CONTRIBUTION TO THE THEORY OF A GAS RING LASER IN A MAGNETIC FIELD

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Submitted April 25, 1970

Zh. Eksp. Teor. Fiz. 59, 1634–1644 (November, 1970)

The equations for generation of a single-mode ring laser in a constant longitudinal magnetic field are derived on the basis of the semiclassical theory. It is shown that, depending on the values of the total moments of the generating levels and on the values of the resonator frequency mismatch, there can exist stable four-wave, two-wave, and single-wave traveling-wave generation regimes. In a non-rotating isotropic ring laser, two four-wave regimes are possible with different polarizations of the total radiation field in the resonator. In an He^3-Ne^{20} laser ($\lambda = 0.63\,\mu$), in a certain range of mismatches, depending on the prior history, the polarization established is either linear or periodic along the laser axis (with period λ). For the case when generation is realized on two opposing waves of equal circular polarization, expressions are obtained for the total intensity and for the frequency of the beats between the waves. The total intensity depends little on the magnetic field **H**. The beat frequency $\Delta\omega$ varies linearly with increasing **H**, with a small proportionality coefficient ($\Delta\omega \sim 10^{-3}$ μgH for an He^3-Ne^{20} laser).

1. INTRODUCTION

THE behavior of standing-wave gas lasers (SWGL) in a magnetic field has been sufficiently well investigated both theoretically and experimentally (see, for example, [1-4]). The same cannot be said, however, of traveling-wave gas lasers (TWGL). This situation can apparently be attributed to the fact that the customarily employed TWGL have a number of features, compared with SWGL, which greatly hinder the experimental investigation of their properties in a magnetic field. Such features, include, first, the presence in the customarily employed TWGL of preferred directions of the polarization plane, i.e., the presence of linear polarization anisotropy, resulting from the oblique incidence of the laser beam on the mirrors of the ring resonator.[5] There are, however, reports (see, for example, [6-7]) that it is possible to produce TWGL with a small linear polarization anisotropy; this opens practical possibilities for a comprehensive study of TWGL generation in a magnetic field.

We assume in this paper that the ring resonator is isotropic, i.e., it has neither amplitude nor phase linear polarization anisotropy. Pressure effects are disregarded. The equations of single-mode generation of a gas laser in a longitudinal magnetic field (i.e., a field directed along the tube axis) are derived under such approximations in Sec. 2.

The physical picture of TWGL generation is much more detailed than that of SWGL. In the TWGL, depending on the type of working transition and on the resonator mismatch, there can exist four-wave, two-wave, and single-wave regimes of the traveling waves. In a non-rotating TWGL in a zero magnetic field, the four traveling waves add up, just as in the SWGL, to form two standing waves with right-hand and left-hand circular polarizations. In the SWGL, however, the nodes (antinodes) of the two waves must coincide by virtue of the requirement that the field vanish on the mirrors, whereas in the TWGI the nodes (antinodes) of the standing wave with one circular polarization may not coincide with the nodes (antinodes) of the wave with the other polarization. The relative phase difference $\Psi/2$ between the standing waves $\sim \cos kz$ and $\sim \cos(kz + \Psi/2)$ is given by the conditions that determine the stability of the generation regime (in the SWGL we always have $\Psi = 0$).

The theory developed in Sec. 3 yields the following results. There exist two four-wave regimes: (a) two standing waves shifted in phase by $\pi/2$; (b) two standing waves with zero phase difference. In case (a) the polarization of the total radiation field in the resonator varies periodically along the resonator axis, with a period equal to the generation wavelength, while regime (b) coincides with the SWGL regime. The region of stability of regime (b) is part of the stability region of regime (a). In the overlap region, the choice of the generation regime is determined by the prior history (hysteresis). The instability regions of the four-wave regimes overlap considerably the stability regions of the regimes with two traveling waves. Two types of stable two-wave regimes are possible: (1) generation is produced by two opposing waves of different "origin" (with identical circular polarization); (2) generation is produced by two opposing waves of the same "origin" (one circularly-polarized standing wave).

In Sec. 4 we consider a case of practical interest, that of a rotating ring laser, when generation is produced on two opposing waves of equal circular polarization. The beat frequency of the opposing waves $\Delta\omega$ depends linearly on the angular velocity of rotation and on the magnetic field. The proportionality coefficient between $\Delta\omega$ and μgH is of the order of the ratio of the resonator line width to the Doppler width of the amplification contour. Since different atomic transitions σ_+ and σ_- contribute to the generation of the opposing waves, the nonlinear interaction between the waves occurs only via the atoms at the Zeeman sublevels common to the transitions σ_+ and σ_-, and consequently it decreases strongly compared with the usual case of interaction of opposing traveling waves of linear polarization in a ring laser with booster windows.

2. EQUATIONS OF TWGL GENERATION IN A MAGNETIC FIELD

1. In the single-mode regime, the field $\mathbf{E}$ in the resonator can be represented in the form

$$\mathbf{E}(z,t)=\sum_{s=\pm 1}\sum_{r=n,\,m}\{(-1)^s\hat{e}_{-s}E_{rs}(t)\exp(-i[\omega_{rs}t-k_{rs}z])+\mathrm{c.c.}\},\quad (1)$$

where

$$\hat{e}_{\pm 1}=\mp\frac{1}{\sqrt{2}}(\hat{e}_x\pm i\hat{e}_y),\quad \hat{e}_0=\hat{e}_z;$$

$E_{rs}(t)$ are the circular components of the complex field amplitudes, which are slowly varying functions of the time; the index $s = \pm 1$ indicates the "origin" of the wave, i.e., the transition σ_+ ($s = +1$) or σ_- ($s = -1$) that contributes to the given wave ($E_{r0} = 0$, since the field in the resonator is assumed transverse); the index $r = n, m$ labels the wave propagation direction (E_{ns} propagates in the $+z$ direction, which coincides with the magnetic-field direction); $k_{ns} = 2\pi q/L_{ns}$, $k_{ms} = -2\pi q/L_{ms}$, q being a large integer ($q > 0$); L_{rs} is the length of the ring resonator for the wave E_{rs}. The opposing-wave generation frequencies ω_{ns} and ω_{ms} can be different, for example, as a result of rotation of the ring resonator.

We shall assume that the Lande g factors of the upper and lower levels of the working transition are equal. Then the magnetic field H, which is directed along the z axis, splits the amplification contour of the active medium into two contours corresponding to σ_+ and σ_- transitions with central frequencies $\omega_{0\pm 1} = \omega_0 \pm \Omega$, where ω_0 is the frequency of the atomic transition at $H = 0$, and $\Omega = \mu g H$ is the frequency difference between the Zeeman sublevels (μ is the Bohr magneton). Owing to the known effect whereby the generation frequencies are pulled towards the center of the amplification contour, the frequencies ω_{rs} and ω_{r-s} differ somewhat, and the electromagnetic field in the TWGL is a superposition of four traveling waves E_{rs} ($r = n, m$; $s = \pm 1$).

2. The field (1) in the TWGL satisfies the following wave equation:

$$\frac{\partial^2\mathbf{E}(z,t)}{\partial t^2}+\frac{\omega}{\hat{Q}}\frac{\partial\mathbf{E}(z,t)}{\partial t}-\frac{c^2}{\varepsilon_0}\frac{\partial^2\mathbf{E}(z,t)}{\partial z^2}=-\frac{4\pi}{\varepsilon_0}\frac{\partial^2\mathbf{P}(z,t)}{\partial t^2}.\quad (2)$$

Here $\hat{Q}$ is the resonator Q-factor tensor (it is assumed diagonal in circular coordinates); ϵ_0 is the dielectric constant of the medium without a preferred transition; $\mathbf{P}(z, t)$ is the macroscopic polarization induced in the medium by the field (1).

From the wave equation (2), in the customary approximations, it is easy to obtain truncated equations that relate the circular components $E_{rs}(t) = |E_{rs}(t)| \times \exp[-i\varphi_{rs}(t)]$ of the slowly-varying complex amplitudes of the field and those of the macroscopic polarization, $P_{rs}(t)$ (see, for example [3]). Substituting in the truncated equations the expression for the component $P_{rs}(t)$, calculated in third order in the field (1) by a procedure similar to that used, say, in [2], we obtain for the complex amplitudes $E_{rs}(t)$ the following system of four nonlinear equations:

$$\frac{dE_{rs}}{dt}=E_{rs}\{\alpha_{rs}+i(\omega_{rs}-\nu_{rs}-\sigma_{rs})-\sum_{s'=\pm s}[(\beta_{rsrs'}+i\rho_{rsrs'})|E_{rs'}|^2$$
$$+(\theta_{rsr's'}+i\tau_{rsr's'})|E_{r's'}|^2]\}-$$
$$-(\xi_{rsr's}+i\vartheta_{rsr's})E_{r-s}E_{r's}E_{r'-s}\exp(i\Phi_{rs});\quad (3)$$
$$r,\ r'=n,\ m,\quad r\neq r',\quad s=\pm 1,$$

where

$$\Phi_{rs}=(\omega_{rs}-\omega_{r-s})t-(\omega_{r's}-\omega_{r'-s})t,$$

and $\nu_{rs} = c|k_{rs}|/\sqrt{\epsilon_0}$ is the resonator frequency for the wave E_{rs}.

We present the coefficients of the system (3) for the case when $\eta = \gamma_{ab}/ku \ll 1$ (inhomogeneous-broadening model) and

$$\left|\frac{\omega_{rs}-\omega_0-s\Omega}{k_{rs}u}\right|\ll 1,$$

neglecting quantities of order η^2 in the nonlinear coefficients:

$$\alpha_{rs}=\frac{\delta\omega_{rs}}{2}\left\{\frac{N_0}{N_{\mathrm{thr}}^{rs}}(1-\eta^2 f^2{}_{rs})-1\right\},\quad \sigma_{rs}=\frac{\delta\omega_{rs}N_0\eta f_{rs}}{\sqrt{\pi}\,N_{\mathrm{thr}}^{rs}};$$
$$\beta_{rsrs}=\frac{4\pi^{3/2}\omega N_0|(j_a\|d\|j_b)|^4 G}{\varepsilon_0\hbar^3|k_{rs}|u\gamma_a\gamma_b}\equiv\beta,\quad \rho_{rsrs}=O(\eta^2);$$
$$\beta_{rsr-s}+i\rho_{rsr-s}=2\beta\left\{\frac{K''\gamma_a}{\gamma_a+2is\Omega}+\frac{K'\gamma_b}{\gamma_b+2is\Omega}\right\};\quad (4)$$
$$\theta_{rsr's}+i\tau_{rsr's}=\frac{\beta}{1-if_{rsr's}};\quad \theta_{rsr'-s}+i\tau_{rsr'-s}=\frac{\beta K_1}{1-if_{rsr'-s}};$$
$$\xi_{rsr's}+i\vartheta_{rsr's}=\frac{\beta\varkappa\gamma_{ab}}{1-if_{rsr's}}\left\{\frac{K''}{\gamma_a+2is\Omega}+\frac{K'}{\gamma_b+2is\Omega}\right\},$$

where $\delta\omega_{rs} = \omega/Q_{rs}$ is the resonator line width for the wave E_{rs}; N_0 is the population difference in the absence of generation;

$$N_{\mathrm{thr}}^{rs}=3\hbar|k_{rs}|u\varepsilon_0/4\pi^{3/2}Q_{rs}|(j_a\|d\|j_b)|^2;$$

u is the average thermal velocity of the atom; $(j_a\|d\|j_b)$ is the reduced dipole-moment matrix element; γ_a and γ_b are the radiative widths of the upper and lower levels:

$$\gamma_{ab}={}^1\!/_2(\gamma_a+\gamma_b),\quad \varkappa=\gamma_a\gamma_b/\gamma_{ab}^2,$$
$$f_{rs}=(\omega_{rs}-\omega_0-s\Omega)\gamma_{ab}^{-1},\quad f_{rsr's'}={}^1\!/_2(f_{rs}+f_{r's'}),$$
$$K_1=\frac{K'(j_a,j_b)\gamma_b+K''(j_a,j_b)\gamma_a}{\gamma_{ab}};$$

$G(j_a, j_b)$, $K'(j_a, j_b)$ and $K''(j_a, j_b)$ are functions of the total angular momenta j_a and j_b of the upper and lower levels,

$$G(j,j+1)=G(j+1,j)=\frac{6j^2+12j+5}{15(j+1)(2j+1)(2j+3)},$$
$$G(j,j)=\frac{2j^2+2j+1}{15j(j+1)(2j+1)},$$
$$K'(j,j+1)=K''(j+1,j)=\frac{(j+2)(2j+5)}{4(6j^2+12j+5)},\quad (5)$$
$$K''(j,j+1)=K'(j+1,j)=\frac{j(2j-1)}{4(6j^2+12j+5)},$$
$$K'(j,j)=K''(j,j)=\frac{(2j-1)(2j+3)}{4(2j^2+2j+1)}.$$

We shall henceforth be interested in stable stationary solutions of the system (3). In the investigation of the stability of four-wave regimes, it is convenient to change over from the system of equations (3) for the complex amplitudes to a system of five amplitude-phase equations for the four intensities $I_{rs} = |E_{rs}|^2$ and for the relative phase

$$\Psi=(\omega_{n1}-\omega_{n-1})t-(\omega_{m1}-\omega_{m-1})t+(\varphi_{n1}-\varphi_{n-1})-(\varphi_{m1}-\varphi_{m-1}).\quad (6)$$

These equations are more general than the equations for the field in the SWGL. To change over to the latter it is necessary, besides satisfying the obvious condition that the intensities and frequencies of generation be independent of the wave propagation direction $I_{ns} = I_{ms}$; $\omega_{ns} = \omega_{ms}$), to put $\Psi = 0$ (see Sec. 1).[1)]

3. ISOTROPIC RING LASER

1. Let us consider a non-rotating ring laser with an isotropic resonator. Neglecting the action of such small effects as birefringence of the resonator mirrors, Faraday rotation, etc., we can assume with sufficient accuracy that the resonator frequencies and the circular components of the Q-tensor do not depend on the wave propagation direction or polarization. Then

$$f_{rs} = f_{rsr's} = (\omega_s - \omega_0 - s\Omega)\gamma_{ab}^{-1} \equiv f_s,$$
$$f_{rsr'-s} = [{}^1/_2\,(\omega_s + \omega_{-s}) - \omega_0]\gamma_{ab}^{-1} \equiv \tilde{f} \qquad (\omega_s \equiv \omega_{rs} = \omega_{r's}) \tag{7}$$

and all the coefficients (4) of the system (3) for the complex amplitudes do not depend on the indices r and r′. We shall henceforth omit these indices.

The stationary solutions of the system (3) will be four-wave, two-wave, and single-wave regimes.[2)] Which of the possible stationary regimes is realized for specified TWGL operating conditions is determined by the stability conditions, and hysteresis takes place in the case when there are several stable regimes.

2. There are two stationary four-wave solutions of the system (3):

$$\text{a) } I_{rs} = I_{r's} \equiv I_s, \quad \Psi = \pi;$$
$$\text{b) } I_{rs} = I_{r's} \equiv I_s, \quad \Psi = 0 \quad (s = \pm 1),$$

where

$$I_s = \frac{a_s b_{-s-s} - a_{-s} b_{s-s}}{b_{ss} b_{-s-s} - b_{s-s} b_{-ss}}; \tag{8}$$

$$b_{ss} = \beta_{ss} + \theta_{ss}, \qquad b_{s-s} = \beta_{s-s} + \theta_{s-s} + \xi_{ss}\cos\Psi. \tag{9}$$

The coefficients b_{s-s} (and, consequently, also the intensities I_s) are different for regimes (a) and (b). Thus, in a TWGL at rest, the four traveling waves (1) combine into two standing waves with opposite circular polarizations:

$$E(z,\ t) = 2\{e^{i\chi}e_1|E_1|e^{i\omega_1 t}\cos\,(kz - \Theta) +$$
$$+ e^{-i\hat{\chi}}e_{-1}|E_{-1}|e^{i\omega_{-1}s}\cos(kz - \Theta + \Psi/2) + \text{к.с.}\}, \tag{10}$$

where

$$\chi = {}^1/_4\,(\varphi_{n1} - \varphi_{n-1} + \varphi_{m1} - \varphi_{m-1}); \qquad \Theta = {}^1/_2\,(\varphi_{n1} - \varphi_{m1}).$$

When $\Psi = 0$, the spatial dependences of both waves coincide and are determined by the function cos $(kz - \Theta)$.

When $\Psi = 0$, the interaction coefficients $b_{ss'}$ coincide with the corresponding coefficients $b''_{ss'}$ for the SWGL from [2]. A TWGL at rest operates when $\Psi = 0$ in analogy with an SWGL, the only difference being that the value of the field on the mirrors in the ring resonator is not fixed and Θ can be arbitrary, whereas in the SWGL we have $\Theta = \pi/2$ (the nodes of the field are on the mirrors). The phase χ corresponds to a certain rotation relative to the unit vectors $\hat{e}_x$ and $\hat{e}_y$, and determines in the case $\Psi = 0$ the orientation of the axes of the polarization ellipse of the total field in the resonator relative to the unit vectors $\hat{e}_x$ and $\hat{e}_y$. In an ideal isotropic resonator there is no preferred direction, and the phase χ is random. In real resonators, however, there always exists a certain preferred direction. We shall assume that the unit vectors $\hat{e}_x$ and $\hat{e}_y$ are chosen such that $\chi = 0$. In the case when $\Psi = \pi$, the operating regime of the TWGL at rest differs appreciably from the operating regime of the SWGL. Which of the two stationary values of the relative phase, $\Psi = 0$ or $\Psi = \pi$, is realized at specified operating conditions of the TWGL is determined by the stability conditions.

3. An investigation of the stability of the stationary solutions of (3) (when the directions are isotropic) against small perturbations leads to two groups of stability conditions. The first group determines the stable value of the relative phase Ψ and the stability of the solution $I_{ns} - I_{ms} = 0$ $(s = \pm 1)$, at which the intensities of the opposing waves of equal "origin" are equal, i.e., the stability of the standing waves against decay into traveling waves:

$$\sum_{s=\pm1}\{p_{ss} - 2\xi_{-s}\cos\Psi\}I_s > 0,$$

$$\begin{aligned}\sum_{s=\pm1}\{&p_{ss}(\xi_{-s}^2 + \tilde{\vartheta}_{-s}^2)I_s^3 + [p_{ss}\xi_{-s}\xi_s + \tilde{\vartheta}_s(\xi_{-s}t_s \\ &- p_{-ss}\tilde{\vartheta}_{-s})]I_s^2 I_{-s} - [\xi_{-s}(\xi_s\xi_{-s} + p_{ss}p_{-s-s} - p_{s-s}p_{-ss}) \\ &+ \tilde{\vartheta}_{-s}(\xi_s\tilde{\vartheta}_{-s} + p_{ss}t_{-s} - p_{s-s}t_s)]I_s^2 I_{-s}\cos\Psi\} > 0,\end{aligned} \tag{11}$$

$$\begin{aligned}\sum_{s=\pm1}\{&4p_{ss}\xi_{-s}^2 I_s^3 + [\xi_{-s}(3p_{-s-s}\xi_{-s} + 8p_{ss}\xi_s + 2t_{-s}\tilde{\vartheta}_{-s} + t_s\tilde{\vartheta}_s) \\ &+ p_{ss}(p_{ss}p_{-s-s} - p_{s-s}p_{-ss}) + \tilde{\vartheta}_{-s}(p_{-s-s}\tilde{\vartheta}_{-s} + p_{-ss}\tilde{\vartheta}_s)]I_s^2 I_{-s} \\ &- [2\xi_{-s}(\xi_{-s}^2 + p_{ss}^2 + \tilde{\vartheta}_{-s}^2)I_s^3 + [\xi_s(7\xi_{-s}^2 + p_{ss}^2 + \tilde{\vartheta}_{-s}^2) \\ &+ \xi_{-s}(4p_{ss}p_{-s-s} - p_{s-s}p_{-ss}) + t_s(p_{ss}\tilde{\vartheta}_s + p_{s-s}\tilde{\vartheta}_{-s})]I_s^2 I_{-s}]\cos\Psi\} > 0,\end{aligned}$$

where

$$p_{ss'} = \beta_{ss'} - \theta_{ss'}, \qquad \vartheta_s = (-1)^{(1-s)/2}\vartheta_{ss},$$

$$\xi_s = \xi_{ss}, \quad t_s = (-1)^{(1-s)/2}(\rho_{ss} - \rho_{-ss} + \tau_{-ss} - \tau_{ss}).$$

The second group of conditions determines the stability of the regime of two standing waves with respect to competition between them. This is the so called "weak-coupling" condition[3)]

$$b_{11}b_{-1-1} - b_{1-1}b_{-11} > 0. \tag{12}$$

1)No account is taken in (3) of the linear phase coupling between the opposing waves (the back-scattering effect [8]). This is valid if the nonlinear phase relation contained in (3) is much stronger than the linear relation determined by the quantity $2\pi\omega|\kappa|m \approx {}^1\!/_2 m\delta\omega$ in [8], where m is the back-scattering coefficient, i.e., when ${}^1\!/_2 m\delta\omega_{rs} \ll \xi_{rsr's}I_s$. It should be noted that the following weakfield condition should be simultaneously satisfied:

$$I_s \ll \hbar^2\gamma_a\gamma_b / |\langle j_a m_a|d_s|j_b m_b\rangle|^2.$$

2)The system of equations (3) does not have stationary three-wave solutions.

3)In the derivation of the stability conditions (11) and (12), we used only the structure of the amplitude-phase equations obtained from (3) (see the first paragraph following the definitions (5)) and the condition that the directions be isotropic. The concrete form of the coefficients of these equations is immaterial. Conditions (11) and (12) are suitable for the determination of the stability of any fourwave generation regime in an isotropic ring laser (in this case $\Phi_{rs} = 0$).

4. Let us consider the stability of relatively small perturbations of regimes (a) ($\Psi = \pi$) and (b) ($\Psi = 0$) in a zero magnetic field.

For H = 0, it follows from (7) that the mismatch is $f_s = \tilde{f} = (\omega - \omega_0)\gamma_{ab}^{-1} \equiv f$ ($\omega \equiv \omega_s = \omega_{-s}$) and the coefficients (4) are independent not only of the direction indices r and r′, but also of the wave "origin" indices s and s′. Substituting the values of the coefficients (4) in the inequalities (11) and (12) and recognizing that $I_1 = L_{-1}$, $\xi_1 = \xi_{-1}$, $t_1 = -t_{-1}$, $-\tilde{\vartheta}_1 = \tilde{\vartheta}_{-1}$, $p_{11} = p_{-1-1}$, $p_{1-1} = p_{-11}$, we obtain the mismatch regions in which the four-wave regimes are stable.[4)] These regions depend strongly on the quantity

$$K_0(j_a, j_b) = K'(j_a, j_b) + K''(j_a, j_b),$$

which is determined by the relative number of the common Zeeman sublevels of the upper and lower states of the working transition for waves of different "origin."

Let us consider the regions of stability of the regime (a) ($\Psi = \pi$) as functions of the type of working transition ($j_a \to j_b$). In the transitions $\frac{1}{2} \to \frac{1}{2}$, $0 \to 1$, $1 \to 0$ and $1 \to 1$ ($K_0 = 0$, $K_0 = \frac{1}{2}$) the left-hand side of the second inequality of (11) vanishes, corresponding to the state of indifferent equilibrium. For the transitions $j \to j+1$ $j+1 \to j$ at $j < 0$ ($0 < K_0 < \frac{1}{2}$), regime (a) is stable for mismatches satisfying the inequality

$$f^2 > \frac{(2K_0 - K_1)\varkappa}{\varkappa K_0 + 2(2K_0 - K_1)} \equiv x_0,$$

here $\kappa = \gamma_a\gamma_b/\gamma_{ab}^2$.

Finally, for the transitions $j \to j$ with $j > 1$ ($K_0 > \frac{1}{2}$), the stability region of regime (a) is given by the inequality

$$f^2 < \min\left\{\frac{\varkappa}{\varkappa + 2};\ \frac{2(1 - K_0)}{2K_0 - 1}\right\}.$$

Thus, in some mismatch region determined by the type of the working transition (K_0), the stable regime is that of two right-circular-polarized and left-circular-polarized standing waves shifted in phase by $\pi/2$ (the regime (a)). In this case the total field in the resonator is

$$\mathbf{E}(z, t) = 2\sqrt{2}|E_1\{[\cos(kz - \Theta) - \sin(kz - \Theta)]\hat{e}_y \sin\omega t - [\cos(kz - \Theta) + \sin(kz - \Theta)]\hat{e}_x \cos\omega t\}.$$

We see that the polarization of the total field varies periodically along the resonator axis z, with a period $\lambda = 2\pi/k$.

We now proceed to the regime (b) ($\Psi = 0$). For the transitions $\frac{1}{2} \to \frac{1}{2}$, $j \to j+1$, $j+1 \to j$ with $j > 0$ ($K_0 < \frac{1}{2}$), the regime (b) is stable if $f^2 > x_1$, where x_1 is the real positive root of the cubic equation

$$\frac{1 - 4K_0^2}{8(2K_0 - K_1)^3}x^3 - \frac{3 + 2K_0 - 2K_0K_1 + K_1}{4(2K_0 - K_1)^2}x^2 + \frac{5 + 4K_1 - 2K_0}{4(2K_0 - K_1)}x - 1 = 0. \tag{13}$$

In the case of generation on the transitions $0 \to 1$, $1 \to 0$, $j \to j$ with $j > \frac{1}{2}$ ($K_0 \geq \frac{1}{2}$), regime (b) is unstable. Thus, in the case $K_0 < \frac{1}{2}$ there is a mismatch region where, besides the regime (a), the regime (b) is also stable when $\Psi = 0$ and the generation is on a standing linearly-polarized wave.[5)] When $K_0 > \frac{1}{2}$, the interaction between the standing waves via the common Zeeman sublevels increases compared with the case $K_0 < \frac{1}{2}$. In the case of generation in regime (b), this leads to suppression of one of the standing waves (the "weak-coupling" conditions (12) are not fulfilled). In case (a), however, the nonlinear interaction between the standing waves is smaller than in the case (b); this follows already from the form of the coefficients b_{s-s} (see (9)) and the "weak-coupling" conditions are satisfied in the mismatch region $f^2 < 2(1 - K_0) \times (2K_0 - 1)^{-1}$.

If we determine the stability of the four-wave regimes only from the equation for the relative phase, neglecting the fluctuations of the intensity difference between the opposing waves, then conditions (11) reduce to the inequality $\cos\Psi < 0$, from which we find immediately that only the regime (a) ($\Psi = \pi$), at which the maximum power is generated, is stable. An exact analysis, however, shows that hysteresis is possible in some mismatch regions. By way of an example, let us find the region of stability of the regimes (a) and (b) for an He^3-Ne^{20} laser generating on the $3S_2$–$2P_4$ line ($\lambda = 0.6328\ \mu$) at $\gamma_a = 18$ MHz, $\gamma_b = 40$ MHz, $\gamma_{ab} = 29$ MHz, and ku = 1010 MHz.[3] Here $j_a = 1$, $j_b = 2$, $K_0 \approx 0.24$; $K_1 \approx 0.32$, and we obtain $x_0 \approx 0.39$; $x_1 \approx 1.23$.

5. Possible stationary solutions of the system (3) are two-wave regimes (two traveling waves out of four produce the generation).

We investigate here the so-called "external" stability of two-wave regimes, i.e., we find the regions of resonator mismatch where the other two (non-generating) waves, produced in the form of fluctuations, are damped. We consider the case H = 0. The "internal" stability of the two-wave regimes in a TWGL—competition between the opposing waves in the magnetic field—is considered in [9]. There are three types of two-wave solutions:

1) Generation is produced by two opposing waves of equal "origin" E_{ns} and E_{ms}, i.e., $E_{ns} \neq 0$, $E_{ms} \neq 0$, $E_{n-s} = E_{m-s} = 0$, s = +1 or −1. Such a circularly-polarized standing-wave regime will have "external" stability in the case $K_0 > \frac{1}{2}$ (the transitions $j \to j$, $j > 1$) at mismatches f satisfying the following inequality of fourth degree relative to f^2:

$$(1 - 2K_0)^4 f^8 + (2K_0 - 1)^2(25K_0^2 - 28K_0 + 8)f^6 + (2K_0 - 1)(112K_0^3 - 201K_0^2 + 120K_0 - 24)f^4 + (212K_0^4 - 538K_0^3 + 504K_0^2 - 208K_0 + 32)f^2 - 4(2K_0 - 1)(1 - K_0)(2 - 3K_0)^2 > 0. \tag{14}$$

The conditions of "internal" stability of such a regime are satisfied in the region (see [9])

$$f^2 > \varkappa\eta^2.$$

2) Generation is produced by two opposing waves of different "origin" E_{ns} and E_{m-s}, i.e., $E_{ns} \neq 0$, $E_{m-s} \neq 0$, $E_{n-s} = E_{ms} = 0$, s = +1 or −1. This regime with opposing waves of equal circular polarization has external stability in the following cases:

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4)In the second inequality of (11), in the case $\Psi = \pi$ (regime (a)), we used the values of the coefficients (4) accurate to terms of order η^2, since the substitution of these coefficients, accurate to terms of order $O(\eta)$, causes the left side of the inequality to vanish.

5)The stability region of regime (a) is larger than that of regime (b), i.e., $x_1 > x_0$. This can be easily verified by substituting x_0 in (13).

a) $K_0 < \frac{1}{2}$ (the transitions $j \to j+1$, $j+1 \to j$, $j > 0$ and $\frac{1}{2} \to \frac{1}{2}$) if $f^2 < (2K_0 - K_1)/(1 - 2K_0)$;

b) $K_0 \geq \frac{1}{2}$ (the transitions $j \to j$, $j \geq 1$, $1 \to 0$, $0 \to 1$) for all mismatches f. The condition of "internal" stability leads to the following inequality:[9]

$$f^2 > (K_1 - 1) + \eta^2 K_0 K_1. \qquad (14a)$$

3) Generation is produced by two waves propagating in the same direction ($E_{r+1} \neq 0$, $E_{r-1} \neq 0$, $E_{r'+1} = E_{r'-1} = 0$; $r, r' = n, m$; $r \neq r'$). The two-wave regime is unstable against the occurrence of opposing waves. Thus, in a TWGL with an isotropic resonator it is possible, without using non-reciprocal elements, to obtain the two-wave generation regimes (1) and (2) only as a result of nonlinear interaction between the waves. For the He^3-Ne^{20} laser ($\lambda = 0.6328\,\mu$) with the same parameters in the mismatch region $f^2 < 0.31$ as in the preceding example, the stable regime is that of two opposing waves of equal circular polarization (regime (2)). In the He^3-Ne^{20} laser at a wavelength $\lambda = 1.523\,\mu$ (the $1 \to 0$ transition), regime (2) is stable at any mismatch.

6. The stationary single-wave regime $E_{rs} \neq 0$; $E_{r-s} = E_{r's} = E_{r'-s} = 0$ is stable in a small region of mismatches near the center

$$f^2 < K_0 \varkappa \eta^2 - (1 - K_0). \qquad (15)$$

We see that such a regime of unidirectional generation of a wave with circular polarization can generate only in the case $K_0 \approx 1$.

4. TWO-WAVE REGIMES IN A RING LASER

1. Generation in a TWGL with two circularly-polarized opposing waves sets in at definite resonator mismatches even in a ring laser with a resonator in which the directions are isotropic.

Two-wave regimes can also be produced in a rotating TWGL in a magnetic field with the aid of non-reciprocal polarization elements. By introducing into the ring resonator a non-reciprocal element producing a resonator-frequency difference, it is possible to obtain, by the frequency-selection method, generation on two out of four opposing waves. Thus, by inserting in the resonator a quartz plate of definite thickness, cut perpendicular to the optical axis of the crystal, it is possible, if the resonator is suitably tuned, to obtain generation on two opposing waves of different "origin" (E_{ns} and E_{m-s} waves). The other two waves, E_{n-s} and E_{ms}, turn out to be in a region where the effective gain is smaller than the losses.[6)] The beat frequency and the total intensity of the opposing waves are given by the following expressions:

$$\omega_{ns} - \omega_{m-s} = \nu_{ns} - \nu_{m-s} - \frac{N_0}{N^{ns}_{\text{пор}}} \frac{\delta\omega_{ns}}{ku} (\nu_{ns} - \nu_{m-s} - 2s\Omega) - \frac{K_1 (\alpha_{ns} - \alpha_{m-s})(\tilde{\omega} - \omega_0)\gamma_{ab}}{\gamma_{ab}^2 (1 - K_1) + (\tilde{\omega} - \omega_0)^2}, \qquad (16)$$

$$I_{ns} + I_{m-s} = \frac{\alpha_{ns} + \alpha_{m-s}}{\beta} \frac{\gamma_{ab}^2 + (\tilde{\omega} - \omega_0)^2}{\gamma_{ab}^2 (1 + K_1) + (\tilde{\omega} - \omega_0)^2}, \qquad (17)$$

where $s = +1$ or -1, and $\tilde{\omega} = \frac{1}{2}(\omega_{ns} + \omega_{m-s})$.

The beat frequency (16) depends linearly on the magnetic field $\sim\Omega$. The proportionality coefficient is of the order of $\delta\omega_{rs}/ku$. Since different atomic transitions σ_+ and σ_- contribute to the generation of the waves E_{ns} and E_{m-s}, the nonlinear interaction between the waves proceeds only via atoms at the Zeeman sublevels common to the transitions σ_+ and σ_-. Consequently, the nonlinear interaction decreases strongly compared with the interaction of two opposing waves of the same "origin" ($K_1 \leq 1$) or with the case of TWGL with a preferred polarization plane in the absence of a magnetic field.[10] Thus, in the case of the transition $j_a = j_b = \frac{1}{2}$ there are no common sublevels and waves of different "origin" do not interact. In generation of an He^3-Ne^{20} laser ($\lambda = 0.6328\,\mu$) and for the same data as in the preceding examples, we get $K_1 = 0.32$.

The total generation intensity (7) as a function of the resonator mismatch has a minimum at $\tilde{\omega} = \omega_0$ and depends little on the magnetic field. To explain this fact, we can use the concept of "own" and "foreign" dips in the amplification contours.[1] The concept of "own" and "foreign" dips can be generalized naturally to include the case of the TWGL, the difference being only that in the TWGL there is no rigid coupling between the opposing waves, and consequently, the dips for the opposing waves of the same "origin" need not necessarily be symmetrical relative to the centers of the amplification contours. The summary intensity is determined in this case by the total number of atoms in the "own" and "foreign" dips for the opposing waves. When these dips coalesce, the total intensity is minimal. The distance between the "own" and "foreign" dips for opposing waves is independent of the magnetic field and is equal to $2(\tilde{\omega} - \omega_0)$.

2. In the other case, when a Faraday cell is introduced into the resonator, it is possible to produce with the aid of frequency selection generation on two opposing waves of equal "origin" (i.e., of different circular polarization), namely E_{ns} and E_{ms}. The nonlinear interaction between the waves can be described formally in this case by the theory developed in [10-12] for the generation of opposing waves in a TWGL with a preferred polarization plane, with the center ω_0 of the amplification contour replaced by $\omega_{0s} = \omega_0 + s\Omega$, which is the center of the amplification contour in a magnetic field ($s = \pm 1$). The dependence of the center of the amplification contour ω_{0s} on the magnetic field makes it possible, by choosing the value of the magnetic field, to effect unidirectional generation in a narrow region (of the order of γ_{ab}^2/ku) at all generation frequencies of the waves E_{ns} and E_{ms}.[9]

The authors are grateful to V. I. Perel' for a discussion of the results.

6)In the absence of linear anisotropy of the resonator, the discrimination is complete and only waves having the same circular polarization generate (E_{ns} and E_{m-s}). The region of internal stability of such a two-wave regime is given by the inequality (14a) with f replaced by $f_{rsr'-s}$ ($r, r' = n, m$; $s = \pm 1$).

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Translated by J. G. Adashko
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