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LOW-TEMPERATURE AND VIRTUAL PHASE TRANSITIONS

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The properties of low-temperature phase transitions are investigated using as an example several two-dimensional lattices of the Ising type. It is shown that in the region $|T - T_c| \ll T_c$ the Nernst theorem does not impose any restrictions, generally speaking, on the behavior of the specific heat. Depending on the structure of the lattice, the coefficient a in the relation $C \approx a \ln|T - T_c|$ may tend to zero, remain constant, or even increase exponentially as $T_c \to 0$. In two of the three models considered, the region of applicability of the usual temperature dependences as $T_c \to 0$ for the spontaneous magnetization, $M \approx |T - T_c|^{1/8}$, and for the heat capacity, $C \sim \ln|T - T_c|$, turns out to be exponentially narrow; outside of this interval different power laws are obtained for M and C. The properties of systems with a "virtual" transition are discussed, i.e., systems in which a slight change of the interaction constant leads to the appearance of a low-temperature transition. For small values of T, in these systems the specific heat has a maximum, imitating a phase transition. In all cases the spin correlations are investigated in detail.

1. INTRODUCTION

It is well known that a number of thermodynamic quantities—specific heat, susceptibility, the derivative of the spontaneous magnetic moment with respect to the temperature, and others—have singularities at the point corresponding to a second-order phase transition. The two-dimensional Ising lattice[1] is an exactly solvable model for the investigation of these singularities. In the three-dimensional case the singularities are described with the aid of considerations based on the hypothesis of scaling (see, for example,[2]), which establishes a connection between the exponents of the powers of the different singularities near T_c.

In the present article the singularities of the low-temperature phase transitions are investigated in the case of several two-dimensional lattices of the Ising type. This may be of interest for several reasons. In the first place, it is of interest to see how the Nernst theorem influences the nature of the transition: as $T \to 0$ the specific heat must tend to zero, whereas from general considerations about transitions[2] the specific heat must tend to infinity as $T \to T_c$. Even if we postulate, for example, that in two-dimensional lattices in some kind of neighborhood of T_c the specific heat C(T) tends to infinity according to a universal logarithmic law, $C(T) = a \ln|T - T_c|$, the question of the dependence of the coefficient a on T_c arises. If as $T_c \to 0$ one also has $a \to 0$, then one can talk about the physical feasibility of the Nernst theorem—the singularity in C(T) as $T_c \to 0$ is increasingly less noticeable. On the other hand, if $a \to$ const as $T_c \to 0$, then one can talk about the stability of the singularity relative to the position of T_c. It turns out that, depending on the structure of the lattice, both the first and the second cases are possible (Secs. 2 and 3), and also a third case, which was very unexpected, is possible—an exponential increase of a as $T_c \to 0$, i.e., the total "suppression" of the Nernst theorem in the vicinity of the transition (Sec. 4).

The stated treatment was also of interest from the point of view of checking the physical resistance to change of the above-mentioned arguments based on the scaling hypothesis. In this approach, the basic assumption is that in the neighborhood of T_c the behavior of all quantities is determined by only a single small parameter—the proximity $\tau = (T/T_c) - 1$ to the transition point. point. But in real systems other small parameters which are essential for the transition frequently exist—for example, a small anharmonicity for structural transitions in crystals, the smallness of the ratio of T_c to the Fermi energy in metals, and so on. The question arises as to how the presence of these additional parameters influences the range of validity of the considerations based on the scaling hypothesis. The quantity T_c itself may be such an additional parameter (its ratio to some characteristic large interaction energy J). It turns out that in certain cases (see Sec. 3 and 4) the region of validity of the results of the scaling hypothesis in this connection becomes exponentially narrow, and in the region $\exp(-2JT^{-1}) \ll \tau \ll 1$ the dependence of the thermodynamic quantities on τ and the decrease of the correlations with distance strongly differ from the usual results.

Finally, we wanted to discuss the properties of the so-called virtual transitions. If for some kind of values of the interaction constants a transition occurs at small values of T_c, then it is possible that a small change of these constants will in general eliminate the given transition, similar to the way in quantum mechanics in which a small change of the interaction constant may change a level from real to virtual. If the degree of the "virtual nature" is small, the transition is close to a real transition, then it is physically obvious that the correlation phenomena associated with the transition, in particular the growth of C with decreasing T, will occur even though at very small values of T the specific heat tends to zero according to Nernst. As a result, a certain "imitation" transition will be observed, for example, a maximum in the specific heat. In two of the three models considered (see Secs. 3 and 4) such virtual transitions are possible, and our investigation illustrates their properties, in particular, the form of the dependence C(T) and the nature of the correlations in the systems.

2. ANISOTROPIC SQUARE ISING LATTICE

As the first, simplest example, let us consider the usual Ising lattice in which the interaction energy J_1 of the spins vertically is much smaller than the interaction energy horizontally, $J_2 \equiv -J$. In this connection, in the zero-order approximation in $\gamma = J_1/J$ the system breaks up into a collection of noninteracting one-dimensional chains, which are ordered only at absolute zero (see, for example,[3]). Therefore, it is clear that with a decrease of J_1 the transition temperature T_c decreases. Let us find the specific heat and the correlations near T_c.

The free energy per cell, F, is given by the well-known expression (see[4])

$$\beta F=\frac{1}{2}\ln\frac{(1-x^2)(1-y^2)}{4}-\frac{1}{2}\iint_0^{2\pi}\frac{d\omega\, d\rho}{(2\pi)^2}\ln[(1+x^2)(1+y^2) - 2x(1-y^2)\cos\omega-2y(1-x^2)\cos\rho] \quad (1)$$

Here $\beta = 1/T$, $x = \tanh \beta J_1$, and $y = \tanh \beta J$; in order to be definite we shall consider the "ferromagnetic" case: $J > J_1 > 0$. The equation for the transition temperature, $x(1 + y) = 1 - y$, gives

$$T_c \approx 2J/\ln(1/\gamma). \quad (2)$$

for small values of γ. From Eqs. (1) and (2) one can see that here the parameter characterizing the nearness to the transition is the quantity $t = (T - T_c)\beta^2 J \approx \tau\beta J$. For $|t| \ll 1$ we have the following result for the specific heat C(T):

$$C(T)=\frac{9}{4\pi}\gamma\ln^2\gamma\ln\frac{1}{|t|}=\frac{9}{\pi}\left(\frac{J}{T_c}\right)^2 e^{-2J/T_c}\ln\frac{1}{|t|}. \quad (3)$$

The over-all behavior of C(T) is shown in Fig. 1. The high-temperature maximum, just like the total behavior of C(T) in the region $T \gg T_c$, corresponds to the thermodynamics of uncorrelated one-dimensional chains for which $C = \beta^2 J^2 \cosh^{-2}\beta J$. One can understand the exponential smallness of the coefficient affiliated with $\ln|t|$ in Eq. (3) by considering the spin correlations.

One can find the correlation functions along horizontals, $G_h(r)$, and along verticals, $G_v(r)$, by using the methods of articles[5-8]. Above T_c in the region $0 \ll t \ll 1$ for large values of r we have

$$G_h(r)=\begin{cases}\dfrac{\text{const}}{(\gamma r)^{1/4}}\left(1-\dfrac{r}{4r_{ch}}\ln\dfrac{r_{ch}}{r}\right), & r\ll r_{ch}=\dfrac{1}{6t}e^{2\beta J}\\ \left(\dfrac{r_{ch}}{8\pi^2\gamma r^2}\right)^{1/4}e^{-r/r_{ch}}, & r\gg r_{ch}\end{cases}; \quad (4)$$

$$G_v(r)=\begin{cases}\dfrac{\text{const}}{r^{1/4}}\left(1-\dfrac{r}{4r_{cv}}\ln\dfrac{r_{cv}}{r}\right), & r\ll r_{cv}=\dfrac{1}{3t}\\ \left(\dfrac{r_{cv}}{2\pi^2 r^2}\right)^{1/4}e^{-r/r_{cv}}, & r\gg r_{cv}\end{cases}. \quad (5)$$

For $\gamma \to 0$ the numerical constants in Eqs. (4) and (5) do not depend on γ. The correlations have a similar form below T_c, but the spontaneous magnetization M for small values of $|t|$ is given by the expression $M \approx |3t|^{1/8}$.

From Eqs. (4) and (5) it is clear that near T_c the correlation radius r_{ch} is exponentially large: r_{ch}

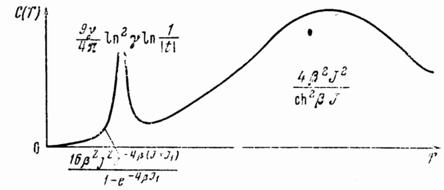

FIG. 1

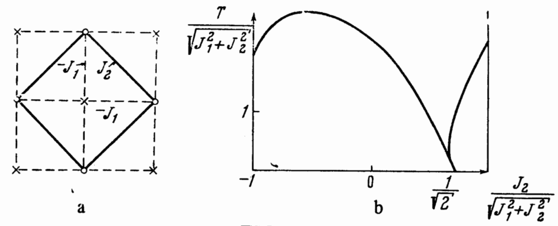

FIG. 2

$\sim \exp(2\beta J) r_{cv}$. This exponential growth corresponds to the usual one-dimensional correlations associated with low temperatures, $T \ll J$, since in this connection each row divides into exponentially long regions of homogeneous magnetization (see[3]). The phase transition at $T = T_c$ consists in the ordering of these regions in different rows as a consequence of the interaction J_1. Therefore, the entropy ΔS of the transition is, roughly speaking, proportional to the number of statistical objects which are ordered during the transition, which is exponentially small compared to the number of particles: $\Delta S \sim N r_{ch}^{-1} \sim N e^{-2\beta J}$, which also leads to small values of the heat capacity $C \sim T\Delta S/\Delta T$ in the transition region. Correct to within this "scaling factor" the picture of the transition and the nature of the temperature dependences near T_c remain the same as in the ordinary Ising lattice. Virtual transitions do not occur here, and the transition occurs for arbitrary values of J_1 and J_2.

3. MODEL WITH AN INTERACTION BETWEEN NON-NEAREST NEIGHBORS (NNN-MODEL)

Let us consider the model proposed in[8] of a plane lattice, consisting of two kinds of spins which are arranged and interacting as indicated in Fig. 2 a. The case of an antiferromagnetic interaction along the diagonals, $J_2 > 0$, is considered. The sign of the interaction along the vertical and horizontal (J_1) is unimportant for what follows; for the sake of definiteness we shall assume $J_1 > 0$. A graph showing the dependence of the Curie temperature T_c on J_1 and J_2 is shown in Fig. 2 b. In the region of values $J_1 > J_2 > 0.94\, J_1$, upon a reduction of T the system experiences three successive phase transitions: from an unordered to an antiferromagnetic state, then back to an unordered state, and finally into a ferromagnetic state. According to Fig. 2 b, the regions of the low-temperature transitions we are interested in correspond to similar values of the constants, when the difference $\delta = J_1 - J_2$ is small relative to the half-sum $J = (J_1 + J_2)/2 \approx J_1, J_2$.

First let us consider the thermodynamics. Expanding the exact expression (see[8]) for the free energy per cell in the region $T \ll J$, to within an unimportant constant and exponentially small terms $\sim e^{-2\beta J}$ we obtain

$$\beta F=-\frac{1}{2}\int_0^{2\pi}\!\!\int\frac{d\omega\, d\rho}{(2\pi)^2}\ln[1+4\xi^2-4\xi\cos\omega\cos\rho]. \qquad (6)$$

Here $\xi = e^{-4\beta\delta}$. The case of real transitions corresponds to $\delta > 0$. In this connection, to within the exponential accuracy being considered, the Curie temperatures T_{c2} and T_{c3} coincide and are equal to

$$T_c=4\delta/\ln 2\equiv 4(J_1-J_2)/\ln 2. \qquad (7)$$

The temperature dependence of the specific heat is depicted by the solid curve on Fig. 2. In the region of transitions $|\tau| \ll 1$ the coefficient in front of the logarithm in the specific heat turns out to be of the order of unity (although numerically it is appreciably smaller than in the square Ising lattice):

$$C=\frac{(\ln 2)^2}{2\pi}\ln\frac{1}{|\tau|}\approx 0{,}1\ln\frac{1}{|\tau|}$$
$$\left(C_{\mathrm{Is}}=\frac{2}{\pi}\ln^2(1+\sqrt{2})\ln\frac{1}{|\tau|}\approx 0{,}5\ln\frac{1}{|\tau|}\right). \qquad (8)$$

If J_1 is identically equal to J_2, that is, $\delta = 0$, then the low-temperature transitions vanish, and the region of transitions $|T - T_c| < T_c$ vanishes, and for $T \ll J$ the specific heat monotonically tends to zero: $C = 6.4\,\beta^2J^2e^{-4\beta J}$.

Finally, if $\delta < 0$ the thermodynamic behavior of the system corresponds to what was called above a virtual transition. In analogy with Eq. (6) one can introduce the temperature

$$T_v=-4\delta/\ln 2=4|\delta|/\ln 2. \qquad (9)$$

Then ξ in Eq. (6) is equal to $\exp(\beta T_V \ln 2) > 1$. The graph of C(T) in this case is qualitatively depicted on Fig. 3 by the dotted curve. For $T = 0.7\ T_V$ the specific heat C(T) has a low-temperature maximum, which is expressed more sharply the smaller the ratio T_V/J. Thus, for $\delta/J = 0.1$ the height of the hump C_{max} exceeds the value of C_{min} at the minimum between T_{c1} and T_V by only tens of percent, but for $\delta/J = 0.05$ one already has $C_{max} \approx 2\ C_{min}$.

In order to understand the thermodynamic behavior, let us again consider the spin correlations. The correlation function G(r) along the diagonals was evaluated in[8]. The answer is given by the usual formula for Ising lattices in the form of a Toeplitz determinant of order r:

$$G(r)=\operatorname{Det}c_{k-l},\quad c_{k-l}=\int_0^{2\pi}\frac{d\omega}{2\pi}f(\omega)\,e^{i\omega(k-l)}, \qquad (10a)$$

$$f(\omega)=\left[\frac{(\alpha_1e^{i\omega}-1)(\alpha_2e^{i\omega}-1)}{(e^{i\omega}-\alpha_1)(e^{i\omega}-\alpha_2)}\right]^{1/2}. \qquad (10b)$$

In the present case

$$\alpha_1=\frac{1-y}{1+y}\,\frac{4x^2-y(1+x^2)^2}{(1-x^2)^2},\quad \alpha_2=-y\,\frac{1+y}{1-y}\,\frac{(1-x^2)^2}{(1+x^2)^2-4x^2y},$$
$$x=\operatorname{th}\beta J_1,\qquad y=\operatorname{th}\beta J_2. \qquad (11)$$

Confining our attention to the region of very large values of r, for the values $T \ll J$, $\delta \ll J$ under consideration, we obtain by using the method of Wu[7] the following result for the antiferromagnetic region

$$G(r)=(-1)^rM^2+\frac{8\xi^2}{r\pi\sqrt{(4\xi^2-1)(4\xi^2+1)^3}}e^{-2r\ln 2\xi},\quad r\ln 2\xi\gg 1. \qquad (12)$$

Here ξ is the same as in Eq. (6), and

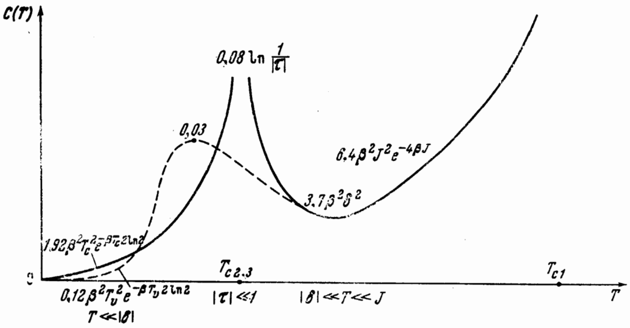

FIG. 3

$M = (4\xi^2 - 1)^{1/4}(4\xi^2 + 1)^{-1/4}$ is the spontaneous magnetization. In the case of a real transition, for the ferromagnetic region we similarly have

$$G(r)=M^2+(-1)^r\frac{8\xi^2e^{2r\ln 2\xi}}{r\pi\sqrt{(1-4\xi^2)(1+4\xi^2)^3}},\quad r|\ln 2\xi|\gg 1;$$
$$M=(1-4\xi^2)^{1/4}(1+4\xi^2)^{-1/4}. \qquad (13)$$

Formulas (12) and (13) enable us to trace the nature of the correlations in the system. In the temperature interval T_c, $T_V \ll T \ll J$ the quantity ξ is close to unity and the system behaves as if thermodynamically "frozen"—neither the magnetization nor the correlations change with T. Consequently the entropy is approximately constant, and the specific heat T(dS/dT) has a minimum. The quantity ξ begins to change appreciably only for $T \sim T_c$, T_V; in the case of a real transition it decreases, and in the case of a virtual transition it increases. As a result the distribution of the spins and the entropy change, and the specific heat has a maximum even in the case of a virtual transition. For a real transition, the change in the distribution of the spins and in the entropy is generally of the same order as that for the transition in an ordinary Ising lattice, and therefore the coefficient of $\ln|\tau|$ in the heat capacity is of the order of unity for both transitions. However, in the region under consideration, $1 \gg |\tau| \gg e^{-2\beta J}$, which is not exponentially close to the transition temperature, the spontaneous magnetization changes like $|\tau|^{1/4}$ instead of the usual $|\tau|^{1/8}$ law for Ising lattices (the latter dependence, as one can show, is valid here in an exponentially narrow neighborhood of T_c, $\tau \ll e^{-2\beta J}$). The unusual dependence of M on τ is obviously related to the "convergence" of the transitions T_{c2} and T_{c3} in a given temperature interval: instead of the usual transition into a disordered phase, the transition within an exponentially narrow interval T goes from the antiferromagnetic state immediately into the ferromagnetic state.

We further note that, according to Eq. (6), in the antiferromagnetic region for T = 0 a residual entropy exists which is equal to ln 2 per cell. This is associated with the fact that upon total antiferromagnetic ordering of the circles in Fig. 2a, each of the crosses is found to be in neutral equilibrium with regard to orientation.

4. TRIANGULAR LATTICE

The low-temperature transitions in the triangular lattice[9] shown in Fig. 4 possess the most unusual

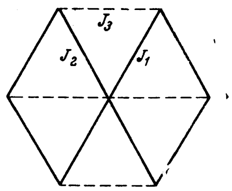

FIG. 4

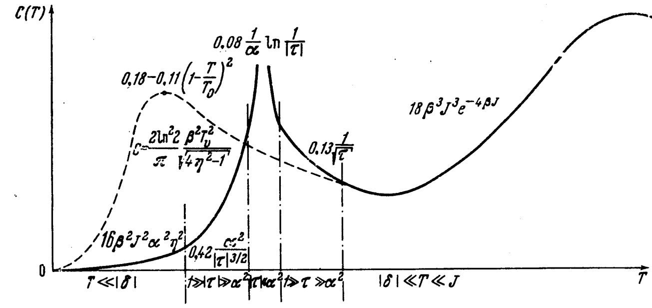

FIG. 5

properties. An expression for F for arbitrary values of J_1, J_2, and J_3 is given by Houptappel.[10] The case of equal antiferromagnetic bonds, $J_1 = J_2 = J_3$, was considered in the article by Wannier.[9] In this connection there is no phase transition, as is obvious from geometrical considerations, and the entropy does not vanish even at absolute zero.

We shall consider a "weakly distorted" Wannier lattice in which the antiferromagnetic constants J_1 and J_2 are equal and slightly different from the third constant J_3:

$$J_1 = J_2 = J, \quad \delta = J_1 - J_3, \quad |\delta| \ll J. \tag{14}$$

According to[10] the exact expression for F with $J_1 = J_2$ is written in the form

$$\beta F = -\frac{1}{2}\iint_0^{2\pi} \frac{d\omega\, d\rho}{(2\pi)^2} \ln[1 + 2\eta^2 + \alpha^4 + 2\eta(1-\alpha^2)(\cos\omega + \cos\rho) + 2(\eta^2 - \alpha^2)\cos(\omega+\rho)], \tag{15}$$

where $\eta = e^{-2\beta\delta}$ and $\alpha = e^{-2\beta J}$.

Just as in Sec. 3, for $\delta > 0$ there is a transition into the antiferromagnetic state in the system. The temperature T_c is determined by the relation

$$1 - 2\eta(T_c) - \alpha^2(T_c) = 0, \tag{16}$$

corresponding in (15) to the point where the value of the logarithm has a minimum: $\cos\omega = \cos\rho = -1$; for small values of δ we have $T_c \ln 2 = 2\delta$. For $\delta < 0$ the transition is absent, it becomes a virtual transition since here the "disruptive order" of influence of J_3 becomes larger than the ordering influence of the bonds J_1 and J_2.

The temperature dependence of the specific heat C(T) is depicted schematically on Fig. 5. For $T \gg \delta$ the specific heat does not depend on δ and has a maximum for $T \sim J$, corresponding to the appearance of "short-range order," a partial ordering of the spins in a structure of the type considered by Wannier[9] (which is not simply ferromagnetic or antiferromagnetic, but corresponds to a more complicated law of alternation of the orientations). With a reduction of T in the region T_c, $T_V \ll T \ll J$ the size of these regions of "short-range" order begins to grow exponentially, similar to the way this occurs, for example, in the case mentioned above of a linear Ising chain, and the specific heat falls exponentially. For $T \lesssim |\delta|$ the growth of C(T) starts again. In the case of a virtual transition the behavior of C(T) in general is close to the case of the NNN-model described above, although the low-temperature anomaly is expressed more strongly (the dotted curve in Fig. 5). Thus, for $|\delta|/J = 0.1$ the height of the maximum C_{max} at the point $T = 0.35\ T_V$ is larger than the value of the minimum lying between T_V and $T \sim J$, roughly by a factor of 2.2. The nature of the dependence of C on T for a real transition turns out to be extremely unexpected. Right up to temperatures which are exponentially close to T_c, $\alpha^2 \ll \tau \ll 1$, instead of the usual logarithmic laws the specific heat C(T) increases like $1/\sqrt{\tau}$; however, in the region $|\tau| \ll \alpha^2$ the coefficient affiliated with the logarithm in the dependence $C \approx a\ln(1/|\tau|)$ is exponentially large: $a \sim e^{2\beta J}$. Below T_c the specific heat also changes according to a power law: $C(T) \sim \alpha^2|\tau|^{-3/2}$, and only for $T \ll T_c$ do the usual dependences $C \sim e^{-4\beta J}$ exist.

In order to understand these results, let us again consider the correlations. By the usual methods[5] we find that the correlation function in the direction of the coupling J_1 (in the general case of different values of J_1, J_2, and J_3) is given by formula (10a) with $f(\omega)$ given by

$$f(\omega) = [4yz(1+x^2) - 2x(1+y^2)(1+z^2) - (e^{i\omega}x^2 - e^{-i\omega})(1-y^2)(1-z^2)] \times \{[(1+x^2)(1+y^2)(1+z^2) - 8xyz]^2 + 4x^2(1-y^2)^2(1-z^2)^2\cos^2\omega - 4(1-x^2)^2[(1-z^2)^2y^2 + (1-y^2)^2z^2] - 4(1+x^2)(1-y^2)(1-z^2) \times [2yz(1+x^2) - x(1+y^2)(1+z^2)]\cos\omega\}^{-1/2}, \tag{17}$$

where $x = \tanh\beta J_1$, $y = \tanh\beta J_2$, and $z = \tanh\beta J_3$. For $z = 0$ Eq. (17) agrees with the expression for correlations along a row, and for $x = 0$, $y = z$ it agrees with the expression for the correlations along a diagonal for the usual square Ising lattice.[11,5]

In the case $J_1 = J_2 = J$ which is being considered, for the correlations in the directions J_1 and J_3, respectively, we have the following results:

$$f_1(\omega) = \left(\frac{e^{i\omega}\eta(1-\alpha)^2 + e^{-i\omega}\eta(1+\alpha)^2 + 1 - \alpha^4}{e^{-i\omega}\eta(1-\alpha)^2 + e^{i\omega}\eta(1+\alpha)^2 + 1 - \alpha^4}\right)^{1/2},$$
$$f_3(\omega) = \left(\frac{(\eta-\alpha)^2e^{i\omega} + (\eta+\alpha)^2e^{-i\omega} - (1 - 2\eta^2 + \alpha^4)}{(\eta-\alpha)^2e^{-i\omega} + (\eta+\alpha)^2e^{i\omega} - (1 - 2\eta^2 + \alpha^4)}\right)^{1/2}. \tag{18}$$

Just as above, we find the asymptotic behavior of $G_1(r)$ and $G_3(r)$ for large values of r by using Wu's method.[7] Above the transition for all values of T except those exponentially close to T_c: $\tau \gg \alpha^2$ (and for all values of T for a virtual transition), from Eq. (18) we have

$$G_1(r) = (-1)^r\sqrt{\frac{2}{\pi}}\frac{\cos\omega_{01}r}{\sqrt{r\sin\omega_{01}}}e^{-2\alpha\tau}, \quad \alpha r \gg 1,$$
$$\cos\omega_{01} = \frac{1+\alpha^2}{2\eta}, \quad \sin\omega_{01} > 0, \tag{19a}$$

$$G_2(r) = (-1)^r\sqrt{\frac{2}{\pi}}\frac{\cos\omega_{03}r}{\sqrt{r\sin\omega_{03}}}e^{-2\alpha r/\eta}, \quad \frac{\alpha r}{\eta} \gg 1,$$
$$\cos\omega_{03} = \frac{2\eta^2 - 1 - \alpha^4}{2(\eta^2 - \alpha^2)}, \quad \sin\omega_{03} > 0. \tag{19b}$$

(We note that the asymptotic expressions (19) indicate the presence of complex singularities for the Fourier

$$G_1(r)=\frac{(-1)^r}{(2\pi^2\ln 2)^{1/4}}\frac{1}{(r^2\tau)^{1/4}}\exp\left(-\frac{r\tau\ln 2}{2\alpha}\right).$$

components of the correlation functions $G(\mathbf{k})$, and thus this possibility should be taken into consideration in the analytical approaches to the theory of correlations near T_c which have been developed recently.[12])

From Eq. (19) it is seen that inside regions of dimension $r_{cl} \approx \alpha^{-1} \gg 1$ the spins are ordered and form something like a periodic "domain" structure with the periods $r_{d1} = 2\pi\omega_{01}^{-1}$, $r_{d3} = 2\pi\omega_{03}^{-1}$. Far away from the transition in the region $|\delta| \ll T \ll J$ the value of α is small, and η is close to unity so that the periods r_d are of the order of unity and the ordering of the spins in "domains" corresponds to the Wannier structures[9] which have been mentioned. But with a reduction of T the periods r_d begin to change. In the case of a real transition, as $T \to T_c$, $\eta \to 1/2$ so that the periods grow like $1/\sqrt{\tau}$. This apparently corresponds to the fact that inside the common exponentially large radius r_{cl} of correlation, "nucleating centers" of a new antiferromagnetic (in the directions J_1 and J_2) phase begin to appear, and these start to displace the phase correlated "according to Wannier". With the approach to T_c the dimensions of the nucleating center grow so that in the important regions $r_d \sim \tau^{-1/2}$ a simple antiferromagnetic ordering still exists, although among themselves these "domains" are no longer correlated in an antiferromagnetic way but somehow "according to Wannier". At a certain temperature $\tau \sim \alpha^2$ which is exponentially close to T_c, the size of the domains is comparable with the correlation radius r_{cl} (formally the point $r_d = \infty$ is determined by the equation $1 + \alpha^2 = 2\eta$). After this the largest correlation dimension is the transition-associated radius of the simple antiferromagnetic correlations $r_d \sim \alpha\tau^{-1}$, and for $\tau \ll \alpha^2$ the functions $G_i(r)$ have the forms

$$G_3(r)=\frac{1}{(8\pi^2\ln 2)^{1/4}}\frac{1}{(r^2\tau)^{1/4}}\exp\left(-\frac{r\tau\ln 2}{\alpha}\right); \quad \frac{r\tau}{\alpha}\gg 1. \tag{20}$$

The asymptotic behavior at small distances can also be found by using the methods of articles[6-8] (as is done in (4) and (5)), and also the correlations below T_c can be found. Thus, in the region $\mu = 1 - 2\eta \gg \alpha^2$ the function $G_1(r)$ has the form

$$G_1(r)=(-1)^r M^2+(-1)^r M^2\frac{e^{-r\sqrt{8\mu}}}{2\mu}\begin{cases}1/16\pi r^2, & \alpha r\gg 1\\ \alpha^2 & 1/\sqrt{\mu}\ll r\ll 1/\alpha\end{cases}, \tag{21}$$

and the spontaneous magnetization M is given by the expression

$$M=\left(\frac{1+\alpha^2}{1-\alpha^2}\right)^{1/4}\left[\frac{(1-\alpha^2)^2-4\eta^2}{(1+\alpha^2)^2-4\eta^2}\right]^{1/8}\approx\begin{cases}(2^{-1}|\tau|\alpha^{-2}\ln 2)^{1/8}, & |\tau|\ll\alpha^2\\ 1-\dfrac{1}{4\ln 2}\dfrac{\alpha^2}{|\tau|}, & \alpha^2\ll|\tau|\ll 1.\end{cases} \tag{22}$$

It is evident that saturation of the magnetic moment occurs here even exponentially close to T_c.

Thus, the unusual square-root growth of the specific heat in the region $\alpha^2 \ll \tau \ll 1$ is here associated with the unusual nature of the transition, consisting in the displacement of one quasi-ordered phase by another in a narrow temperature interval (for low-temperature transitions such a situation may not even be very rare). Using the result about the size of the antiferromagnetic regions in the interval $\alpha^2 \ll \tau \ll 1$: $r_d \sim \tau^{-1/2}$, one can simply obtain this square-root law with the aid of dimensional arguments, similar to those used in the methods of scaling (see[2]). Let us start from the fact that the free energy associated with an expansion in powers of the small quantity r_d^{-1} is represented in the form of a sum of terms corresponding to each of the correlation regions:

$$F\sim\sum_i f_i\left(\frac{1}{r_d}\right)\sim\frac{1}{r_d^2}f\left(\frac{1}{r_d}\right). \tag{23}$$

In the zero order approximation in r_d^{-1} the dependence (23) would in general lead to the law $F \sim A\tau\ln\tau$, since the logarithmic terms cannot be detected by dimensional arguments. But as $\tau \to 0$ this would give a logarithmically divergent value of the entropy, so that the constant A must be equal to zero. Therefore, the expansion of F must begin with the next power of r_d^{-1} (which corresponds, possibly, to taking "surface" boundary effects into consideration). Then $F \sim \tau^{3/2}$ and $C \sim \tau^{-1/2}$. An exponentially large value of the coefficient affiliated with $\ln|\tau|$ in the region $|\tau| \ll \alpha^2$ is obtained as a result of joining the square-root formula for C with the logarithmic formula at the limit of applicability of the law $r_d \sim \tau^{-1/2}$, i.e., for $r_d \sim r_{cl} \sim e^{2\beta J}$. Physically the exponentially large magnitude of the specific heat evidently reflects the fact that there is a substantial change in the structure of the state and in the entropy in a very small, exponentially narrow, temperature interval.

Let us again discuss the nature of the correlations associated with a virtual transition. In this connection in formulas (15) and (18) the parameter $\eta = \exp(\beta T_V \ln 2) > 1$, so that one can neglect terms $\sim \alpha$ everywhere except in the argument of the exponential in (19). With an increase of η from $\eta \approx 1$ to $\eta \to \infty$ the period of the correlations r_{d1} with regard to J_1 decreases (from 12 to 4) whereas in the direction of the larger coupling J_3 instead of Wannier correlations total antiferromagnetic ordering begins. The maximum in the specific heat at $T \sim T_V$ reflects this change of the structure, the appearance of simple antiferromagnetic ordering along the filaments of the direction J_3.

5. CONCLUSION

From the examples which have been considered it is clear that a reduction of the transition temperature T_c has a universal effect only on the width of the region associated with the transition of correlation phenomena: $|T - T_c| \lesssim T_c$. However, the scale and character of these phenomena may be very different, and the specific heat near T_c may be either exponentially small, of the order of unity, or exponentially large. The presence of a small parameter T_cJ^{-1} may also lead to a deviation of the temperature dependences of thermodynamic quantities from those predicted by the scaling hypothesis in the region $e^{-2\beta J} \ll \tau \ll 1$ which is not exponentially close to the transition—the law $M \sim |\tau|^{1/4}$ in Sec. 3 and the laws $C \sim \tau^{-1/2}$ and $C \sim |\tau|^{-3/2}$ above and below T_c in Sec. 4. In the case of virtual transitions a "partial ordering" of the system occurs in the region $T \sim T_V$, and there is a maximum in the specific heat. As mentioned, this should be taken into consideration in connection with the experimental investigation of low-temperature transitions: a maximum in the dependence of C on T may not actually indicate a real transition, but it may only correspond to some kind of change in the short-range order (although at the low temperatures under consideration the corresponding radii of ordering may turn out to be extremely large).

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