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CONTRIBUTION TO THE EVOLUTION OF HOMOGENEOUS COSMOLOGICAL MODELS

I. M. KHALATNIKOV and V. L. POKROVSKIĬ

L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences

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We investigate the evolution of homogeneous metrics of type VIII and IX. For a long era, when two of the metric coefficients oscillate and one decreases in time, we obtain an analytic solution of Einstein's equations; this solution is valid for arbitrary oscillation amplitudes.

THE general solution of the gravitation equation near a singularity in time was investigated in[1,2]. A characteristic property of the metric is its oscillatory behavior, which is quite similar to that revealed by the metric in the homogeneous models of type VIII and IX after Bianchi[3,4]. The evolution of the metric in these cases consists of long periods (or eras), during the course of which two out of the three metric coefficients oscillate against the background of a smooth decrease. In the interval between the long periods, the metric has a Kasner character. The behavior of the metric during a long era, for the case of small-amplitude oscillations, was investigated in[3].

In the case of large oscillations, the solution was obtained by the "joining-together" method[5], and in each section between the minimum and the maximum the metric was approximated by a Kasner solution.

In the present paper we obtain an analytic expression for the metric of type IX or VIII; this solution is valid for a long era at arbitrary oscillation amplitudes. In limiting cases, the obtained solution gives the results of[3,5].

We write the homogeneous metric in a synchronous reference frame in the form

$$-ds^2 = -dt^2 + (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\, dx^\alpha dx^\beta, \tag{1}$$

where a, b, and c depend only on the time t, and the vectors l, m, and n depend only on the spatial coordinates x^α. These vectors are subject to the conditions

$$\begin{array}{lll} \mathbf{l}\,\mathrm{rot}\,\mathbf{l} = \lambda, & \mathbf{m}\,\mathrm{rot}\,\mathbf{l} = 0, & \mathbf{n}\,\mathrm{rot}\,\mathbf{l} = 0, \\ \mathbf{l}\,\mathrm{rot}\,\mathbf{m} = 0, & \mathbf{m}\,\mathrm{rot}\,\mathbf{m} = \mu, & \mathbf{n}\,\mathrm{rot}\,\mathbf{m} = 0, \\ \mathbf{l}\,\mathrm{rot}\,\mathbf{n} = 0, & \mathbf{m}\,\mathrm{rot}\,\mathbf{n} = 0, & \mathbf{n}\,\mathrm{rot}\,\mathbf{n} = \nu, \end{array} \tag{2}$$

$$\mathbf{l}[\mathbf{m}\times\mathbf{n}] = 1, \quad \lambda,\ \mu,\ \nu = \mathrm{const.}$$

The solution of (2) is given in the appendix of[3].

The Einstein equations in empty space $R_{ik} = 0$ reduce to ordinary differential equations for the functions a, b, and c. It is convenient to introduce a variable τ by means of the relation

$$d\tau = dt / abc. \tag{3}$$

The system of equations for a, b, and c is

$$\begin{aligned} (\ln a^2)_{\tau\tau} &= (\mu b^2 - \nu c^2)^2 - \lambda^2 a^4, \\ (\ln b^2)_{\tau\tau} &= (\nu c^2 - \lambda a^2)^2 - \mu^2 b^4, \\ (\ln c^2)_{\tau\tau} &= (\lambda a^2 - \mu b^2)^2 - \nu^2 c^4; \end{aligned} \tag{4}$$

$$(\ln a^2)_\tau (\ln b^2)_\tau + (\ln b^2)_\tau (\ln c^2)_\tau + (\ln c^2)_\tau (\ln a^2)_\tau = \lambda^2 a^4 + \mu^2 b^4 + \nu^2 c^4 - 2\lambda\mu a^2 b^2 - 2\mu\nu b^2 c^2 - 2\nu\lambda c^2 a^2. \tag{5}$$

Equation (5) constitutes in essence one of the first integrals of Eqs. (4).

Long eras are characterized by the fact that one of the functions, say c, become small compared with the other two. Neglecting the value of c in the first two equations of the system (4), and introducing the notation

$$a^2 = e^\chi, \quad b^2 = e^\varphi, \quad c^2 = e^\psi, \tag{6}$$

we obtain from (4) and (5)

$$(\chi + \varphi)_{\tau\tau} = 0, \tag{7}$$

$$(\chi - \varphi)_{\tau\tau} = 2\lambda^2 (e^{2\varphi} - e^{2\chi}), \tag{8}$$

$$\psi_\tau (\varphi_\tau + \chi_\tau) = -\varphi_\tau \chi_\tau + \lambda^2 (e^\chi \mp e^\varphi). \tag{9}$$

Here we put $|\lambda| = |\mu| = |\nu|$ without loss of generality. The minus sign in (9) pertains to the metric of type IX and the plus sign to the metric of type VIII.

From (7) it is obvious that $\chi + \varphi$ is a linear function of τ, which we write in the form

$$\chi + \varphi = p(\tau - \tau_0), \tag{10}$$

where we assume, for concreteness, that the constant $p > 0$. We introduce the variable ξ in the following manner:

$$\xi = \frac{4\lambda}{p} e^{1/2\, p(\tau - \tau_0)}. \tag{11}$$

Then we obtain for the function $q = \chi - \varphi$ the equations

$$q_{\xi\xi} + \frac{1}{\xi} q_\xi + \mathrm{sh}\, q = 0. \tag{12}$$

The equation for ψ is

$$\psi_\xi = -\frac{1}{2\xi} + \frac{1}{8}\xi (q_\xi^2 + 2\,\mathrm{ch}\, q \mp 2). \tag{13}$$

The metric coefficients a^2 and b^2 of interest to us are expressed in terms of the function q as follows:

$$a^2 = \frac{p\xi}{4\lambda} e^{q/2}; \quad b^2 = \frac{p\xi}{4\lambda} e^{-q/2}. \tag{14}$$

Equation (12) was first derived in[3], where it was investigated for the case of small q. During a long era, ξ changes from large values to zero. We investigate the case of large ξ without limitations on the value of q. In this case we employ for the solution of (12) a method similar to the WKB method.

We consider first the equation

$$q_{\xi\xi} + \mathrm{sh}\, q = 0. \tag{15}$$

Its solution q_0 is determined by the relation

$$\mathrm{ch}\, q_0 = 1 + \frac{2k^2}{k'^2} \mathrm{cn}^2 u, \quad u = \frac{\xi}{k'} + \alpha. \tag{16}$$

Here $k \leq 1$ and α are two constants that determine the solution, $k'^2 = 1 - k^2$, cn u is the elliptic Jacobi function (see, for example,[6]). We shall seek the solution of (12) by the method of varying the constants

$$q = q_0(\xi, k, \alpha), \tag{17}$$

With k and α assumed to be functions of ξ. At large ξ, these functions will be slow. We stipulate satisfaction of the relation

$$dq / d\xi = q_{0\xi}. \tag{18}$$

This requirement, together with Eqs. (12), leads to a system of equations for k and α:

$$q_{0\alpha}\frac{d\alpha}{d\xi} + q_{0k}\frac{dk}{d\xi} = 0, \quad q_{0\alpha\xi}\frac{d\alpha}{d\xi} + q_{0k\xi}\frac{dk}{d\xi} = -\frac{1}{\xi}q_{0\xi}, \tag{19}$$

which can be solved with respect to the derivatives

$$\frac{d\alpha}{d\xi} = -\frac{q_{0\xi}q_{0k}}{\xi\Delta}, \tag{20}$$

$$\frac{dk}{d\xi} = \frac{q_{0\xi}q_{0\alpha}}{\xi\Delta}, \tag{21}$$

$$\Delta = q_{0\alpha\xi}q_{0k} - q_{0k\xi}q_{0\alpha} = -\frac{4k}{k'^3}. \tag{22}$$

In calculating Δ,[1)] and also for further calculations, we shall need the explicit form of the derivatives that enter in the formulas

$$q_{0\xi} = -\frac{2k}{k'}\operatorname{cn}u, \quad q_{0\alpha} = -2k\operatorname{sn}u, \tag{23}$$
$$q_{0\alpha\xi} = -\frac{2k}{k'}\operatorname{cn}u\operatorname{dn}u;$$

$$q_{0k} = -\frac{2k^2\xi}{k'^3}\operatorname{sn}u + \frac{2}{k'^2}\frac{\operatorname{cn}u}{\operatorname{dn}u} + \frac{2k}{\operatorname{dn}u}\frac{\partial}{dk}(\operatorname{cn}u),$$
$$q_{0k\xi} = -\frac{2k^2\xi}{k'^4}\operatorname{cn}u\operatorname{dn}u - \frac{2}{k'^3}\operatorname{sn}u - \frac{2k}{k'}\frac{\partial}{\partial k}(\operatorname{cn}u). \tag{24}$$

Substituting formulas (23) and (24) in (20) and (21), we obtain

$$\frac{dk}{d\xi} = -\frac{kk'^2}{\xi}\operatorname{sn}^2 u, \tag{20$'$}$$

$$\frac{d\alpha}{d\xi} = \frac{k^2}{k'}\operatorname{sn}^2 u - \frac{1}{\xi}\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u} - \frac{kk'^2}{\xi}\frac{\operatorname{sn}u}{\operatorname{dn}u}\frac{\partial}{\partial k}(\operatorname{sn}u). \tag{21$'$}$$

Equation (20′) and (21′) are exact. In the case of large ξ, they can be solved approximately by using the fact that the period of the oscillations of the functions cn u, sn u, and dn u is much smaller than ξ. It is therefore natural to separate the non-oscillating part from Eqs. (20′) and (21′). In integrating (20′) and (21′), the smooth parts accumulate with increasing ξ, so that the order of the oscillating increments of α and k, compared with the separated smooth function, is $1/\xi$. We thus neglect the dependence of k and α on u and average formulas (20′) and (21′) over the period. We obtain

$$\frac{d\alpha}{d\xi} = \frac{k^2}{k'}\frac{\mathrm{D}}{\mathrm{K}}, \tag{25}$$

$$\frac{dk}{d\xi} = -\frac{kk'^2}{\xi}\frac{\mathrm{D}}{\mathrm{K}}. \tag{26}$$

Here K and D are complete elliptic integrals. Equation (26) can be integrated:

$$\mathrm{D}k^2 / k' = C / \xi \tag{27}$$

(C is the integration constant).

We note that following the averaging indicated above, the second and third terms of (24) give identically zero. Since the first term contains the large factor ξ, the oscillatory corrections to k turn out to be very small.

The oscillatory corrections to α, however, are appreciable, since they change the argument of sn u, cn u, and dn u by an amount on the order of unity when k, $k' \sim 1$. We shall show, however, that in this order the total corrections to the argument u do not contain oscillating terms. In fact, by definition,

$$\frac{du}{d\xi} = \frac{1}{k'} - \frac{\xi}{k'^2}\frac{dk'}{d\xi} + \frac{d\alpha}{d\xi}.$$

Using the equations (20′) and (21′), we therefore obtain

$$\frac{du}{d\xi} = \frac{1}{k'}. \tag{21$''$}$$

Thus, the "frequency" $du/d\xi$ coincides formally with the parameter $1/k'$, just as in the unperturbed solution.

The phase α can vary quite rapidly as a function of ξ (faster than ξ/k' if C is large). Using formulas (14) and (16), we obtain final expressions for the coefficients a^2 and b^2:

$$a^2 = \frac{p\xi}{4\lambda k'}(\operatorname{dn}u + k\operatorname{cn}u),$$
$$b^2 = \frac{p\xi}{4\lambda k'}(\operatorname{dn}u - k\operatorname{cn}u), \tag{28}$$
$$u = \xi/k' + \alpha,$$

where k and k' are determined by (27), and α by (25).

It remains to obtain an expression for the third coefficient $c^2 = e^{\psi}$. Equation (13), which defines the function ψ, can be solved in quadratures:

$$\psi = -\frac{1}{2}\ln\xi - \frac{C^2}{2}\int\frac{\mathrm{K}\,dk}{k'^2k^3\mathrm{D}^3} \quad \text{(type IX)}. \tag{29}$$

$$\psi = -\frac{1}{2}\ln\xi - \frac{C^2}{2}\int\frac{\mathrm{K}\,dk}{k'^2k^5\mathrm{D}^3} \quad \text{(type VII)}. \tag{30}$$

In the derivation of (29) and (30) we used the relations (16), (23), (26), and (27).

Formulas (28)—(30) solve our problem completely at large values of ξ. Let us investigate the limiting cases. We consider first the case $C/\xi \ll 1$. From (27) we obtain in this case

$$k = (4C / \pi\xi)^{1/2}, \; k' \approx 1; \tag{31}$$

the elliptic functions sn u, cn u, and dn u go over respectively into sin u, cos u, and 1. From (25) we obtain the phase

$$\alpha \approx (2C / \pi)\ln\xi + \alpha_0. \tag{32}$$

Equation (29) and (30) reduce in this case to

$$\psi = \frac{2C}{\pi}\xi + \left(\frac{C^2}{\pi^2} - \frac{1}{2}\right)\ln\xi + \text{const} \quad \text{(type IX)}, \tag{33}$$

$$\psi = \frac{\xi^2}{4} + \frac{2C}{\pi}\xi - \left(\frac{1}{2} + \frac{7C^2}{\pi^2}\right)\ln\xi + \text{const} \quad \text{type VIII)}. \tag{34}$$

Thus, for small-amplitude oscillations, the formulas that give the metric coefficients at large ξ take the form

1)The fact that Δ is independent of ξ is no accident, but results from the form of Eq. (15); $q\xi\xi + f(q) = 0$. Differentiating this equation with respect to the parameters α and k, multiplying the resulting equations by q_k and q_α, respectively, and subtracting one from the other, we get $\partial\Delta/\partial\xi = 0$.

$$a^2 = \frac{p\xi}{4\lambda}\left[1 + \sqrt{\frac{4C}{\pi\xi}}\cos\left(\xi + \frac{2C}{\pi}\ln\xi + \alpha_0\right)\right], \quad (35)$$

$$b^2 = \frac{p\xi}{4\lambda}\left[1 - \sqrt{\frac{4C}{\pi\xi}}\cos\left(\xi + \frac{2C}{\pi}\ln\xi + \alpha_0\right)\right], \quad (36)$$

$$c^2 = c_0{}^2\xi^{C^2\pi^2 - 1/2}e^{2C\xi/\pi} \quad \text{(type IX)}, \quad 37)$$

$$c^2 = c_0{}^2\xi^{-(1/2 + 7C^2/\pi^2)}\exp\{2C\xi/\pi + {}^1/_4\xi^2\} \quad \text{(type VIII)}. \quad (38)$$

Formulas (35)—(37) coincide in the assumed approximation with the results obtained in[3], and formula (38) was obtained earlier in[4].

A more interesting case is that of large-amplitude oscillations, $C \gg \xi$. In this case it follows from (27) that k is close to unity, and k' is close to zero. The approximate solution of (27) and (25) is

$$k' = \frac{\xi}{C}\ln\frac{C}{\xi}, \quad (39)$$

$$\alpha = \alpha_0 - C\ln\ln\frac{C}{\xi}. \quad (40)$$

The period $4K$ of the Jacobi elliptic functions behaves like $4\ln(C/\xi)$. Therefore the period in terms of the argument ξ is approximately equal to $4\xi L^2/C$, where L stands for $\ln(C/\xi)$. We emphasize that this period is small compared with ξ, thus confirming the correctness of the assumed approximation.

Using the properties of the elliptic functions at small k', we can reduce the expressions for a^2, b^2, and c^2 to the form

$$a^2 = \frac{p\xi}{2\lambda k'}\left[\theta\left(\frac{u}{2K}\right)\operatorname{dn}u + \left(1 - \theta\left(\frac{u}{2K}\right)\right)\frac{k'^2}{4\operatorname{dn}u}\right], \quad (41)$$

$$b^2 = \frac{p\xi}{2\lambda k'}\left[\left(1 - \theta\left(\frac{u}{2K}\right)\right)\operatorname{dn}u + \theta\left(\frac{u}{2K}\right)\frac{k'^2}{4\operatorname{dn}u}\right], \quad (42)$$

where

$$c^2 = c_0{}^2\xi^{-1/2}\exp\{C^2/2\ln(C/\xi)\}, \quad (43)$$

$$\theta(x) = {}^1/_2(1 + (-1)^{[x + 1/2]}), \quad (44)$$

$$K = \ln\frac{1}{k'}; \quad u = \frac{C}{\ln(C/\xi)} - C\ln\ln(C/\xi) + \alpha_0. \quad (45)$$

The square brackets in (44) denote the integer part of the number. We present an approximate parametric expression for the function dn u at small k':

$$\operatorname{dn}u \approx \sqrt{\cos^2\varphi + k'^2},$$
$$u = \left(1 + \left[\frac{\varphi}{\pi}\right]\right)K - \operatorname{Arsh}\left[\frac{2}{k'}\operatorname{tg}\left(\frac{\pi}{4} - \frac{\varphi}{2} + \left[\frac{\varphi}{\pi}\right]\frac{\pi}{2}\right)\right]. \quad (46)$$

We note that formula (43) for c^2 is equally valid for metrics of type VIII and IX in the approximation of large C/ξ.

From (46) in the interval $0 < \varphi\ \pi/2$ it follows that $\operatorname{dn}u = 1/\operatorname{ch}u$ (see also[7]), and formulas (41) and (42) take the form

$$a^2 = \frac{pC}{2\lambda\ln(C/\xi)}\frac{1}{\operatorname{ch}u}, \quad (47)$$

$$b^2 = \frac{p\xi^2\ln(C/\xi)}{8\lambda C}\operatorname{ch}u, \quad (48)$$

whereby they describe the change of the metric coefficients in one half-period (cf.[3], formulas (2.12) and (2.13)).

In the case of extremely large C, when $\ln C \gg \ln\xi$, we can expand in (47), (48), and (43) in terms of the small quantity $\ln\xi/\ln C$. We thus obtain the asymptotic formulas

$$a^2 \sim \xi^{-C/\ln C}, \quad b^2 \sim \xi^{C/\ln C}, \quad c^2 \sim \xi^{1/2(C/\ln C)^2}. \quad (49)$$

We have obtained a Kasner solution with a set of indices $p_1(s)$, $p_2(s)$, and $p_3(s)$[3], corresponding to the Kasner parameter

$$s = C/2\ln C. \quad (50)$$

We can trace further that on going over to the next quarter of the period, the functions a and b change places, as it were, and the corresponding Kasner index s decreases by unity (see formulas (41) and (42)).

We have thus succeeded in solving completely the problem of the behavior of matrics of type IX and VIII during a long era at large values of the variable ξ. At the end of the long era, when $\xi \ll 1$, the term sinh q in (12) becomes negligibly small, and the solution is described by power-law asymptotic formulas of the Kasner type, as was demonstrated in[3].

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205