ACOUSTIC FLOW IN A SUPERFLUID LIQUID

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Acoustic flows arising in the propagation of first and second sound in a superfluid liquid are considered with an accuracy to the second approximation.

THE propagation of a sound wave in an absorbing medium is always accompanied by constant flows, inasmuch, along with the energy losses, the sound wave loses part of its momentum and, as a consequence of the law of momentum conservation, this loss is compensated by the appearance of a unidirectional movement of the medium—acoustic flow.

In the consideration of acoustic flow, the equations of hydrodynamics are usually solved in the second approximation, taking the equations of linear acoustics as the first approximation.[1,2]

As a basis of further discussion, we introduce the set of equations proposed by Khalatnikov:[3]

$$\partial\rho/\partial t+\operatorname{div}\mathbf{j}=0,$$

$$\frac{\partial j_i}{\partial t}+\frac{\partial\Pi_{ik}}{\partial x_k}=\frac{\partial}{\partial x_k}\left\{\eta\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_{nl}}{\partial x_l}\right)\right\}+\frac{\partial}{\partial x_i}\{\xi_1\operatorname{div}(j-\rho v_n)+\xi_2\operatorname{div}v_n\},\tag{1}$$

$$\dot{v}_s+\nabla(\mu+v_s^2/2)=\nabla\{\xi_3\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)+\xi_1\operatorname{div}\mathbf{v}_n\},$$

$$T\{\dot{s}+\operatorname{div}(\rho s\mathbf{v}_n+\varkappa\nabla T/T)\}=\xi_3[\operatorname{div}(\mathbf{j}-\rho\mathbf{v}_n)+\xi_2(\operatorname{div}\mathbf{v}_n)^2+2\xi_1\operatorname{div}(j-\rho v_n)\operatorname{div}\mathbf{v}_n+\frac{1}{2}\left(\frac{\partial v_{ni}}{\partial x_k}+\frac{\partial v_{nk}}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial v_{nl}}{\partial x_l}\right)^2+\frac{\varkappa}{T}(\nabla T)^2,$$

where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the velocities of the normal and superfluid components, respectively, j is the current density of the liquid, Π_{ik} the momentum flux tensor, η the coefficient of "first" viscosity, ξ_1, ξ_2, ξ_3 the coefficients of "second" viscosity, κ the thermal conductivity, ρ the density, and T the temperature of the liquid.

For a solution of the set (1), we use the method of successive approximations. The solution of the set (1) in first approximation was found in[3]. It has the form

$$v_{1s}=A_1\cos(\omega t-k_1z)\exp(-\alpha_1 z),$$
$$v_{2s}=A_2\cos(\omega t-k_2z)\exp(-\alpha_2 z),\tag{2}$$

where k_1 and k_2 are respectively equal to ω/c_1 and ω/c_2, A_1 and A_2 are the corresponding velocities of these waves, α_1 and α_2 the absorption coefficients:

$$\alpha_1=\frac{\omega^2}{2\rho c_1^3}\left\{\frac{4}{3}\eta+\xi_2\right\},\quad \alpha_2=\frac{\omega^2}{2\rho c_2^3}\left\{\frac{4\eta\rho_s}{3\rho_n}+\frac{\rho_s}{\rho_n}(\xi_2-2\rho\xi_1+\rho^2\xi_3)\right\}\tag{3}$$

The values of the quantities A_1 and A_2 depend on the method of sound excitation. For simplicity, we have assumed the thermal expansion coefficient $\beta = 0$ everywhere below (see[1]); then, for sound excited by a piston executing vibrations in a direction perpendicular to its plane (first sound),

$$\mathbf{v}_n=\mathbf{v}_s,\ A_1=v_0,\ A_2=0.\tag{4}$$

In the case in which sound is excited from a surface with a periodically changing temperature (second sound)

$$\mathbf{v}_n=-(\rho_s/\rho_n)\mathbf{v}_s,\ A_1=0,\ A_2=-T's/c_2.\tag{4'}$$

The time average of (2) is zero, which means that, in first approximation, there are no acoustic currents in the superfluid liquid.

In second approximation, time-independent terms appear in the expressions for the velocities. We obtain a set of equations for these quantities in the usual fashion (see, for example,[2]), leaving the terms in (1) of second order of smallness and averaging the resultant equations in time (the smallness of the acoustic and flow Reynolds numbers $\mathrm{Re}_a = v_1\lambda/\nu$, $\mathrm{Re} = ud/\nu$) is assumed here. The resultant sets of equations will differ slightly for first and second sound.

For first sound, we obtain

$$\operatorname{div}\langle\mathbf{v}_{2s}\rangle=0,\ \operatorname{div}\langle\mathbf{v}_{2n}\rangle=0,\ \eta\Delta\langle\mathbf{v}_{2n}\rangle-\nabla\langle p_2\rangle=\mathbf{F},$$
$$\operatorname{rot}\langle\mathbf{v}_{2s}\rangle=0,\ \nabla\langle p_2\rangle=\rho_0 s_0\nabla\langle T_2\rangle,\tag{5}$$

where $\langle v_{2s}\rangle$ and $\langle v_{2n}\rangle$ are the time-constant flow velocities in the second approximation, the index 0 or 1 denotes the zeroth or first approximation of the corresponding quantities.

For flow velocities arising in the propagation of second sound, we obtain the following set of equations:

$$\operatorname{div}\langle\mathbf{v}_{2s}\rangle=0,\ \operatorname{div}\langle\mathbf{v}_{2n}\rangle=0,\ \operatorname{rot}\langle\mathbf{v}_{2s}\rangle=0,$$
$$\eta\Delta\langle\mathbf{v}_{2n}\rangle-\nabla\langle p_2\rangle=-\mathbf{\Phi},\ \nabla\langle T_2\rangle=\mathbf{\Phi}/\rho_0 s_0.\tag{6}$$

Both sets of equations are valid for the assumption

$$\alpha L\ll 1,\tag{7}$$

where L is the maximum dimension of the region in which the currents are observed (for example, the length of the tank); thus, this condition denotes small sound absorption over the wavelength.

The condition for radiation of a plane wave:

$$(kr_1)^2\gg 1,\tag{8}$$

where r_1 is the characteristic dimension of the radiator. Further,

$$\alpha\nu\ll\rho_0 r_1,\ \alpha_2\nu\ll\rho_n c_2/\rho_s,\ \nu=\eta/\rho_0\ll c_2 r_1\tag{9}$$

It is also assumed that the product of second-order quantities by the dissipation coefficients or the thermal conductivity has a higher order of smallness.

As an example, we now consider currents that arise in the propagation of plane sound beam produced by a circular radiator of radius r_1 in a cylindrical pipe of radius r_0, the ends of which are closed by films that are sound-transparent. We shall be interested in the flow in the central part of the pipe, neglecting the effects of its ends; we shall also assume that the radius of the beam is equal to the radius of the radiator (the vibrational velocity outside the beam is equal to zero).

Let us consider the propagation of first sound in the pipe. We must solve the system (5) with boundary conditions that require the vanishing of the tangential components of normal flow. We introduce a cylindrical set of coordinates with the z axis directed along the axis of the pipe. This boundary condition means that

$$v_{2nz}(r_0) = 0. \tag{10}$$

In addition, the condition of conservation of mass over the cross section of the pipe must be satisfied:

$$\int_0^{r_0} (\rho_n\langle v_{2n}\rangle + \rho_s\langle v_{2s}\rangle)\, r\, dr = 0. \tag{11}$$

We now consider the current in the central part of the pipe and neglect the change in the quantities of interest to us along the z axis. We take the curl of the third equation of the set (5). After integration of this equation, use of the boundary conditions (10), the condition of the finiteness of the velocity for r = 0 and the matching of the solution and its derivative at the point $r = r_1$, we determine four of the five integration constants for the quantity $u_{nz} = \langle v_{2nz}\rangle$. Here it must be taken into account that the quantity F in the system (5) has a discontinuity and is equal to

$$F = \begin{cases} \alpha_1\rho_0 A_1^2/2 = F_1 + F_2, & 0 \leqslant r < r_1, \\ 0, & r_1 < r \leqslant r_0, \end{cases} \tag{12}$$
$$F_1 = \alpha_1\rho_{0n}A_1^2/2,\quad F_2 = \alpha_1\rho_{0s}A_1^2/2.$$

We determine the last integration constant from the following considerations. It is evident that the quantity $\langle v_{2s}\rangle$ can only be a constant in order to satisfy the corresponding equations of the system (5), and for $\rho_s = 0$ (i.e., above the λ point) $\langle v_{2s}\rangle \equiv 0$ and $F_2 = 0$. Using this circumstance and the condition (11), we determine the missing constant in the expression for u_{nz}. Further, from the condition (11) we determine the quantity $\langle v_{2s}\rangle$ = const below the λ point; we get

$$\langle v_{2sz}\rangle = \frac{F_2 r_1^2}{\rho_{0s}}\left(1 - \frac{r_1^2}{2r_0^2}\right) = \frac{\alpha_1 A_1^2 r_1^2}{2}\left(1 - \frac{r_1^2}{2r_0^2}\right).$$

However, this value of $\langle v_{2sz}\rangle$ does not vanish for $\rho_0 = 0$ and the solution of the set (5) will not be identical with the solution determining the acoustic current in an ordinary liquid for $\rho_s = 0$.[2] It is evident that $\langle v_{2s}\rangle$ can only be equal to zero.

Thus we have, finally,

$$\langle v_{2nr}\rangle = \langle v_{2sr}\rangle = \langle v_{2sz}\rangle = 0,$$

$$u_{nz} = \begin{cases} \dfrac{Fr_1^2}{\eta}\left[\dfrac{1}{2}\left(1 - \dfrac{x^2}{y^2}\right) - \left(1 - \dfrac{y^2}{2}\right)(1 - x^2) - \ln y\right], & y \geqslant x \geqslant 0 \\ -\dfrac{Fr_1^2}{\eta}\left[\left(1 - \dfrac{y^2}{2}\right)(1 - x^2) + \ln x\right], & y < x \leqslant 1 \end{cases} \tag{13}$$
$$x = r/r_0,\quad y = r_1/r_0.$$

It can be shown that for propagation of second sound in the pipe, $\langle v_{2nr}\rangle = \langle v_{2sr}\rangle = 0$ and $\nabla\langle p_2\rangle = 0$. The solution for the velocity $\langle v_{2nz}\rangle$ is then found by simple integration of the fourth equation of the set (6); with account of the boundary conditions it has the form

$$\langle v_{2nz}\rangle = \begin{cases} \dfrac{\Phi(r^2 - r_1^2)}{4\eta} + \dfrac{\Phi r_1^2}{2\eta}\ln\dfrac{r_1}{r_0}, & 0 \leqslant r < r_1, \\ \dfrac{\Phi r_1^2}{2\eta}\ln\dfrac{r}{r_0}, & r_1 \leqslant r \leqslant r_0. \end{cases} \tag{14}$$

The velocity $\langle v_{2sz}\rangle$ is found from the condition (11):

$$\langle v_{2sz}\rangle = \frac{\rho_s}{\rho_n}\frac{\Phi r_1^2}{4r_0^2\eta}\left(r_0^2 - \frac{r_1^2}{2}\right). \tag{15}$$

The quantity Φ in (14) and (15) is equal to

$$\Phi = \begin{cases} -\dfrac{\alpha_2}{2}\dfrac{\rho_0\rho_{0s}A_2^2}{\rho_{0n}}, & 0 \leqslant r < r_1, \\ 0, & r_1 \leqslant r \leqslant r_0. \end{cases}$$

We note that in this case, when the radius of the source in equal to the radius of the pipe, we get

$$\langle v_{2sz}\rangle = \Phi(r^2 - r_1^2)/4\eta,$$

i.e., the Poiseuille solution results.

[1] L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Fluid Mechanics) (GITTL, 1954) [Addison-Wesley, 1958].

[2] C. Eckart, Phys. Rev. **73**, 68 (1948).

[3] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **23**, 169 (1952).

[4] I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **23**, 265 (1952).

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