OPTICAL PROPERTIES OF CHOLESTERIC LIQUID CRYSTALS

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The propagation of light in a cholesteric medium is considered. Solutions of Maxwell's equations are obtained. It follows from these solutions that there is a region of total reflection of the incident light. The conclusions of the paper agree with the available experimental data.[1]

1. INTRODUCTION

CHOLESTERIC liquid crystals have highly unique optical properties.[1] First among them is the very strong "rotating ability." Whereas in quartz, for example, the rotation of the plane of polarization is 24 deg/mm, in crystals of this type this value can reach several thousand degrees. In addition, in liquid crystals of the cholesteric type there is a region of total reflection of circularly polarized light. At the same time, the other circularly polarized component of the incident ray is transmitted entirely into the medium. The electric vectors of the incident and reflected waves are rotated in opposite directions. (We note that in reflection from "ordinary" crystals the direction of the circular polarization remains unchanged.)

The group of problems connected with these phenomena was first considered by de Vries.[2] He solved Maxwell's equations in a cholesteric medium. To this end, all of space was broken up into thin layers having different orientations of the electric axes. The reflected wave was obtained as the resultant of all the waves reflected from the individual layers. Although the method of breaking up into layers did not enter in the final formulas, account was taken in their derivation of additional (actually nonexistent) waves reflected from the boundaries of these hypothetical layers. Therefore the results obtained in this paper are quantitatively incorrect. Conners[3] investigated a similar problem, but considered only normal incidence of the beam, and all the equations were solved only in the total-reflection region, when the wavelength of the light was exactly equal to the pitch of the cholesteric helix.

We present here a complete investigation of this problem for arbitrary incidence angles. It turns out here that in the case of normal incidence (see Sec. 2), Maxwell's equations can be solved exactly for arbitrary ratios of the parameters of the problem. In Sec. 3 we investigate oblique incidence of the light. In this case, an exact solution can no longer be obtained. In this section, the resonance region (total reflection) and the regions off resonance, where ordinary perturbation theory is already valid, are considered separately. All the deductions of the paper are in good agreement with the available experimental data.[1]

2. NORMAL INCIDENCE OF LIGHT

Maxwell's equations in a medium can be written in the following form:

$$\gamma_{ijk}\frac{\partial H_k}{\partial x_j}-\frac{1}{c}\frac{\partial D_i}{\partial t}=0,\qquad \gamma_{ijk}\frac{\partial E_k}{\partial x_j}+\frac{1}{c}\frac{\partial H_i}{\partial t}=0. \tag{1}$$

In these formulas γ_{ijk} denotes a completely antisymmetrical unit tensor, $D_i = \epsilon_{ij}E_j$, and the magnetic permeability is set equal to unity. As usual, summation over repeated indices is implied.

A feature of a cholesteric medium is the dependence of the dielectric tensor ϵ_{ij} on one of the coordinates, say, z. This is connected with the helix-like symmetry of the medium (the electric axes along the x and y directions rotate relative to each other as functions of the coordinate z). We shall assume throughout that the z axis is directed along the axis of the cholesteric helix. We refer the dielectric tensor in the $z = 0$ plane to the principal axes

$$\varepsilon_{ij}^0=\begin{Bmatrix}\varepsilon_1 & 0 & 0\\ 0 & \varepsilon_2 & 0\\ 0 & 0 & \varepsilon_3\end{Bmatrix} \tag{2}$$

For other values of z, the tensor ϵ_{ij} will already have off-diagonal elements. The value of ϵ_{ij} at a certain value of z is obtained from ϵ_{ij}^0 by rotating through an angle proportional to z:

$$\varepsilon_{ij}(z)=A_{ik}A_{jl}\varepsilon_{kl}^0, \tag{3}$$

where A_{ij} is the rotation matrix

$$A_{ij}=\begin{Bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{Bmatrix},\quad \theta=\alpha z, \tag{4}$$

The value of α depends on the pitch p of the cholesteric helix, $\alpha = 2\pi/p$. From (3) we easily obtain the components of the dielectric tensor

$$\begin{aligned}&\varepsilon_{11}=\bar\varepsilon(1+\delta\cos 2\theta),\\ &\varepsilon_{22}=\bar\varepsilon(1-\delta\cos 2\theta),\\ &\varepsilon_{12}=\varepsilon_{21}=\bar\varepsilon\delta\sin 2\theta,\\ &\varepsilon_{33}=\varepsilon_3,\\ &\varepsilon_{13}=\varepsilon_{31}=\varepsilon_{32}=\varepsilon_{23}=0.\end{aligned} \tag{5}$$

We use here the following notation: $\bar\epsilon = \frac{1}{2}(\epsilon_1 + \epsilon_2)$; $\delta = (\epsilon_1 - \epsilon_2)/(\epsilon_1 + \epsilon_2)$ (for concreteness we assume that $\epsilon_1 > \epsilon_2$ and $\delta > 0$).

Substituting (5) in (1) and eliminating the magnetic field in the usual manner, we obtain the following equations for electromagnetic waves propagating in the z direction

$$\frac{\partial^2 E_1}{\partial z^2}=\frac{1}{c^2}\left(\varepsilon_{11}\frac{\partial^2 E_1}{\partial t^2}+\varepsilon_{12}\frac{\partial^2 E_2}{\partial t^2}\right), \tag{6}$$

$$\frac{\partial^2 E_2}{\partial z^2} = \frac{1}{c^2}\left(\varepsilon_{12}\frac{\partial^2 E_1}{\partial t^2} + \varepsilon_{22}\frac{\partial^2 E_2}{\partial t^2}\right). \quad (7)$$

An important factor in the derivation of these equations was the fact that by virtue of the symmetry of the problem, the waves normally incident (along the z axis) on the cholesteric medium had no dependence whatever on the coordinates x and y. We now write out explicitly the time dependence

$$E_1(z,t) = F_1(z)e^{i\omega t}, \quad E_2(z,t) = F_2(z)e^{i\omega t}.$$

Introducing a new symbol $k^2 = \omega^2\bar{\varepsilon}/c^2$, we obtain in lieu of (6) and (7)

$$\frac{d^2F_1}{dz^2} + k^2[F_1 + \delta\cos 2\alpha z F_1 + \delta\sin 2\alpha z F_2] = 0,$$
$$\frac{d^2F_2}{dz^2} + k^2[F_2 - \delta\cos 2\alpha z F_2 + \delta\sin 2\alpha z F_1] = 0. \quad (8)$$

All the optical properties of cholesteric media for light propagating along the z axis are contained in the system (8). This system was solved in [3] only for $k = \alpha$. However, as will be shown later, this equation can also be solved exactly in the general case. We note also that in real cholesteric materials δ is small ($\delta \sim 0.01$).

We introduce the circular components of the field

$$F_+ = F_1 + iF_2, \quad F_- = F_1 - iF_2.$$

In terms of these components, the equations in (8) are written in the form

$$d^2F_+/dz^2 + k^2[F_+ + \delta e^{2i\alpha z}F_-] = 0,$$
$$d^2F_-/dz^2 + k^2[F_- + \delta e^{-2i\alpha z}F_+] = 0. \quad (9)$$

We seek a solution of these equations in the form

$$F_+ = A_+e^{i(\alpha+\beta)z}, \quad F_- = A_-e^{i(\beta-\alpha)z}. \quad (10)$$

From (9) and (10) we obtain a biquadratic equation for the determination of β:

$$\beta^4 - 2(k^2+\alpha^2)\beta^2 + [(k^2-\alpha^2)^2 - k^4\delta^2] = 0$$

or

$$\beta^2 = k^2 + \alpha^2 \pm k\sqrt{4\alpha^2 + k^2\delta^2}. \quad (11)$$

The two solutions for β^2 mean two circularly-polarized waves propagating with different velocities in the cholesteric medium (and the sign of β itself denotes the direction of wave propagation along the z axis). Substituting (11) in (10), we find the exact solution of Maxwell's equations in a cholesteric medium under normal propagation of the waves (along the z axis).

It is interesting to consider several different particular cases:

A. $\alpha^2(1-\delta) < k^2 < \alpha^2(1+\delta)$,

$$\beta_1^2 = k^2 + \alpha^2 + k^3\delta^2/4\alpha, \quad \beta_2^2 = k^2 + \alpha^2 - k^3\delta^2/4\alpha. \quad (12)$$

For the first wave $A_+/A_- = \alpha^2\delta^2/8k^2\delta \sim \delta$. Thus, in the first wave we have

$$E_+ \approx \delta e^{i(\alpha+\beta_1)z} = \delta e^{i(k+2\alpha)z}, \quad E_- \approx e^{i(-\alpha+\beta_1)z} = e^{ikz}.$$

This means (since $\delta \ll 1$) that the first wave is circularly polarized in a counterclockwise direction.

B. $k \leqslant \alpha/(1-\delta)^{1/2} \approx \alpha(1 + {}^1/_2\delta)$,

$$|\beta_1| = k + \alpha + k^3\delta^2/8\alpha(k+\alpha), \quad \beta_2^2 = (k-\alpha)^2 - k^3\delta^2/4\alpha. \quad (13)$$

Here we must consider two additional cases:

1) $|k-\alpha| \gg \alpha\delta$,

$$|\beta_2| = \alpha - k - k^3\delta^2/8\alpha(\alpha-k). \quad (14)$$

We then have for the first wave

$$A_+/A_- = k^2\delta/4\alpha(k+\alpha) \ll 1,$$

and for the second wave

$$\frac{A_+}{A_-} = \frac{k^2\delta(\alpha-k)4\alpha}{k^4\delta^2} = \frac{4\alpha(\alpha-k)}{k^2\delta} \sim \frac{\alpha-k}{\alpha\delta} \gg 1.$$

This means that the two waves are polarized in opposite directions. In the first wave $F_- = A_- \times \exp\{i[k + k^3\delta^2/8\alpha(k+\alpha)]\}$, and in the second $F_+ = A_+\exp\{i[k + k^3\delta^2/8\alpha(\alpha-k)]\}$. Thus there exists a phase difference

$$\Delta k = \frac{k^3\delta^2}{8\alpha(k+\alpha)} - \frac{k^3\delta^2}{8\alpha(\alpha-k)} = \frac{k^4\delta^2}{4\alpha(\alpha^2-k^2)}. \quad (15)$$

2) $|k-\alpha| \leq {}^1/_2\alpha\delta$.

Then $A_+/A_- \sim 1$ in the second wave, corresponding to an almost plane-polarized wave. If $A_+ = A(1+\lambda)$ and $A_- = A(1-\lambda)$, where $\lambda \ll 1$, then

$$F_x = Ae^{i|\beta_2|z}(\cos\alpha z + i\lambda\sin\alpha z),$$
$$F_y = Ae^{i|\beta_2|z}(\sin\alpha z - i\lambda\cos\alpha z). \quad (16)$$

C. $k \geqslant \alpha/(1-\delta)^{1/2} \approx \alpha(1 + {}^1/_2\delta)$,

$$|\beta_1| = k + \alpha + k^3\delta^2/8\alpha(k+\alpha), \quad \beta_2^2 = (k-\alpha)^2 - k^3\delta^2/4\alpha.$$

Let again $|k-\alpha| \gg \alpha\delta$. Then, just as in case B, there is a phase difference between the waves (rotation of the plane of polarization)

$$\Delta k = -k^4\delta^2/4\alpha(k^2-\alpha^2). \quad (17)$$

This quantity is shown in the figure as a function of the wave number k. The shaded area corresponds to case A.

D. $k^2\delta^2 \gg 4\alpha^2$ or $k \gg 2\alpha/\delta$.

This case corresponds to a quasiclassical analysis of Eqs. (9)

$$\beta_{1,2}^2 = k^2 + \alpha^2 \pm k^2\delta. \quad (18)$$

For the first wave we then have

$$A_+/A_- = -k^2\delta/k^2\delta = -1,$$
$$F_{1+} = A_{1+}e^{ik(1+\delta/2)z}e^{i\alpha z};$$
$$F_{1-} = A_{1-}e^{ik(1+\delta/2)z}e^{-i\alpha z}, \quad A_{1-} = -A_{1+}.$$

This wave is plane-polarized but is modulated with the pitch of the cholesteric helix

$$E_{x1} = A_{1+}e^{ik(1+\delta/2)z}\sin\alpha z,$$
$$E_{y1} = -A_{1+}e^{ik(1+\delta/2)z}\cos\alpha z. \quad (19)$$

Analogously, for the second wave ($A_+/A_- = 1$)

$$E_{x2} = A_2e^{ik(1-\delta/2)z}\cos\alpha z,$$
$$E_{y2} = A_2e^{ik(1-\delta/2)z}\sin\alpha z. \quad (20)$$

By way of an example of the use of the derived formulas, let us solve the problem of the reflection of normally incident light from a half-space filled with a cholesteric medium (vacuum at $z > 0$). We have here an incident wave

$$F_x = f_x\exp(-iQz), \quad H_x = -F_y,$$
$$F_y = if_y\exp(-iQz), \quad H_y = F_x$$

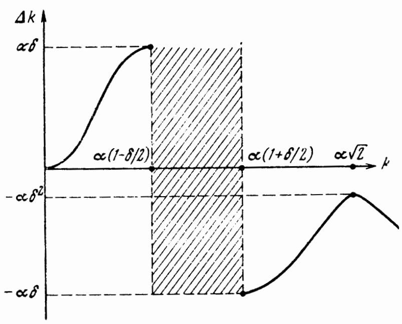
and a reflected wave

$$F_x' = f_x' \exp(iQz), \quad H_y' = -F_x',$$
$$F_y' = if_y' \exp(iQz), \quad H_x' = F_y'.$$

The wave in the medium, on the other hand, will be written in the form

$$F_x'' = A\exp(-i\gamma z), \quad F_y'' = iAf\exp(-i\gamma z), \tag{21}$$

where the quantities A, f, and γ are determined in obvious fashion from (9) and (10). f characterizes the ellipticity of the wave.



The magnetic fields in the medium are obtained from Maxwell's equations

$$\frac{1}{c}\frac{\partial H_y}{\partial t} = -\frac{\partial F_x}{\partial z}, \quad \frac{1}{c}\frac{\partial H_x}{\partial t} = \frac{\partial F_y}{\partial z}.$$

The quantities F_x and F_y in the right-hand side are obtained from the corresponding values at z = 0 (F_ξ and F_η) by rotating through an angle αz:

$$F_x(z) = F_\xi \cos\alpha z - F_\eta \sin\alpha z,$$
$$F_y(z) = F_\xi \sin\alpha z + F_\eta \cos\alpha z,$$

F_ξ and F_η are given by formulas (21). Hence

$$\frac{i\omega}{c}H_y = \alpha E_\eta + i\gamma E_\xi, \quad \frac{i\omega}{c}H_x = \alpha E_\xi - i\gamma E_\eta$$

at z = 0. Equating the tangential components of F and H at z = 0, we obtain

$$f_x + f_x' = A,$$
$$f_y + f_y' = Af,$$
$$f_x - f_x' = -\frac{ic}{\omega}A(f\alpha + i\gamma),$$
$$f_y - f_y' = -\frac{ic}{\omega}A(\alpha - i\gamma f).$$

Thus, for example, the "reflection coefficient" for the amplitude along the x axis is

$$R_x = \frac{f_x'}{f_x} = \frac{1 + ic(f\alpha + i\gamma)/\omega}{1 - ic(f\alpha + i\gamma)/\omega}. \tag{22}$$

Along the y axis

$$R_y = \frac{f_y'}{f_y} = \frac{f + ic(\alpha - i\gamma f)/\omega}{f - ic(\alpha - i\gamma f)/\omega}. \tag{23}$$

Formulas (22) and (23) solve our problem. For example, in the "resonance" region, i.e., at k = α, if left-polarized light with amplitude $|A_l|$ and wavelength in vacuum λ_0 ($Q = 2\pi/\lambda_0$) is incident on a cholesteric medium, then it is easy to obtain from (22), (23), and (12) the amplitude of the reflected left-polarized light $|A_l'|$:

$$\left|\frac{A_l'}{A_l}\right| = \frac{4Q}{k}\left(\frac{Q}{k}+1\right)^{-2} \tag{24}$$

and the amplitude of the reflected right-polarized light

$$\left|\frac{A_r'}{A_l}\right| = \left(\frac{Q}{k}-1\right)\left(\frac{Q}{k}+1\right)^{-1}. \tag{25}$$

In the derivation of (24) and (25) we have omitted terms of order δ and higher throughout. It follows from these expressions that $|A_l'/A_l| \sim 0.96$, $|A_r'/A_l| = 0.04$. Thus we see that actually in practically all of the left-polarization is reflected.

Let us summarize briefly the optical properties under normal incidence.

1) At $|k - \alpha| \le \alpha\delta/2$ we have resonance (total reflection).

2) At $|k-\alpha| \gg \alpha\delta/2$ we have rotation of the plane of polarization, given by the formula $\Delta k = k^4/4\alpha(\alpha^2 - k^2)$.

3) At $|k - \alpha| \ge \alpha\delta/2$ we have a standing wave

$$E_x = \cos\omega t \sin\alpha z, \quad E_y = \cos\omega t \cos\alpha z.$$

4) At $k^2\delta^2 \gg \alpha^2$ there are two plane-polarized wave waves, but the polarization vector rotates with a pitch αz.

3. OBLIQUE INCIDENCE OF LIGHT

In the case of oblique incidence of light it is necessary to take into account the dependence of the fields on the transverse coordinates x and y, and also the field components along the z axis. By virtue of the symmetry we can effect a Fourier transformation with respect to the transverse coordinates:

$$E_1(x, y, z) = \int E_1(z)\, e^{i(q_x x + q_y y)}\, dq_x\, dq_y,$$
$$E_2(x, y, z) = \int E_2(z)\, e^{i(q_x x + q_y y)}\, dq_x\, dq_y.$$

In addition, we choose the y axis such as to make it parallel to the vector **q**. Then Eqs. (1) take the form

$$\begin{aligned} &d^2F_1/dz^2 + k^2[(1 - q^2/k^2)F_1 + F_1\delta\cos 2\alpha z + F_2\delta\sin 2\alpha z] = 0,\\ &d^2F_2/dz^2 + k^2(1 - q^2/k_z^2)[F_2 - F_2\delta\cos 2\alpha z + F_1\delta\sin 2\alpha z] = 0. \end{aligned} \tag{26}$$

Here $k_z^2 \equiv c^{-2}\epsilon_3\omega^2$.

The quantity q characterizes the angle of incidence. At q = 0, the light is normally incident and Eqs. (26) naturally go over into (8). In all cholesteric media, ϵ_3 is equal to one of the transverse components of the dielectric tensor. Let, for example, $\epsilon_3 = \epsilon_1$. Then $k_z^2 = (1 + \delta)k^2$, and Eqs. (26) take the form (the choice of $q \| X$ or $\epsilon_3 = \epsilon_2$ is equivalent to a rotation of the axis and reduces to the substitution $F_1 \to F_2$):

$$\begin{aligned} &d^2F_1/dz^2 + k_1^2F_1 + k_1^2\delta(F_1\cos 2\alpha z + F_2\sin 2\alpha z) = 0,\\ &d^2F_2/dz^2 + k_2^2F_2 + k_2\delta(F_1\sin 2\alpha z - F_2\cos 2\alpha z) = 0. \end{aligned} \tag{27}$$

Here

$$k_1^2 = k^2 - q^2/(1-\delta), \quad k_2^2 = k^2 - q^2. \tag{28}$$

The system (27) can no longer be solved exactly. It can be investigated in the resonant region, however, by the methods developed in the theory of parametric resonance,[4] and off resonance by perturbation theory. It should be noted here that there are now two resonant regions (when k_1 or k_2 is close to α), unlike the normal incidence.

A. Resonant region ($k^2 \to \alpha^2$). We seek solutions of (27) in the form

$$F_1 = f_{1+}e^{i\alpha z+isz} + f_{1-}e^{-i\alpha z+isz},$$
$$F_2 = f_{2+}e^{i\alpha z+isz} + f_{2-}e^{-i\alpha z+isz}. \qquad (27')$$

Upon substitution of (28) in (27) we neglect all the "nonresonant exponentials." As a result of the calculations we have the following algebraic system for the determination of s:

$$[k_1^2 - (\alpha+s)^2]f_{1+} + k_1^2\delta f_{1-}/2 - ik_1^2\delta f_{2-}/2 = 0,$$
$$[k_1^2 - (\alpha-s)^2]f_{1-} + k_1^2\delta f_{1+}/2 + ik_1^2\delta f_{2+}/2 = 0, \qquad (29)$$
$$[k_2^2 - (\alpha+s)^2]f_{2+} - k^2\delta f_{2-}/2 - ik^2\delta f_{1-}/2 = 0,$$
$$[k_2^2 - (\alpha-s)^2]f_{2-} - k^2\delta f_{2+}/2 + ik^2\delta f_{1+}/2 = 0.$$

Equating the determinant of this system to zero, we obtain an equation of the eighth degree for s. This equation reduces to two biquadratic equations and can easily be solved. In the general case the resultant formulas are quite cumbersome and difficult to interpret, and will therefore not be given here. At $k_1^2 = \alpha^2$, for example, these roots are

$$s_1^2 = 0, \quad s_2^2 = 4\alpha^2, \quad s_3^2 = 4\alpha^2, \quad s_4^2 = -(\alpha\delta - q^2\delta\sqrt{3}/2\alpha + q\delta\sqrt{5}/2)^2.$$

If a left-polarized light in this region is incident from vacuum at an angle q, then the fraction of the left-hand polarization in the reflected light will be

$$\left|\frac{A_l'}{A_l}\right| = \frac{4Q}{\sqrt{k^2-q^2}}\left(\frac{Q}{\sqrt{k^2-q^2}}+1\right)^{-2},$$

and the fraction of the right-hand polarization

$$\left|\frac{A_r'}{A_l}\right| = \left[\frac{Q}{\sqrt{k^2-q^2}}-1\right]\left[\frac{Q}{\sqrt{k^2-q^2}}+1\right]^{-1}$$

(The incident light was only left-polarized.)

B) Perturbation-theory region. Here it is more convenient to use the circular components of the fields. The equations for them can readily be obtained from (27)

$$\frac{d^2F_+}{dz^2} + \bar{k}^2F_+ + \delta\varkappa^2e^{2i\alpha z}F_- - \delta a^2e^{-2i\alpha z}F_+ - \delta a^2F_- = 0, \qquad (30)$$
$$\frac{d^2F_-}{dz^2} + \bar{k}^2F_- + \delta\varkappa^2e^{-2i\alpha z}F_+ - \delta a^2e^{2i\alpha z}F_- - \delta a^2F_+ = 0.$$

We have introduced here the notation

$$\bar{k}^2 = k^2 - \frac{q^2}{2}\frac{2-\delta}{1-\delta}, \quad \varkappa^2 = k^2 - \frac{q^2}{2(1-\delta)},$$
$$a^2 = \frac{q^2}{2(1-\delta)}, \quad \varkappa^2 = k^2 - a^2.$$

As the zeroth solutions we choose F_{+0} = A exp (ikz) and F_{-0} = B exp (ikz). Then in first order of perturbation theory in δ we have

$$F_+ = Ae^{ikz} + Ce^{i(k+2\alpha)z} + De^{i(k-2\alpha)z},$$
$$F_- = Be^{ikz} + Ee^{i(k+2\alpha)z} + Ge^{i(k-2\alpha)z}, \qquad (31)$$

where, as follows from (30),

$$C = B\frac{\delta\varkappa^2}{(k+2\alpha)^2-k^2}, \quad D = -A\frac{\delta a^2}{(k-2\alpha)^2-k^2}, \qquad (32)$$
$$E = -B\frac{\delta a^2}{(k+2\alpha)^2-k^2}, \quad G = A\frac{\delta\varkappa^2}{(k-2\alpha)^2-k^2}.$$

In turn, substituting (31) and (32) in (30), we obtain a homogeneous system for the determination of A and B:

$$A\left\{\bar{k}^2 - k^2 + \frac{\delta^2\varkappa^4}{(k-2\alpha)^2-k^2}\right\} + B\left\{-\delta a^2 - \frac{\delta^2a^2\varkappa^2}{(k+2\alpha)^2-k^2}\right\} = 0,$$
$$A\left\{-\delta a^2 - \frac{\delta^2a^2\varkappa^2}{(k-2\alpha)^2-k^2}\right\} + B\left\{\bar{k}^2 - k^2 + \frac{\delta^2\varkappa^4}{(k+2\alpha)^2-k^2}\right\} = 0.$$

Its determinant, as can readily be verified, is equal to zero. The quantity A/B characterizes the ellipticity of the wave

$$\frac{F_+}{F_-} = \frac{A}{B} = \frac{2a^2}{\delta\varkappa^2(D_- - D_+) \pm [\delta^2\varkappa^4(D_+ + D_-)^2 + 4a^4]^{1/2}}, \qquad (33)$$
$$D_+ = \frac{\varkappa^2}{(k+2\alpha)^2-k^2}, \quad D_- = \frac{\varkappa^2}{(k-2\alpha)^2-k^2}.$$

The two signs mean that there are two waves. We note here also that all the formulas of perturbation theory are valid when the denominators $(k \pm 2\alpha)^2 - k^2$ are not small, i.e., far from resonance.

Since $\delta \ll 1$, we get from (33) $F_+/F_- = \pm 1$, i.e., there are two plane-polarized waves in this region.

The formulas for the reflection coefficient can be obtained in this region in analogy with the preceding. We present the answer only for the reflection coefficient of the "amplitude" along the X axis:

$$R_x = |\mathscr{A}/\mathscr{B}|,$$
$$\mathscr{A} = A_+\left\{1 - \frac{kc}{\omega}\right\} + C_+\left\{1 - (k+2\alpha)\frac{c}{\omega}\right\} + D_+'\left\{1 - (k-2\alpha)\frac{c}{\omega}\right\},$$
$$\mathscr{B} = A_+\left\{1 + \frac{kc}{\omega}\right\} + C_+\left\{1 + (k+2\alpha)\frac{c}{\omega}\right\} + D_+'\left\{1 + (k-2\alpha)\frac{c}{\omega}\right\},$$

where

$$A_+ = {}^1/_2(A+iB), \quad C_+ = {}^1/_2(C+iE), \quad D_+' = {}^1/_2(D+iG).$$

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