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HYDRODYNAMIC FLUCTUATIONS NEAR THE CONVECTION THRESHOLD

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We consider hydrodynamic fluctuations near the mechanical-equilibrium stability boundary of a liquid heated from below. We show that when the boundary is approached either from the equilibrium side or from the side of the stationary convective motion, the velocity and temperature fluctuations increase. The correlation time and radius depend on the difference between the Rayleigh number R and its threshold value R_0 like $\tau_c \sim |R_0 - R|^{-1}$ and $r_c \sim |R_0 - R|^{-1/2}$.

THE mechanical equilibrium of a liquid heated from below is stable if the temperature gradient in the liquid does not exceed a certain limiting value[1]. When the critical gradient is exceeded, flow is produced in the liquid and leads to a redistribution of the temperature. It is natural to expect the velocity and temperature fluctuations to reach appreciable values near the convection threshold, and the correlation time of the fluctuations to increase strongly. In fact, the change of stability means that so long as the Rayleigh number R (the dimensionless temperature gradient) does not exceed a critical value R_0, all the small equilibrium perturbations attenuate in time, and when $R > R_0$ there exists at least one growing perturbation that disrupts the stability. Since all types of perturbations are excited in the liquid as a result of the fluctuations, the growth of the fluctuations in the vicinity of R_0 is ensured precisely by this critical perturbation. In this sense one can say that fluctuation perturbations lead to a change in the character of the motion.

Velocity and temperature fluctuations are brought about by different random actions (thermal fluctuations, random jolts, unevenness in the heating, etc.). The conclusion drawn in this article, namely that the fluctuations increase on approaching the conduction threshold, does not depend, however, on the nature of the fluctuations and remains valid for practically all random forces.

The start of convection is the simplest example of hydrodynamic instability, so that there is every reason for assuming that the instability of stationary flows is connected with the increase in the intensity of the hydrodynamic fluctuations when the critical Reynolds numbers are approached. In general, hydrodynamic instability, just as the instability of any stationary state that is not in thermodynamic equilibrium, is analogous to the loss of stability of equilibrium in second-order phase transitions or at the critical point.

1. STABILITY OF EQUILIBRIUM AND OF STATIONARY MOTION IN THE VICINITY OF R_0

Let us consider the stability of the equilibrium of a liquid heated from below. The temperature gradient at equilibrium is $\nabla\vartheta = -A\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}$ is the unit vector directed upward. If the liquid fills a cavity whose height is not too large, then the liquid can be regarded as incompressible. In this case nonstationary small perturbations of the velocity and of the temperature are described by the equations

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla\frac{p}{\rho} + \nu\Delta\mathbf{v} + g\beta T\boldsymbol{\gamma}, \quad \frac{\partial T}{\partial t} = \chi\Delta T + A\boldsymbol{\gamma}\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0 \quad (1)$$

(the notation is the same as used in the book of Landau and Lifshitz[1]). For the coordinate parts (amplitudes) of perturbations that vary in time like $e^{-\lambda t}$, we obtain from this

$$-\lambda\mathbf{v} = -\nabla p + \Delta\mathbf{v} + D\boldsymbol{\gamma}T,$$
$$-\lambda PT = \Delta T + D\boldsymbol{\gamma}\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0. \quad (2)$$

All the quantities here are dimensionless; the units of length, time, velocity, and temperature are respectively the characteristic height of the cavity l, l^2/ν, ν/l, and $\nu l^{-1}(A\nu/g\beta\chi)^{1/2}$. Equations (2) contain the dimensionless parameters $R \equiv D^2 = g\beta A l^4/\nu\chi$ (the Rayleigh number) and $P = \nu/\chi$ (the Prandtl number). The velocity and temperature perturbations vanish on the boundaries of the cavity:

$$\mathbf{v} = 0, \quad T = 0. \quad (3)$$

The boundary-value problem (2)–(3) is self-adjoint; its eigenfunctions satisfy the orthonormalization conditions[2]

$$\int\{\mathbf{v}_m\mathbf{v}_n + PT_mT_n\}\,dV = \delta_{mn}, \quad (4)$$

and the eigenvalues (the decrements λ_n) are real. The spectrum of λ_n is, generally speaking, discrete. If, however, at least one of the dimensions of the cavity is large compared with the others, then the spectrum of the decrements condenses, and under certain conditions it can be regarded as continuous.

So long as $R < R_0$, all the decrements are positive, i.e., the equilibrium perturbations are attenuated monotonically. When R approaches R_0, one of the decrements (with n = 0) tends to zero in accordance with

$$\lambda_0(R) = \frac{R_0 - R}{R_0}\int(\operatorname{rot}\mathbf{v}_0)^2\,dV. \quad (5)$$

(Such a result is obtained in first order of perturbation theory, if the initial approximation is taken to comprise the proper solutions of the problem (2)–(3) at $R = R_0$.)

When $R > R_0$, the perturbation with n = 0 increases, and this leads in final analysis to establishment of a new stationary regime, in which the velocity U of the liquid differs from zero, and the temperature ϑ is no longer a linear function of the vertical coordinate. In the quasilinear approximation, which is valid at a small excess of R over R_0, we obtain

$$\mathbf{U}=\varepsilon\mathbf{v}_0(\mathbf{r}),\quad \vartheta=-DP^{-1}z+\varepsilon T_0(\mathbf{r}),$$
$$\varepsilon^2=\frac{R-R_0}{R_0}\Big[\sum_{n\neq 0}\lambda_n^{-1}H_{n0}^2\Big]^{-1}\int(\operatorname{rot}\mathbf{v}_0)^2\,dV, \tag{6}$$
$$H_{n0}=-H_{0n}=\int\{\mathbf{v}_n(\mathbf{v}_0\nabla)\mathbf{v}_0+PT_n(\mathbf{v}_0\nabla)T_0\}\,dV.$$

Thus, the motion resulting from the instability of the mechanical equilibrium has the structure of a critical perturbation $\mathbf{v}_0(\mathbf{r})$ with an amplitude proportional to the square root of $R - R_0$.[1)]

The investigation of the stability of the obtained motion is analogous to that in the case of equilibrium. It turns out again that the perturbation that attenuates most slowly is of the type $\mathbf{v}_0$, T_0, the decrement of which is now

$$\lambda_0(R)=2\frac{R-R_0}{R_0}\int(\operatorname{rot}\mathbf{v}_0)^2\,dV. \tag{7}$$

2. HYDRODYNAMIC FLUCTUATIONS IN THE CASE OF A DISCRETE DECREMENT SPECTRUM

The fluctuation velocities and temperatures excited in an immobile liquid by random forces are described by the system of inhomogeneous equations

$$\frac{\partial v_\alpha}{\partial t}=-\frac{\partial p}{\partial x_\alpha}+\Delta v_\alpha+D\gamma_\alpha T+\frac{\partial s_{\alpha\beta}}{\partial x_\beta},$$
$$P\frac{\partial T}{\partial t}=\Delta T+D\gamma\mathbf{v}-\operatorname{div}\mathbf{q},\quad \operatorname{div}\mathbf{v}=0, \tag{8}$$

where $s_{\alpha\beta}$ is the "extraneous stress tensor," $\mathbf{q}$ is the vector of "extraneous heat flux"; as shown in[5], if additional terms are introduced in this manner into the equations of hydrodynamics, the fluctuations of $s_{\alpha\beta}$ and $\mathbf{q}$ are not correlated with one another.

Let us consider first the case of δ-correlated extraneous forces. For these we have[2)]

$$\langle s_{\alpha\beta}(\mathbf{r}_1,t_1)s_{\gamma\delta}(\mathbf{r}_2,t_2)\rangle=2\Theta(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})\delta(\mathbf{r}_2-\mathbf{r}_1)\delta(t_2-t_1),$$
$$\langle q_\alpha(\mathbf{r}_1,t_1)q_\beta(\mathbf{r}_2,t_2)\rangle=2\Phi\delta_{\alpha\beta}\delta(\mathbf{r}_2-\mathbf{r}_1)\delta(t_2-t_1),$$
$$\langle q_\alpha(\mathbf{r}_1,t_1)s_{\beta\gamma}(\mathbf{r}_2,t_2)\rangle=0. \tag{9}$$

In particular, for thermal fluctuations we have

$$\Theta=\frac{kT}{\rho\nu^2 l},\quad \Phi=\frac{g\beta kT^2}{\rho c_p\nu^2 Al}, \tag{10}$$

where T is the average temperature of the liquid.

Fluctuations arise not only in the liquid, but also in the solid that bounds it. We shall, however, neglect the fluctuation displacements of the boundaries of the cavity. In addition, we assume as before that the temperature perturbations vanish on the walls of the cavity (the boundaries have infinite thermal conductivity). Allowance for the finite thermal conductivity of the solids surrounding the liquid leads to the same result with inessential changes in the calculations. Thus, the velocity and temperature fluctuations in the liquid will satisfy the boundary conditions (3). Taking this into account, let us expand the solution of the inhomogeneous system (8) in the complete system of eigenfunctions of the homogeneous problem (2)–(3):

$$\{\mathbf{v},T,p\}=\sum_n c_n(t)\{\mathbf{v}_n,T_n,p_n\}. \tag{11}$$

Substituting the expansion (11) in (8) and recognizing that $\mathbf{v}_n$, T_n, and p_n satisfy Eqs. (2), we get

$$\sum_n(\dot c_n+\lambda_n c_n)v_{n\alpha}=\frac{\partial s_{\alpha\beta}}{\partial x_\beta},\quad P\sum_n(\dot c_n+\lambda_n c_n)T_n=-\operatorname{div}\mathbf{q}. \tag{12}$$

We multiply the first equation by $v_{m\alpha}$ and the second by T_m, add them, and integrate over the volume of the cavity. Taking into account the orthogonality conditions (4), we obtain

$$\dot c_n+\lambda_n c_n=\int\Big(v_{n\alpha}\frac{\partial s_{\alpha\beta}}{\partial x_\beta}-T_n\operatorname{div}\mathbf{q}\Big)dV. \tag{13}$$

The integral in the right-hand side is transformed into

$$\oint(v_{n\alpha}s_{\alpha\beta}-T_nq_\beta)df_\beta-\int\Big(s_{\alpha\beta}\frac{\partial v_{n\alpha}}{\partial x_\beta}-\mathbf{q}\nabla T_n\Big)dV.$$

The surface integral vanishes by virtue of the boundary conditions, so that

$$\dot c_n+\lambda_n c_n=\int\Big(\mathbf{q}\nabla T_n-s_{\alpha\beta}\frac{\partial v_{n\alpha}}{\partial x_\beta}\Big)dV. \tag{14}$$

From this we obtain for the components of the Fourier coefficients $c_n(t)$

$$(-i\omega+\lambda_n)c_n(\omega)=\int\Big\{q_\alpha(\omega)\frac{\partial T_n}{\partial x_\alpha}-s_{\alpha\beta}(\omega)\frac{\partial v_{n\alpha}}{\partial x_\beta}\Big\}dV. \tag{15}$$

Using this expression and the correlation function of the components $q_\alpha(\omega)$ and $s_{\alpha\beta}(\omega)$, we get

$$\langle c_m(\omega)c_n^*(\omega')\rangle=4\pi(\Theta J_{mn}+\Phi K_{mn})\frac{\delta(\omega-\omega')}{(-i\omega+\lambda_m)(i\omega+\lambda_n)},$$
$$J_{mn}=\int\operatorname{rot}\mathbf{v}_m\operatorname{rot}\mathbf{v}_n\,dV,\quad K_{mn}=\int\nabla T_m\nabla T_n\,dV. \tag{16}$$

Let us return to the correlation relations for the coefficients $c_n(t)$:

$$\langle c_m(t)c_n(t+\tau)\rangle=\pi^{-1}(\Theta J_{mn}+\Phi K_{mn})\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}\,d\omega}{(\omega+i\lambda_m)(\omega-i\lambda_n)}. \tag{17}$$

For $\tau > 0$ we find, closing the integration contour in the upper half-plane

$$\langle c_m(t)c_n(t+\tau)\rangle=2(\Theta J_{mn}+\Phi K_{mn})\frac{e^{-\lambda_n\tau}}{\lambda_m+\lambda_n}. \tag{18}$$

With the aid of (18) and (11) we can calculate the correlation relations for the temperature and for arbitrary velocity components. For example,

$$\langle T(\mathbf{r}_1,t)T(\mathbf{r}_2,t+\tau)\rangle=\sum_{m,n}\langle c_m(t)c_n(t+\tau)\rangle T_m(\mathbf{r}_1)T_n(\mathbf{r}_2). \tag{19}$$

The behavior of the function (19) depends on the value of τ and turns out to be essentially different for the

[1)]This result, predicted by Landau on the basis of rather general considerations [3], was obtained in the theory of thermal convection by Sorokin [4].

[2)]Relations (9), which were obtained in [5], are valid, strictly speaking, only for a liquid that is in thermodynamic equilibrium. We can, however, also employ these formulas for small deviations from equilibrium, when each physically infinitesimally small volume is in the state of local equilibrium. We note that the use of the macroscopic equations of motion of the liquid already presupposes satisfaction of this condition.

discrete and continuous decrement spectra. In this section we consider the case of a discrete spectrum.

If the neighboring levels λ_0 and λ_1 are separated by a finite interval, as is certainly the case for cavities whose linear dimensions are of the same order, then the relation $(\lambda_1 - \lambda_0)\tau' = 1$ determines the characteristic time τ'. Near the equilibrium-stability boundary, where the critical-perturbation decrement is small, the main contribution to a sum such as (19) at times $\tau > \tau' \approx \lambda_1^{-1}$ is made by the term with $m = n = 0$, for which

$$\langle c_0(t)c_0(t+\tau)\rangle = (\Theta J_{00} + \Phi K_{00})e^{-\lambda_0\tau}/\lambda_0. \tag{20}$$

For values $\tau < \tau'$, and particularly when calculating the simultaneous correlation, it is necessary to sum the infinite series (19). It is easily understood, however, that consideration of simultaneous fluctuations at different points of an incompressible liquid has no physical meaning at all, since the signal propagation velocity in such a medium would be infinitely large. Even the incompressibility condition div v = 0 implies that the velocity fluctuations are correlated in the entire volume of the liquid. For the same reason, the simultaneous temperature fluctuations, which are connected with the velocity fluctuations by the general system of Eqs. (8), are likewise not δ-correlated. We note that in the absence of heating (R = 0), when the temperature fluctuations are determined independently from the equation

$$P\frac{\partial T}{\partial t} = \Delta T - \operatorname{div}\mathbf{q}, \tag{8'}$$

we obtain for the simultaneous correlation[3)]

$$\langle T(\mathbf{r}_1)T(\mathbf{r}_2)\rangle = \Phi P^{-1}\delta(\mathbf{r}_2 - \mathbf{r}_1).$$

For the thermal fluctuations we obtain from this, returning to dimensional notation

$$\langle T(\mathbf{r}_1)T(\mathbf{r}_2)\rangle = \frac{kT^2}{\rho c_p}\delta(\mathbf{r}_2 - \mathbf{r}_1),$$

which coincides with the usual expression for the thermodynamic temperature fluctuations.

It is clear from the foregoing that the correlation relations of the type (19) are meaningful only for values of τ such that the condition $c\tau \gg |\mathbf{r}_2 - \mathbf{r}_1|$, is satisfied, where c is the velocity of sound in the liquid.

Let us continue the analysis of the fluctuations at $\tau > \tau'$. At Rayleigh numbers close to critical, we can put $R = R_0$ in the integrals $J_{00}(R)$ and $K_{00}(R)$. Then $J_{00} = K_{00}$, as follows directly from equations (2) at $\lambda_0 = 0$:

$$\int (\operatorname{rot}\mathbf{v}_0)^2\,dV = \int (\nabla T_0)^2\,dV = D_0\int \gamma\mathbf{v}_0T_0\,dV.$$

Substituting expression (5) for λ_0 in (20), we get

$$\langle c_0(t)c_0(t+\tau)\rangle = \frac{R_0(\Theta+\Phi)}{R_0 - R}\exp\left\{-\frac{J_{00}(R_0 - R)\tau}{R_0}\right\}, \tag{21}$$

after which the correlation functions for the temperature and for the velocity components take the form[4)]

$$\langle T(\mathbf{r}_1,t)T(\mathbf{r}_2,t+\tau)\rangle = \frac{R_0(\Theta+\Phi)}{R_0 - R}e^{-\tau/\tau_c}T_0(\mathbf{r}_1)T_0(\mathbf{r}_2), \tag{22}$$

$$\langle v_\alpha(\mathbf{r}_1,t)v_\beta(\mathbf{r}_2,t+\tau)\rangle = \frac{R_0(\Theta+\Phi)}{R_0 - R}e^{-\tau/\tau_c}v_{0\alpha}(\mathbf{r}_1)v_{0\beta}(\mathbf{r}_2),$$

$$\tau_c = R_0/J_{00}(R_0 - R).$$

As seen from these formulas, the amplitude of the fluctuations occurring in an immobile liquid increases in inverse proportion to the distance from the critical point; the correlation time τ_c increases in accordance with the same law.

For the thermal fluctuations, the correlation relations simplify somewhat, since the ratio

$$\Phi/\Theta = g\beta T/c_pA_0$$

is always small in the case of an incompressible liquid. This means that the thermal fluctuations of the velocity and of the temperature are determined mainly by the tensor $s_{\alpha\beta}$, and the random heat fluxes q_α in Eqs. (8) can be omitted, Neglecting in (22) the value of Φ compared with Θ, we write out the correlation functions in dimensional form:

$$\langle T(1)T(2)\rangle = \frac{kTA_0\nu}{\rho l^3 g\beta\chi}\frac{R_0}{R_0 - R}e^{-\tau/\tau_c}T_0(\mathbf{r}_1)T_0(\mathbf{r}_2),$$

$$\langle v_\alpha(1)v_\beta(2)\rangle = \frac{kT}{\rho l^3}\frac{R_0}{R_0 - R}e^{-\tau/\tau_c}v_{0\alpha}(\mathbf{r}_1)v_{0\beta}(\mathbf{r}_2). \tag{23}$$

It is clear, therefore, that growth of the fluctuations near R_0 is inevitable even if the external actions that disturb the equilibrium of the liquid are very carefully eliminated. A crisis-free transition through $R = R_0$, the possibility of which is sometimes discussed in the literature, is not realizable to the extent that the thermal fluctuations cannot be suppressed.

If R exceeds R_0 slightly, the thermodynamic fluctuations arise against the background of the stationary motion (6) and are described by the equations

$$\frac{\partial v_\alpha}{\partial t} + \varepsilon\left(v_\beta\frac{\partial v_{0\alpha}}{\partial x_\beta} + v_{0\beta}\frac{\partial v_\alpha}{\partial x_\beta}\right) = -\frac{\partial p}{\partial x_\alpha} + \Delta v_\alpha + D\gamma_\alpha T + \frac{\partial s_{\alpha\beta}}{\partial x_\beta},$$

$$P\frac{\partial T}{\partial t} + P\varepsilon(\mathbf{v}\nabla T_0 + \mathbf{v}_0\nabla T) = \Delta T + D\gamma\mathbf{v} - \operatorname{div}\mathbf{q},\quad \operatorname{div}\mathbf{v} = 0. \tag{24}$$

It is convenient to seek the solution of this system, as before, in the form of the expansion (11). The amplitudes $\{\mathbf{v}_n, T_n, p_n\}$, in terms of which it is now necessary to carry out the expansion, are determined as the eigenfunctions of the homogeneous boundary-value problem corresponding to the system (24). At nonzero

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3) Let us show how this result is obtained. We expand T in (8′) in the eigenfunctions of the equation $-\lambda PT = \Delta T$, which satisfies the orthonormalization conditions

$$P\int T_mT_n\,dV = \delta_{mn},\qquad K_{mn} = \lambda_n\delta_{mn}.$$

We now have in place of (19)

$$\langle T(\mathbf{r}_1,t)T(\mathbf{r}_2,t+\tau)\rangle = 2\Phi\sum_{m,n}K_{mn}\frac{e^{-\lambda_n\tau}}{\lambda_m+\lambda_n}T_m(\mathbf{r}_1)T_n(\mathbf{r}_2)$$
$$= \Phi\sum_n e^{-\lambda_n\tau}T_n(\mathbf{r}_1)T_n(\mathbf{r}_2).$$

Putting here $\tau = 0$ and recognizing that the $T_n(\mathbf{r})$ form a complete system of functions with normalization (a), we obtain the formula given in the text.

4) We assume here that the ground level λ_0 is not degenerate.

values of ϵ, this problem becomes non-self-adjoint, and therefore we have in lieu of the orthogonality condition (4)

$$\int \{\mathbf{v}_m \tilde{\mathbf{v}}_n^* + PT_m \tilde{T}_n^*\} dV = \delta_{mn}, \quad (25)$$

where $\tilde{\mathbf{v}}$ and $\tilde{T}$ comprise the solution of the adjoint problem. Calculations analogous to those made above yield in place of (18)

$$\langle c_m(t) c_n^*(t+\tau)\rangle = 2(\Theta \tilde{J}_{mn} + \Phi \tilde{K}_{mn}) \frac{e^{-\lambda_n^* \tau}}{\lambda_m + \lambda_n^*},$$
$$\tilde{J}_{mn} = \int \operatorname{rot} \tilde{\mathbf{v}}_m \operatorname{rot} \tilde{\mathbf{v}}_n^* \, dV, \quad \tilde{K}_{mn} = \int \nabla \tilde{T}_m \nabla \tilde{T}_n^* \, dV. \quad (26)$$

Near R_0, the correlator $\langle c_0(t) c_0^*(t+\tau)\rangle$ is maximal, and, as can be seen from formula (7), the decrement λ_0 is real and $\mathbf{v}_0(R)$ and $\tilde{\mathbf{v}}_0(R)$ coincide, accurate to terms of order ϵ, with $\mathbf{v}_0(R_0)$. Therefore $\tilde{J}_{00}(R) \approx \tilde{K}_{00}(R) \approx J_{00}(R_0)$ and

$$\langle c_0(t) c_0(t+\tau)\rangle = \frac{R_0(\Theta+\Phi)}{2(R-R_0)} e^{-\tau/\tau_c}, \quad \tau_c = \frac{R_0}{2J_{00}(R-R_0)}. \quad (27)$$

The correlation functions for any pair of physical quantities are determined in the transcritical region by formulas of the type (22).

If the random forces are not δ-correlated, then relations (9) are replaced by

$$\langle s_{\alpha\beta} s_{\gamma\delta}\rangle = 2\Psi_1(\mathbf{r}_2 - \mathbf{r}_1, t_2 - t_1)(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}),$$
$$\langle q_\alpha q_\beta\rangle = 2\Psi_2(\mathbf{r}_2 - \mathbf{r}_1, t_2 - t_1)\delta_{\alpha\beta}, \quad \langle q_\alpha s_{\beta\gamma}\rangle = 0. \quad (28)$$

The functions Ψ_1 and Ψ_2 decrease when their arguments $r = |\mathbf{r}_2 - \mathbf{r}_2|$ and $t = |t_2 - t_1|$ increase. Let r_1, τ_1, and r_2, τ_2 be the characteristic radii and correlation times for the quantities $s_{\alpha\beta}$ and q_α. The random forces can obviously be regarded as δ-correlated if the times τ_1 and τ_2 are small compared with the fluctuation correlation time τ_c, and the radii r_1 and r_2 are small compared with the correlation radius r_c. As seen from the foregoing, the fluctuation correlation time is determined by the decrement of the critical perturbation $\tau_c = \lambda_0^{-1} \sim (R_0 - R)^{-1}$, and the correlation radius is $r_c \sim l$. Therefore, at sufficiently small $|R_0 - R|$, the conditions $\tau_1, \tau_2 \ll \tau_c$ are satisfied for all finite τ_1 and τ_2. If in addition $r_1, r_2 \ll l$, then practically any random action can be regarded as δ-correlated, and Φ and Θ should be taken to mean

$$\Theta = \int \Psi_1(\mathbf{r}, t)\, d\mathbf{r}\, dt, \quad \Phi = \int \Psi_2(\mathbf{r}, t)\, d\mathbf{r}\, dt. \quad (29)$$

3. FLUCTUATIONS IN THE CASE OF A CONTINUOUS DECREMENT SPECTRUM

By way of an example of a problem with a continuous eigenvalue spectrum, let us consider a flat horizontal layer of a liquid heated from below. If the horizontal layer dimensions $L_x = L_y = L$ are much larger than its thickness l, then the dependence of the perturbation amplitudes on x and y is determined by the formulas

$$\mathbf{v}_n = L^{-1}\mathbf{u}_\nu(z, k) e^{i\mathbf{kr}}, \quad T_n = L^{-1} T_\nu(z, k) e^{i\mathbf{kr}}, \quad (30)$$

where $\mathbf{k}$ and $\mathbf{r}$ are vectors in the (x, y) plane, the index ν denotes the vertical structure of the perturbation, and n stands for the aggregate $(\nu, \mathbf{k})$. The system of functions (30) satisfies the orthogonality conditions

$$\int \{\mathbf{v}_m \mathbf{v}_n^* + PT_m T_n^*\} dV = \delta_{\mu\nu}\delta_{\mathbf{kk}'}. \quad (31)$$

The correlation relations for the coefficients $c_n(t) \equiv c_{\nu\mathbf{k}}(t)$ are determined by the previous formula (18), where now

$$J_{mn} = I_{\mu\nu}\delta_{\mathbf{kk}'}, \quad I_{\mu\nu} = \int \left(\frac{d\mathbf{u}_\mu}{dz}\frac{d\mathbf{u}_\nu^*}{dz} + k^2 \mathbf{u}_\mu \mathbf{u}_\nu^*\right) dz, \quad (32)$$

and expression (19) for the temperature fluctuations takes the form

$$\langle T(\mathbf{r}_1, z_1, t) T(\mathbf{r}_1 + \mathbf{r}, z_2, t+\tau)\rangle$$
$$= \frac{2\Theta}{L^2} \sum_{\mu,\nu} \sum_{\mathbf{k}} I_{\mu\nu}(k) \frac{e^{-\lambda_\nu(k)\tau}}{\lambda_\mu(k) + \lambda_\nu(k)} e^{-i\mathbf{kr}} T_\mu(z_1) T_\nu^*(z_2). \quad (33)$$

For brevity, we shall henceforth assume $\Theta \gg \Phi$, as in the case, for example, of thermal fluctuations.

The equilibrium crisis sets in at $R = R_0$, when the decrement $\lambda_0(k_0)$ goes through zero. At R close to R_0 and at k close to k_0, the corrections to $\lambda_0(R_0, k_0)$ can be taken into account independently:

$$\lambda_0(R, k) = \frac{R_0 - R}{R_0} I_{00}(k_0) + \beta(k - k_0)^2. \quad (34)$$

This formula reveals the characteristic region of spatial dispersion

$$\Delta k = [I_{00}(R_0 - R) / \beta R_0]^{1/2}.$$

The decrement spectrum can be regarded as continuous if L is large compared with the other characteristic horizontal dimensions, i.e., $L \gg k_0^{-1}$ and $L \gg (\Delta k)^{-1}$. Since $k_0 \sim 1$, the first inequality simply means that the length of the layer should be much larger than its height. The second condition is less trivial. When written in the form

$$(R_0 - R) / R_0 \gg \beta / L^2 I_{00}, \quad (35)$$

it limits the closeness of R to R_0. When condition (35) is violated, the spectrum $\lambda_\nu(k)$ should be regarded as discrete. One can then use the results of the preceding section, taking into account the degeneracy with respect to the direction of the vector $\mathbf{k}_0$.

Assuming the condition (35) to be satisfied, we change over in (33) to integration with respect to $\mathbf{k}$. Retaining in the sum over $\mu = \nu = 0$, we obtain

$$\langle T(\mathbf{r}_1, z_1, t) T(\mathbf{r}_1 + \mathbf{r}, z_2, t+\tau)\rangle$$
$$= \frac{\Theta}{4\pi^2} I_{00}(k_0) T_0(z_1) T_0^*(z_2) \int \frac{e^{-\lambda_0\tau}}{\lambda_0} e^{-i\mathbf{kr}} d\mathbf{k}. \quad (36)$$

We have taken the functions $I_{00}(k)$ and $T_0(z, k)$ at the point k_0 outside the integral sign, since only a small vicinity of k_0 is important in the integration. Let us examine this integral. Using expression (34) for $\lambda_0(k)$ and introducing the notation

$$r_c = \left[\frac{\beta R_0}{I_{00}(R_0 - R)}\right]^{1/2}, \quad \tau_c = \frac{R_0}{I_{00}(R_0 - R)}, \quad (37)$$

we obtain

$$\tau_c e^{-\tau/\tau_c} \int_0^\infty k\, dk \frac{e^{-\beta\tau(k-k_0)^2}}{1 + r_c^2(k-k_0)^2} \int_0^{2\pi} e^{-ikr\cos\varphi} d\varphi$$
$$= 2\pi\tau_c e^{-\tau/\tau_c} \int_0^\infty J_0(kr) \frac{e^{-\beta\tau(k-k_0)^2}}{1 + r_c^2(k-k_0)^2} k\, dk.$$

For values of r satisfying the inequality $k_0 r \gg 1$, we can use the asymptotic representation of the Bessel function $J_0(kr)$. Introducing a change of variables $\xi = r_c(k - k_0)$ and taking into account the rapid convergence of the integral, we obtain after simple transformations

$$\frac{4\pi k_0\tau_c}{r_c} e^{-\tau/\tau_c} J_0(k_0 r) \int_0^\infty \frac{e^{-\xi^2\tau/\tau_c}}{1+\xi^2} \cos\left(\frac{\xi r}{r_c}\right) d\xi.$$

The last integral is known[6]. We present the final expression for the temperature correlation function (36)

$$\langle T(1)T(2)\rangle = \frac{\Theta I_{00} k_0 \tau_c}{2r_c\sqrt{\pi}} J_0(k_0 r) T_0(z_1) T_0^*(z_2) \tag{38}$$

$$\times\left\{e^{-r/r_c}\operatorname{Erfc}\left[\sqrt{\frac{\tau_c}{\tau}}\left(\frac{\tau}{\tau_c}-\frac{r}{2r_c}\right)\right] + e^{r/r_c}\operatorname{Erfc}\left[\sqrt{\frac{\tau_c}{\tau}}\left(\frac{\tau}{\tau_c}+\frac{r}{2r_c}\right)\right]\right\},$$

where

$$\operatorname{Erfc}(x) = \frac{\sqrt{\pi}}{2}\left(1 - \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt\right).$$

The character of the correlation function depends essentially on the ratio of r and τ. For small τ, when $r/\tau \gg r_c/\tau_c$, we obtain from the general formula (38), using the asymptotic form of the error integral[7],

$$\langle T(1)T(2)\rangle = {}^1\!/_2 \Theta k_0 r_c e^{-r/r_c} J_0(k_0 r) T_0(z_1) T_0^*(z_2). \tag{39}$$

This expression does not contain τ and describes the exponential damping of the correlation with the distance. For large values of τ, i.e., at $r/\tau \ll r_c/\tau_c$, the attenuation of the correlation becomes asymptotic in time

$$\langle T(1)T(2)\rangle = \frac{1}{2}\Theta \frac{R_0}{R_0 - R} \frac{k_0 e^{-\tau/\tau_c}}{\sqrt{\pi\beta\tau}} J_0(k_0 r) T_0(z_1) T_0^*(z_2). \tag{40}$$

In this limiting case the attenuation of the spatial correlation is determined only by the slowly decreasing Bessel function.

As seen from the last two formulas, the quantities r_c and τ_c, defined in (37), have the meaning of the correlation radius and correlation time. As R approaches R_0, they increase without limit. We note that the condition (35), as expected, means that $L \gg r_c$.

To determine the constants in (38) it is necessary to solve the corresponding boundary-value problem. The simplest solution, which was obtained already by Rayleigh[8], is obtained for a layer with free boundaries maintained at specified temperatures. In this case

$$R_0 = 27\pi^4/4, \quad k_0 = \pi/\sqrt{2},$$

and for the dimensional values of the radius and the correlation time we obtain

$$r_c = \frac{2l}{\pi}\left(\frac{2R_0}{R_0 - R}\right)^{1/2}, \quad \tau_c = \frac{3\pi^2 l^2 \chi}{2\nu(\nu+\chi)} \frac{R_0}{R_0 - R}.$$

In conclusion we note that the results presented in the article cannot be extended to the immediate vicinity of the point R_0. When $|R_0 - R|$ is close to zero, the fluctuations become large, and it becomes necessary to take into account the nonlinear terms in the hydrodynamic equations. More important, however, is the fact that near the convection threshold, where the radius and the correlation time increase strongly, the macroscopic equations themselves may turn out to be inapplicable.

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