum of the oscillations is determined by the dependence of ν on θ' and ϵ near the point $\theta' = \theta_0'$ and $\epsilon = \epsilon_0$. It is clear that in this region of values of θ' and ϵ there should be satisfied the conditions (26) and (31). On the other hand, as noted in Sec. 4 under condition (26) the inhomogeneity of the plasma does not exert an appreciable influence on the oscillations. This means that in expression (35) for ν it is possible to regard (purely formally) μ as a small parameter. Since, in addition, the value of ϵ is also small (see (31)), the integral I which enters in (35) can be expanded in powers of μ and ϵ (the parameters of the expansion are discussed more exactly below), and in accordance with the statements made above, the zeroth term of the expansion should be equal to Λ[6]. In order to realize the indicated expansion, we change over in the integral I to a new integration variable $\eta = \xi/\Delta\theta(z')$. As a result we get

$$I=\int_{\varepsilon/\Delta\theta(z)} d\eta\,\eta G(\eta)\left[\frac{\mu}{\theta'}\Delta\theta+\frac{\eta}{2}\frac{d}{dz'}\Delta\theta^2\right]^{-1}, \qquad (36)$$

where $\Delta\theta$ and $d\Delta\theta^2/dz'$ are regarded as functions of η. In the derivation of the last relation we have assumed that $\epsilon > -\mu z/\theta'$, since in the region $\epsilon < -\mu z/\theta'$ the quantity ν is negative and the energy of the oscillations is close to thermal.

The presented form of the integral I makes it possible to conclude that in the limit $\mu, \epsilon \to 0$ it has a finite value independent of z only when $\Delta\theta(z) = A\sqrt{z}$, where A is a constant. Simple calculations show that in this case

[6] Strictly speaking, the maximum of the function ν should be equal to Λ. But under conditions (26) and (31) this maximum is close to the zeroth term of the expansion of I: the difference between them is of the order of the first term of the expansion of I.

$$I_{\mu,\varepsilon\to 0}=\frac{2}{A^2}\int_0^\infty G(\eta)d\eta=\frac{2\pi}{A^2\Gamma(^7/_5)}.$$

Determining the constant A from the condition $I_{\mu,\epsilon\to 0}=\Lambda$, we obtain ultimately that

$$\Delta\theta(z)=\sqrt{\frac{2\pi z}{\Gamma(^7/_5)\Lambda}}=2{,}7\sqrt{\frac{z}{\Lambda}}. \tag{37}$$

Thus, from the condition $I_{\mu,\epsilon\to 0}=\Lambda$ we have found the dependence of the angle spread of the distribution function on the coordinate z. As noted in Sec. 4, the relaxation terminates when $\Delta\theta\sim(\mu\Lambda)^{-1}$. This value of $\Delta\theta$ is reached at $z\sim(\mu^2\Lambda)^{-1}$. With further increase of z, the angle spread remains unchanged.

It is easy to write down the next term of the expansion of I in terms of the parameters ϵ and μ. Using the $\Delta\theta(z)$ relation obtained above, we obtain from (36) (for the time being, exactly):

$$I=\frac{2}{A^2}\int_{\varepsilon/\Delta\theta(z)}^{\infty}\eta G(\eta)\left\{\eta^2+\frac{4\mu z}{\theta'\Delta\theta(z)}\left[\frac{\varepsilon}{\Delta\theta(z)}+\frac{\mu z}{\theta'\Delta\theta(z)}\right]\right\}^{-1/2}d\eta. \tag{38}$$

We see first of all that actually the parameters of the expansion are the quantities $\mu z/\theta'\Delta\theta(z)$ and $\epsilon/\Delta\theta(z)$. We shall verify later that they are really small. Expanding I in terms of these parameters with accuracy to terms of first order, we arrive at the following result:

$$I=\Lambda\left[1-\frac{5\Gamma(^9/_5)}{2\pi\Delta\theta(z)}\left(\varepsilon+\frac{2\mu z}{\theta'}\right)\right],\quad \varepsilon>-\frac{\mu z}{\theta'}.$$

Turning to formula (35), we find that in the region of values of ϵ and θ' of interest to us

$$\nu(z,\theta',\varepsilon)\approx\Lambda\left[1-\frac{\theta'^2}{2}-\frac{5\Gamma(^9/_5)}{2\pi\Delta\theta(z)}\left(\varepsilon+\frac{2\mu z}{\theta'}\right)\right],\quad \varepsilon>-\frac{\mu z}{\theta'}.$$

Recognizing that the quantity ν is positive only when $\epsilon>-\mu z/\theta'$, we obtain from this

$$\theta_0'=\left[\frac{5\Gamma(^9/_5)\mu z}{2\pi\Delta\theta(z)}\right]^{1/3},\quad \varepsilon_0=-\frac{\mu z}{\theta'}+0.$$

The presence of the large parameter Λ in the formula for ν enables us to state that the spectrum of the oscillations is concentrated near the maximum of the function ν, i.e., near the point θ_0', ϵ_0.

At small deviations of θ' and ϵ from θ_0' and ϵ_0, we have

$$\nu-\nu_0=-\frac{3}{2}\Lambda(\theta'-\theta_0')^2-\Lambda\frac{5\Gamma(^9/_5)}{2\pi\Delta\theta(z)}\left(\varepsilon+\frac{\mu z}{\theta'}\right),\quad \varepsilon>-\frac{\mu z}{\theta'}.$$

It follows from the last relation that the width of the spectrum of the oscillations with respect to θ' is of the order $\Lambda^{-1/2}$, and with respect to ϵ of the order $\Delta\theta/\Lambda$. It can be easily verified that in this region of the variables the parameters $\mu z/\theta'\Delta\theta(z)$ and $\epsilon/\Delta\theta(z)$, with respect to which the expansion was carried out in (38), are indeed small up to the very end of the relaxation (i.e., up to $z\sim(\mu^2\Lambda)^{-1}$).

In accordance with formula (27), the spectrum of the oscillations can be written in the form

$$W=W_0(z)\exp\left[-\frac{3}{2}\Lambda(\theta'-\theta_0')^2+\Lambda\frac{5\Gamma(^9/_5)}{2\pi\Delta\theta(z)}\left(\varepsilon+\frac{\mu z}{\theta'}\right)\right]\quad \varepsilon>-\frac{\mu z}{\theta'},$$
$$W=0,\quad \varepsilon<-\mu z/\theta', \tag{39}$$

where the normalization factor $W_0(z)$ is determined from the relations (32), (34), and (37). Knowing the spectrum of the oscillations, we can calculate their energy density U:

$$U=2\pi\int_0^\infty\theta'^2d\theta'\int_{-\infty}^{+\infty}W d\varepsilon,$$

which properly speaking is the purpose of our paper. Leaving out the cumbersome intermediate steps, we present only the final result:

$$U(z)=\frac{2\pi\Gamma(^9/_5)}{3\Gamma^2(^7/_5)}\frac{1}{\mu\Lambda}\frac{\psi[a(z)]}{\chi[a(z)]}, \tag{40}$$

where

$$a(z)=\sqrt{\frac{3\Lambda}{2}}\theta_0'(z),$$
$$\psi(a)=a^3\left[\left(a^2+\frac{1}{2}\right)I(a)+\frac{a}{2}e^{-a^2}\right],$$
$$\chi(a)=a[^4/_9a^6-{}^4/_3a^4+{}^4/_3a^2+3]I(a)+[^2/_9a^6-{}^2/_9a^4+a^2+1]e^{-a^2},$$
$$I(a)=\int_{-a}^{\infty}e^{-\xi^2}d\xi.$$

An analysis shows that with increasing z the function U(z) first increases in proportion to $z^{1/2}$, reaches a maximum at $z=z_{max}=82/\mu^2\Lambda^4$, after which it decreases like $z^{-1/3}$:

$$U=\frac{1}{\mu\Lambda}\begin{cases}0{,}57\,(\mu^2\Lambda^4z)^{1/2}, & z\ll z_{max},\\ 3{,}76, & z=z_{max},\\ 8{,}31(\mu^2\Lambda^4z)^{-1/3}, & z\gg z_{max}.\end{cases} \tag{41}$$

An interesting feature of the obtained solution is that the relaxation proceeds in two stages. During the first stage, when $z<z_{max}$, the oscillations are excited by the beam, while during the second stage, when $z>z_{max}$, the oscillations are absorbed to a considerable degree by the beam. This phenomenon is characteristic of an inhomogeneous plasma. In the case of a nonrelativistic beam, an effect of this kind was described in[3].

As already noted above, the relaxation terminates at $z\lesssim(\mu^2\Lambda)^{-1}$. The corresponding value of the energy density of the oscillations is of the order of $10(\mu\Lambda^2)^{-1}$. With further increase of z, the energy of the oscillations no longer changes. Thus, the beam releases in the inhomogeneous plasma only a very insignificant fraction of its energy, $\sim 10(\mu\Lambda^2)^{-1}$ (we recall that in dimensionless variables the initial beam energy is equal to unity).

It is possible that some increase of the energy release can be attained by artificially suppressing the relaxation in the region $z>z_{max}$. This can be attained, in particular, by making the plasma in this region strongly inhomogeneous. But even then, as seen from (40), the energy released by the beam will be insignificant (on the order of $3(\mu\Lambda)^{-1}$).

In concluding this section, we point out one more feature of the obtained solution. As can be seen from (39), when $z\gg z_{max}$ the spectral density of the oscillation energy turns out to be exponentially small if $|\epsilon|\ll|\epsilon_0|$. Formally this means that the coefficient of quasilinear diffusion is exponentially small when $\theta\ll|\epsilon_0|$, i.e., in the region of angles $\theta<|\epsilon_0|$ there is no quasilinear relaxation, and a sharp peak of width $\sim|\epsilon_0|\ll\Delta\theta_0$ appears on the distribution function. Simple estimates show that the presence of this peak leads to excitation of oscillations with $|\epsilon|\lesssim|\epsilon_0|$, but the energy of these oscillations remains much smaller (by a factor $(\Delta\theta/|\epsilon_0|)^5$ than the value of U obtained above.

6. SUMMARY OF RESULTS

We have shown that the influence of the inhomogeneity of the plasma on the quasilinear relaxation of an ultrarelativistic electron beam is characterized completely by the product $\mu\Lambda$, where

$$\mu=\frac{1}{2}\frac{c}{L\omega_p}\frac{p_0}{mc}\frac{n_0}{n_0'},$$

and Λ is a quantity on the order of the Coulomb logarithm.

When $\mu\Lambda \lesssim 1$, the relaxation is insensitive to the influence of the inhomogeneity, the energy released by the beam in the plasma is of the order of the initial beam energy, and the scale of the relaxation is equal to

$$z\sim\Lambda\frac{c}{\omega_p}\frac{v_T^2}{c^2}\frac{p_0}{mc}\frac{n_0}{n_0'}\Delta\theta_0^2$$

(we are using here the dimensional variables).

When $\mu\Lambda \gg 1$, the role of the inhomogeneity is very important, and there are two possibilities. If the initial angle spread of the beam $\Delta\theta_0$ is sufficiently large, $\Delta\theta_0 \gtrsim (\mu\Lambda)^{-1}$, then there is no relaxation at all. On the other hand, if the spread $\Delta\theta_0$ is small, $\Delta\theta_0 < (\mu\Lambda)^{-1}$, then relaxation is possible, but the energy lost by the beam is only a small fraction $(\sim 10/\mu\Lambda^2)$ of its initial energy. This occurs over a scale

$$z\sim\frac{1}{\mu^2\Lambda}\frac{c}{\omega_p}\frac{v_T^2}{c^2}\frac{p_0}{mc}\frac{n_0}{n_0'}$$

APPENDIX

We are using here the dimensionless variables defined by relations (6).

The instability increment is

$$\gamma(k,\theta')=\frac{1}{2k^3}\int_{\theta_1}\frac{(\cos\theta-k\cos\theta')\,\partial g/\partial\theta-2g\sin\theta}{\sqrt{(\cos\theta_1-\cos\theta)(\cos\theta-\cos\theta_2)}}d\theta. \quad (A.1)$$

Here

$$g=2\pi\int_0^\infty fp\,dp,\quad \cos\theta_{1,2}=\frac{1}{k}(\cos\theta'\pm\sin\theta'\sqrt{k^2-1}).$$

If $\theta'=0$, then

$$\gamma=\gamma(k)=\frac{\delta}{2k^2}\left(\frac{\partial}{\partial\cos\theta}\sin^2\theta g\right)\Big|_{\cos\theta=k^{-1}}. \quad (A.2)$$

If $\theta'\gg\Delta\theta$ is the angle spread of the beam, then

$$\gamma=\frac{\delta}{2k^2}\int_{|\delta|}^\infty\frac{d\theta}{\sqrt{\theta^2-\delta^2}}\frac{\partial g}{\partial\theta}, \quad (A.3)$$

where $\delta=(1-k\cos\theta')/k\sin\theta'$. If furthermore $\theta'\ll 1$ (but $\theta'\gg\Delta\theta$), then

$$\gamma=\frac{\varepsilon}{2}(1-\theta'^2)\int_{|\varepsilon|}^\infty\frac{d\theta}{\sqrt{\theta^2-\varepsilon^2}}\frac{\partial g}{\partial\theta}, \quad (A.4)$$

where

$$\varepsilon=\theta'-\sqrt{2(k-1)} \quad (A.5)$$

The diffusion tensor is

$$\left.\begin{matrix}\mathcal{D}_{pp}\\ \mathcal{D}_{p\theta}\\ \mathcal{D}_{\theta\theta}\end{matrix}\right\}=2\pi\int_1^\infty\frac{dk}{k}\int_{\theta_1'}^{\theta_2'}\frac{\sin\theta' W(k,\theta')\,d\theta'}{\sqrt{(\cos\theta_1'-\cos\theta')(\cos\theta'-\cos\theta_2')}}\begin{pmatrix}1\\ \Theta\\ \Theta^2\end{pmatrix}. \quad (A.6)$$

Here

$$\Theta=\frac{\cos\theta-k\cos\theta'}{\sin\theta},\quad \cos\theta_{1,2}'=\frac{1}{k}(\cos\theta\pm\sin\theta\sqrt{k^2-1}).$$

If $W=\widetilde{W}(k)\delta(1-\cos\theta')$, and $\theta\ll 1$, then

$$\left.\begin{matrix}\mathcal{D}_{pp}\\ \mathcal{D}_{p\theta}\\ \mathcal{D}_{\theta\theta}\end{matrix}\right\}=\pi^2\begin{pmatrix}1\\ -\theta\\ \theta^2\end{pmatrix}\widetilde{W}|_{k=1+\theta^2/2}. \quad (A.7)$$

If $\theta\ll\Delta\theta'$, where $\Delta\theta'$ is the angle width of the spectrum of the oscillations, with $\Delta\theta'\ll 1$, then

$$\left.\begin{matrix}\mathcal{D}_{pp}\\ \mathcal{D}_{p\theta}\\ \mathcal{D}_{\theta\theta}\end{matrix}\right\}=2\pi\int_0^\infty\theta'd\theta'\int_{-\theta}^{\theta}\frac{W\,d\varepsilon}{\sqrt{\theta^2-\varepsilon^2}}\begin{pmatrix}1\\ \varepsilon\theta'/\theta\\ \varepsilon^2\theta'^2/\theta^2\end{pmatrix}, \quad (A.8)$$

where ϵ is defined by the relation (A.5).

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